

**Structural Knowledge about Shapes:  
A Case Study of Young Children  
Describing, Constructing and Reflecting on Squares**

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I hereby declare that, except where explicit attribution is made, the work presented in this thesis is entirely my own.

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## **Abstract**

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The consensus in existing literature is that children's limited, and often appearance-based descriptions of shapes indicate that children view shapes as a whole and lack understanding of shape structure. This study approaches children's understandings of shapes from a different perspective, based on an alternative and more dynamic interpretation of the van Hiele model and with the acknowledgement that there might be multiple ways of knowing and expressing mathematical knowledge.

This study examines the understandings young children have about the structure of shapes. It studies how this knowledge is expressed, and how it can be used in the process of constructing squares. Fifty-two children were engaged in three phase naturalistic task-based interviews. In Phase A (Description Task) the children were involved in classification and shape recognition activities. In Phase B (Construction Task) the children were asked to construct squares with the use of sticks and, in Phase C (Reflection Task) the children were asked to reflect on the construction process of Phase B. Even though during Phase A, the children, as supported by existing research, exhibited limited understanding about the structure of squares, through their involvement in Phase B, they exhibited much richer intuitive structural understandings. In Phase C, children tended to express structural understandings about squares. However the structural understanding that they exhibited was expressed in diverse and inventive ways.

These findings challenge the view that children's limited verbal descriptions of shapes indicate lack of structural understanding. In the process of the interviews, the children articulated through the 'language' provided, existing intuitive structural knowledge of squares and at the same time they were able to situate their abstractions in the context of construction. Overall the findings indicate that, provided sufficiently sensitive techniques are employed, it is possible for children to express structural knowledge in diverse and often unconventional ways.

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# CHAPTER 1

## INTRODUCTION

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*Walk upstairs, open the door gently, and look in the crib. What do you see? Most of us see a picture of innocence and helplessness, a clean slate. But, in fact, what we see in the crib is the greatest mind that has ever existed, the most powerful learning machine in the universe.*

*(The Scientist in the Crib, Copnik, et al, 1999)*

### 1.1. Background of the study: Unravelling a riddle

Since I had decided that the subject of this thesis would be young children's understandings of shapes, at a very early stage of reviewing the literature, as one would expect, I came across the van Hiele model<sup>1</sup> of geometric thinking. That was when my brother had his first child. So at about the same time I had the opportunity to observe my first niece learn in that rapid way babies learn, I found myself reading 'Structure and Insight' (van Hiele, 1986). That is partly why I rebelled when I got to the point where van Hiele (1986) states that *'thinking without words is not thinking'* (p.9).

How can there be learning without thinking and therefore how could my niece be learning when she was not thinking, oblivious as she was of any words? Now I have a daughter of my own. During infancy, most of the time I had no idea what she was thinking, but I was certain that she was indeed thinking because I watched her learn new things day in day out. I watched her make decisions, intentionally choosing and using objects during her play. I even witnessed her ability to invent ways to play. She was thinking and she was learning from day 0. Babies are beyond a doubt the most expert, innate scientists, even though they cannot express or think in words.

When my daughter was two years old, she could use language to communicate, but still there was so much evidence that often she was not thinking in words. One of her favourite games at the time was a Farm Felt Set where she could use different figures to create a farm scene. All

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<sup>1</sup> Two mathematics teachers, Pierre van Hiele and Dina van Hiele-Geldof from the Netherlands, developed the van Hiele model of geometric thinking in the late 1950's based on their own practice. As reported by Crowley (1987) 'the van Hiele model of geometric thought emerged from the doctoral works of Dina van Hiele-Geldof and Pierre van Hiele, which were completed simultaneously at the University of Utrecht. Since Dina died shortly after finishing her dissertation, it was Pierre who classified amended and advanced the theory' (p.1).

figures were provided four times. There were four cows, four horses, four farmers, four dogs and so on. What she enjoyed doing at the time was finding the same. When she would find three dogs she would say 'Where is the other one?', or 'There is one missing'. When she would find the fourth figure she would stop searching. My daughter, as any other two year-old, could not count at that point (at least not in the conventional way). Or better say, she was in no position to think in words: 'There are four dogs all together. I have three. There is one missing'.

Because of my everyday experiences with babies and young children in my personal and work environment, I could never agree with the statement that thinking without words is not thinking. Nevertheless, I will not simply reject it. It is interesting to understand what exactly this statement is saying, and where it comes from. For me there is definitely a paradox in van Hiele's statement. He somehow admits that it is possible to think without words ('thinking without words') but he just cannot capture this kind of thinking as ..... thinking, as 'actual' thinking. I think even for van Hiele this paradox was vivid since, though he supports the statement that thinking without words is not thinking, he also states that 'an opponent may maintain that it is only a matter of choice'. This paradox of the existence of thinking without words that is not 'actual' thinking was just a riddle that van Hiele 'chose' not to work out in 'Structure and Insight'.

But it just happened to be that this was the riddle that made me want to look into young children's understanding of shapes in the first place. A riddle that puzzled me long before I came across 'the van Hiele paradox'. The riddle arose from the numerous opportunities I had as a teacher-educator, on a daily basis, to observe preschool children (three to six year-olds) interact with (student) teachers and knowledge within school settings. A paradigmatic incident is the one of the preschool child constructing a square with accuracy and without 'actually thinking' about it.

Preschool children often construct shapes with accuracy but 'have no knowledge' (if we measure knowledge by verbal language) of concepts like angle, right angle, parallel lines, etc. Many times I have seen children make the right choices when constructing a square; like 'intuitively' selecting four equal sticks out of a set of numerous sticks of different lengths (the term 'intuitively' is intentionally used here as an alternative to the phrase 'without real thinking' that others might have used in this case). Some of these children would fail to describe the square as the shape with four equal sides and four right angles even after having actually constructed it.

This inconsistency between what children can do and what they can say is even more obvious in younger children. When my niece was two years and ten months old, I asked her to construct a

square. All I knew was that she was able to recognise the square among other shapes. I gave her wooden sticks of various lengths from which you could find many sets of four equal ones. She had no difficulty picking up four of them that were equal and constructing a square. It was something I did not expect her to be able to do. Even for me, for whom it's been said, more often than not, that I have high expectations and think too highly of young children, this was striking. So I asked my niece to construct three squares in a row to make sure it was not accidental. All of her attempts were successful and swift. Even though I was perfectly aware of her vocabulary repertoire, I did try to make her put what she did in words, but it was impossible for her<sup>2</sup>.

So what is the answer to the riddle? Let us deal with 'the van Hiele paradox', as a play on words. How can the following sentence gain some sense?

'Thinking without words is not thinking.'

It can gain sense only if what is before the 'not' is different, in some way, to what comes after it. Just like the sentence 'an apple tree is not an orange tree' makes sense because it is referring to two different types of trees. Similarly, van Hiele's statement would make sense if he were referring to two different kinds of thinking. So the answer could be in a missing word between 'not' and 'thinking'

'Thinking without words is not \_\_\_\_\_ thinking'

One way of completing the sentence is by using words like 'actual', 'real', 'true'. We could, therefore, say that 'thinking without words is not real thinking' as van Hiele (1986) did himself following a discussion he had with Freudenthal on the matter. As van Hiele (1986) reports, Freudenthal's point of view was that 'one can do adequate things without words but real thinking is arguing and that you can't do without words' (p.10). But if we accept this claim, then we would be talking about the existence of at least two ways of thinking, of which only one is acceptable, correct or of any significance in the process of learning and concept formation. In relation to Freudenthal's claim, there are two points that are debatable: the existence of 'real thinking' and that one can argue only with words. Nevertheless, I will agree with Freudenthal that one 'can do adequate things without words'.

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<sup>2</sup> I must admit that my niece had exceptional ('visual') abilities from a very early age. Before the age of four she could solve number problems mentally without the use of real objects. But despite her exceptional abilities, there was still a gap between what she could do and what she could say.

Alternatively to filling in the gap by using the word 'real', I would prefer, at this point, to leave the gap that seems to exist in van Hiele's claim empty, since for me the only reasonable, obvious and 'harmless' way to fill in the gap is the following:

'Thinking without words is not verbal thinking!

This way of filling in the gap does not diminish non-verbal thinking in comparison to (verbal) thinking, but at the same time it does not say much, besides the most obvious. At this point, I shall restrict this discussion simply to the acknowledgment that there might be several ways of thinking, an acknowledgement which opens new doors and perspectives to understanding 'understanding'.

Even though my first reaction when I got to page 9 of 'Structure and Insight', where van Hiele claims that 'thinking without words is not (real) thinking', was to close the book, put it aside and refuse to read any more of it, I soon realised that I could not claim to be studying children's understandings of shapes without reading the whole thing. Almost every existing study on geometry refers to the van Hiele model of geometric thinking. I could, therefore, not ignore it and base my study solely on other researchers' interpretations. So I returned to 'Structure and Insight' and read beyond that page. This turned out to be a wise move, since I surprisingly discovered some really interesting aspects of the van Hiele model which were absent from existing van Hiele-based studies. I gradually realised that the theory unfolded by van Hiele beyond page 9 had nothing to do with his opening claim that 'thinking without words is not thinking' and that van Hiele had important things to say and add to the discussion concerning conceptual change.

All theories on conceptual change refer to the role of prior knowledge in the process of learning. Despite the widespread agreement that children are not blank slates for teachers to write on, there is a variation concerning the frame of mind with which different cultures face children's existing knowledge. Different theories use different words when referring to children's prior knowledge and it is interesting to see what the choice of words implies. We could roughly say that some of the terms used carry a negative connotation (it is interesting to note that in a thesaurus, the word 'negative' is a synonym of the word 'unconstructive') while others are more on the positive side (similarly, in a thesaurus 'positive' is a synonym of 'constructive').

One common word used for children's prior knowledge is the word 'misconception', a term which definitely carries a negative feeling, the feeling of wrongness. Research cultures that adopt this



negative feeling include the use of phrases like 'real thinking' and the description of hierarchies which are made of lower and higher levels of thinking. At a first look, van Hiele can fit in this research culture. His reference to the existence of real thinking and his proposed model of thinking levels, for which he is mostly known, have, as we shall see further on (Chapter 2), led researchers to fit van Hiele in this 'misconception' research culture.

On the other hand, the word intuition is used when referring to children's existing knowledge. Even in everyday language, the word intuition bears a sense of power, a sense of having a unique gift, an exceptional ability that leads to correct and constructive answers. Teachers often refer to the intuitive student, who has a unique ability, an insight into solving problems. Whenever we are faced with abilities which we cannot really describe or determine, 'foggy' understandings of which we cannot identify the origin, but that nevertheless seem to be correct, we tend to talk of 'intuition'. Within a constructive approach<sup>3</sup>, only a positive attitude towards children's existing knowledge can be fruitful.

Thus, one would not expect of van Hiele, who originally placed at the centre of his theory the conviction that 'thinking without words is not thinking', to have anything to say about non-verbal thinking. Isn't it a paradox then that in the same book ('Structure and Insight', 1986) van Hiele, on the one hand underestimates non-verbal thinking and, on the other, glorifies the importance of intuition in the process of thinking? Despite his opening claim that 'thinking without words is not thinking' (p.9) van Hiele devotes a substantial part of his book to intuition and defines it as a way of 'seeing the solution to a problem directly, but without being able to tell' (p.76). These paradoxes in van Hiele did not seem to bother van Hiele-based research. The so called van Hiele-based research chose the van Hiele version that led straight to misconception theory, a theory based on the conviction that knowledge, in general, consists of predefined 'truths' and that mathematical knowledge, in particular, is subjective.

Besides the tendency within research concerning geometric understanding to assess children, that favoured the conviction of the existence of such a thing as 'real thinking' or 'real knowledge', there is a related methodological issue to be considered. Most existing research used verbal data to 'assess' children regardless of their age, experiences or individual differences and learning styles. As a result, we had the assignment of children to levels according to their verbal claims.

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<sup>3</sup> The phrase 'constructive approach' refers to the effort of finding constructive ways of using children's existing knowledge in the process of learning.

Among van Hiele-based research there is widespread acceptance of the existence of a visual level. The visual level (or base level, or Level 0) is the first of the five levels proposed by van Hiele. According to van Hiele (1986) himself, at the first level children will simply say: 'This is a square', without any further explanation and without 'being able to mention even one property of' (p.62) the shape. Consequently, van Hiele-based research has defined Level 0 as the level where children simply 'visually' recognise and describe shapes based on their appearance ('This is a square because it looks like a window'), see shapes as a whole and pay no attention to shape properties. As we shall see in Chapter 2, this interpretation of visual recognition is widely accepted by research concerned with young children and shapes. The question raised is whether visual recognition can simply be the ability to recognise a shape as a whole. This implies that it is possible for someone to be able to see and recognise a shape without having any awareness of the shape's structure. Is this truly possible?

Personal experiences of observing children while attempting to construct shapes have provided me with interesting insight in relation to the previous question. As described earlier in this chapter, often children that are 'simply able of visually recognising a square', as in the case of my niece (p.3), are able to make all the right choices when constructing a square while being, at the same time, totally unable to verbally refer to any of the shape's properties. My experience in relation to children's attempts to construct shapes led me, initially, to the hypothesis that **judging a shape by its appearance does not exclude structural understanding**. On the contrary, children base their constructions and their attempt to imprint their mental representations (how a shape looks) on specific structural knowledge.

Observing children express their understandings in alternative ways (through construction, for example) led me to the hypothesis that visual understanding and shape recognition might not be so 'purely' lower forms of knowing, which exclude structural understanding. Therefore, in the same way I suggested earlier the existence of multiple ways of thinking, I would also like to suggest the existence of multiple ways of expressing. In a foreword to a book on 'The Hundred Languages of Children' (C. Edwards, L. Gandini & G. Forman (eds), 1998), Howard Gardner shares the opinion that early childhood teachers should 'know how to listen to children, how to allow them to take the initiative, and yet how to guide them in productive ways' (p.xvii). One can interpret this point of view in different ways; just like the word 'listen' would normally make one think about verbal communication. But having in mind that Gardner expresses this point of view in a book entitled 'The Hundred Languages of Children', one can imagine how many different meanings the word 'listen' and the phrases 'take the initiative' and 'guide' can actually acquire.

Mathematical discourse (within a didactical or research setting, if one can distinguish the two), which allows an effective interaction between teacher and/or researcher, child and knowledge, should not only be about asking and answering in words. An effective mathematical discourse should allow (a) the teacher and/or researcher to better understand children's understandings and, thus (b) the children to express their existing understandings in different ways, in their attempt to build on them and transform old knowledge into new knowledge.

Besides computer-based research (which usually involves older children), most existing studies saw no meaning in asking children to construct shapes in order to describe their geometric thinking. Verbal expression was considered as an adequate means of evaluating children's understanding. If a child cannot refer to any of the shape's properties when s/he (verbally) describes it, then s/he has no knowledge of them. Thus, this study builds on a more sensitive approach to how young children think and the different ways they use to express their thinking. Even though this study is not concerned with computers, it relies on the conviction that by giving children powerful tools you provide them with access to powerful ideas. As supported by Noss and Hoyles (1996), the computer has allowed 'glimpses to new epistemologies' and 'opened new windows on the construction of meanings'.

The fact that the study which unfolds in the following chapters has to do with young children is very closely related to the epistemological and methodological choices. There is an important issue to consider when dealing with young children's verbal reasoning, in particular, and the way they express knowledge in general. In 'Children's minds' Margaret Donaldson (1979) supports that as far as young children are concerned, adults should be careful with '*what is said and what is meant*'. As a teacher-educator for many years, I have the opportunity to observe children communicate knowledge in school settings on a daily basis. I have an enormous number of examples to support Margaret Donaldson's portrait of children's minds. Following is an example from my 'treasure box'.

In a preschool classroom with four to five year-olds, one of my student teachers was 'teaching' squares. After a construction activity, where the children used sticks to construct squares, she was trying to 'make' them describe their choices<sup>4</sup>. Even though in most cases the children had constructed squares with no difficulty, she was not getting the 'correct' answer – the answer she was looking for ('four equal sticks'). This is part of the conversation she had with one of the

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<sup>4</sup> As one might realise later on, this teaching incident has many commonalities with the research study I will describe in this thesis. As this study was evolving parallel to my years as a teacher-educator, my practice has influenced my research and vice versa.

children, Michael. For the purposes of this discussion, only the child's verbal utterances are included in the description below.

Student teacher: *Why didn't you use these sticks to make a square (she shows him two sets of equal sticks)?*

Michael: *It won't fit.*

Student teacher: *Do you think you will be able to make a square?*

Michael: *No I think I will not be able. **It will be tall!***

For an adult (teacher and/or researcher) looking for 'correct' answers the child's response would be wrong and meaningless. But as stated by Bruner (1983), 'it requires a sensitive teacher to distinguish an intuitive mistake – an interestingly wrong leap – from a stupid and ignorant mistake' (p.244). For a preschool teacher (and for a researcher investigating young children's thinking) no answers should be hastily considered wrong or meaningless. A way out of this difficult mission (of understanding what young children are saying) is to offer children alternative ways of communicating. Like for example the previous 'square construction' incident. As we shall see below, the use of manipulatives and construction available in the setting in which the incident took place allowed the child to better communicate and support his understandings. So this is what actually happened:

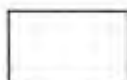
Student teacher: *Why didn't you use these sticks to make a square (she shows him two sets of equal sticks)?*

Michael: *It won't fit. Let's try!*

Student teacher: *Try then. Do you think you will be able to make a square?*

Michael: *No I think I will not be able.*

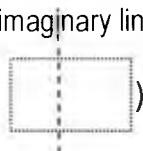
(He constructs a rectangle )



**It will be tall!** In order for it to be a square it had to be like this ...

(He creates an imaginary line by putting his hand vertical in the rectangle)

he constructed )



At some point in the past I had supported Clements' et al (1999) statement that 'because young children expose limited verbalisations and consequently there is ambiguity in the meaning of their utterances, there is a need for research using manipulatives and construction tasks'. The truth is that, at present, I can only support the methodological part of this statement and express a more reluctant position towards the epistemological part of it. I will support the fact that there is a need for research using manipulatives and construction tasks, a need which, after all, I aspire to meet with this study. But I fail to support any belief that refers to children's limitations especially when

referring to young children's ability to communicate. Children may have a limited vocabulary, but this does not necessarily mean that they have a limited ability to communicate.

On the contrary, my experiences have shown me that young children's communication ability is both inventive and rich. In many cases, they might not know how to say something in a way that an adult might understand (school language), but they can realise the adults' difficulty to understand what they are trying to say and exhibit an amazing ability of finding many ways of 'saying' or, better say, communicating the same thing. Adults often exhibit a more restricted and less flexible way of communicating. This leads to another powerful idea expressed by Margaret Donaldson; the idea that it is not only children that are egocentric but, on the contrary, egocentricity is also a characteristic of adults. In the previous incident, Michael exhibits the ability to 'decentre' when he suggests: 'Let's try!'. Michael feels that his verbal reasoning might not be clear to the adult so he uses construction to explain what he means. With construction activities, children find ways to communicate and support their understandings not only because adults encourage them to do so, but also because often they acknowledge themselves that they need to support their verbal reasoning using other means.

So my personal experiences have urged me to go beyond the simple (and rather obvious) realisation that .....

..... thinking without words is not verbal thinking.

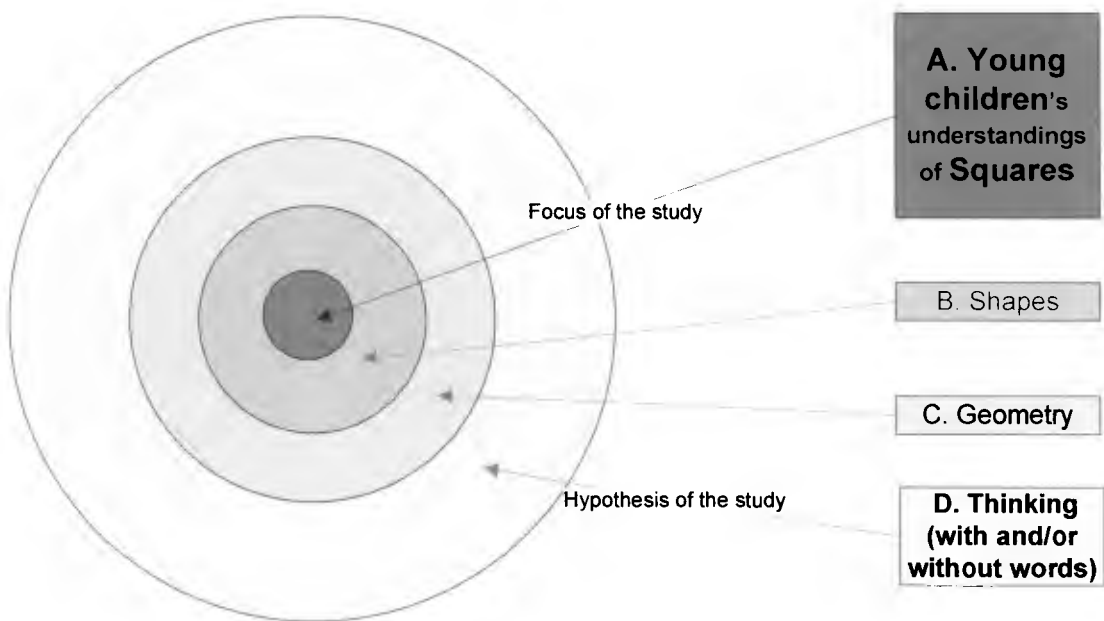
The question is not 'what thinking without words is not' but rather 'what thinking without words is'. In addition to the question of 'what thinking without words is' we also need to investigate the following question: 'what is its role in the more general process of thinking and learning'. My general aim, therefore, to describe and analyze young children's understandings of shapes is supported by the more general point of view that there might be multiple ways of knowing, thinking and expressing which have an equally important (not linear) role to play in the process of mathematical learning. So the observations previously discussed in this chapter have guided me through a fascinating trip into children's minds, leading to the realisation of how grand and complicated understanding is.

Thus, the need to investigate the hypothesis that children might think in alternative ways, and challenge the idea that thinking depends on language (thinking in words) has led to the study I will describe in the following chapters of this thesis. As a consequence of the observations that

originally cultivated my interest in this subject (some of which have been described in this chapter), this investigation was based on the case of squares.

The relationship between the original hypothesis which set this study off and the more focused investigation carried through, for the purposes of this thesis, is illustrated in Figure 1.

**Figure 1: Hypothesis and focus of the study**



So even though this study is shaped as an investigation of young children's understandings of squares (Figure 1, A), it also addresses the broader issue of understanding 'thinking' (Figure 1, D) which might not depend on words. Consequently, the investigation of children's understandings of squares will hopefully contribute to the broader domain of learning geometry (Figure 1, C) and, to be more precise, to the discussion concerning young children's understandings of shapes (Figure 1, B). The following part of this chapter provides an overview of the aims and design of this study.

## **1.2. Overview of the study**

In the previous section of this chapter, I have described the personal story, which led to the need for this study; the need for a study that aims to describe and analyze young children's understandings of shapes through an investigation of squares. Through the course of this study, this main aim has been analysed and re-stated at different levels. This is a procedure that will unfold through the following chapters of this thesis. At a preliminary stage, I had to transform this

aim into a set of research questions. As a result, a more focused purpose of this study was to investigate what knowledge young children have about the structure of simple shapes, how this knowledge is expressed, and how it is used in the process of constructing shapes.

Thus, fifty-two children were engaged in task-based interviews, consisting of a three-phase framework (Description-Construction-Reflection). In Phase A (Description Task), the children were involved in classification and shape recognition activities. This opening phase enabled subsequent data to be evaluated in comparison to those of existing research. In Phase B (Construction Task), the children were given wooden sticks of various lengths and were asked to construct squares. This allowed the children to express their understandings of shapes in alternative ways. Finally, in Phase C (Reflection Task), the children were involved in a process of reflecting on the construction process of Phase B. Consequently, a comparison between the ways the children expressed themselves about squares and the knowledge they expressed before the Construction Task could be made with the ways in which the children expressed themselves and the knowledge they expressed after the Construction Task.

Special attention was paid in relation to designing a research appropriate for the age of the subjects. During the research design, I had to constantly keep in mind the fact that this study was both a study within the domain of mathematics education as well as within that of early childhood education. Early childhood education research points towards the need of investigating children within their natural learning settings, while interacting with familiar adults. Therefore, in an effort to enable children to communicate in an authentic way, special arrangements were made that allowed naturalistic elements into the setting of the research design.

The interviews were conducted by a group of thirteen student teachers that I had trained for this purpose as part of a teacher training course they were attending at the University of Cyprus. The interviews were conducted in two public schools where the student teachers were 'working' as pre-service teachers. This allowed the inclusion of naturalistic elements in the research design and enabled the children interviewed to express their understandings while being involved in activities with familiar adults and in familiar settings. It also allowed for a far greater sample than would have been possible otherwise.

I hope that by the end of this thesis I will have demonstrated the richness that emerges from acknowledging the cognitive and epistemological resources that children bring to activities and the importance of identifying these resources.

### 1.3. Structure of the thesis

This thesis is organised into ten chapters. Chapter 2, following this introductory chapter, is a critical review of existing literature concerning young children's geometric thinking with an extensive review of van Hiele and van Hiele-based research. The aim is to provide a critical review of van Hiele-based research and give an alternative and more constructive interpretation of the van Hiele model. Chapter 2 concludes with important remarks concerning gaps in the existing literature on young children's geometric thinking, which point towards specific methodological suggestions.

Based on the concluding remarks of Chapter 2, in Chapter 3 we formulate the aims of the research study described for the empirical part of this thesis. An overview of the research study where fifty-two five to six year-olds were involved in naturalistic task-based interviews is described. A discussion in relation to the rationale supporting the research design described is also provided. Chapter 4 details the designing of (a) the task sequence used in the interviews of the main study, (b) the interview script tool used for facilitating the adult-child interaction and (c) the training programme for preparing the student teachers for the interviews, through the course of a piloting procedure which was completed in three phases. Following consideration of the evidence and findings provided by the piloting procedure, in Chapter 5 the research questions are reformulated in a more investigable manner. Parallel to this procedure of reformulating the research questions, the process of defining categories and sub-categories among the data is also described. Based on this categorising process, the coding scheme that was used for analysing the data of the main study is developed and foci for analysis are identified.

Chapters 6, 7 and 8 contain a detailed analysis of the data collected during the main study. This analysis is organised by means of the foci described in Chapter 5. As explained earlier (§1.2.), the task sequence consisted of a three-phase framework (Description-Construction-Reflection). Chapters 6, 7 and 8 correspond to each one of these phases respectively. This analysis is discussed in Chapter 9 and the research questions are addressed. Finally, Chapter 10 draws conclusions from the discussion and areas for further research are identified and developed by highlighting the main aspects, findings and implications for learning and teaching geometry.



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## **CHAPTER 2**

### **CHILDREN AND SHAPES: DYNAMIC ALTERNATIVE INTERPRETATIONS OF VAN HIELE**

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#### **2.1. Introduction**

In an effort to set the epistemological foundations of this thesis, this chapter provides a review of existing research and literature concerning young children's understandings of shapes. Even though it addresses a variety of both specific and broader issues in relation to children's thinking about geometric shapes it builds on a critical discussion of the van Hiele theory, mostly in relation to van Hiele-based research.

Although the van Hiele theory is a model synthesising a number of powerful ideas, the van Hieles are mostly remembered for their model of levels of geometric thinking. This was the aspect of the van Hiele theory that researchers, such as Burger & Shaughnessy (1986), Flores (1993), Gutierrez et al (1991), Gutierrez & Jaime (1998), Mayberry (1983), Teppo (1991), Fuys (1985), were mostly concerned with. As stated by Hoffer (1983), the work of the van Hieles has been communicated 'in terms of a stratification of human thought as a sequence of levels of thinking into which people may be classified' (p.205).

The question of whether knowledge consists of 'truths' goes way back in time. Every single theory, research or study on learning or thinking and mathematical knowledge can fit into this discussion. If we were to attempt to formulate a historical portrait of the epistemological query of whether mathematical knowledge is subjective or objective, one should embark on a voyage starting from the Greek philosophers, Plato and Aristotle, and moving on to the more recent supporters of the computer as a means for mathematical meaning-making, making stops in between, at key stations like 'misconception theory', 'ethnomathematics', 'street mathematics', etc. One such voyage is successfully attempted by Noss & Hoyles (1996). Extensive discussions on this epistemological query can also be found in Sierpiska & Lerman (1996). This chapter attempts, partly, to investigate the place of van Hiele in this discussion.

Studying van Hiele and van Hiele-based research, I came to the realisation that the van Hiele theory is characterised by certain paradoxes, which create uncertainties in relation to its epistemological background. The aim is to address these paradoxes and analyse how they have influenced van Hiele-based research. As an extension to identifying these paradoxes, I aim to emphasise significant dynamic aspects of the van Hiele theory, which were eliminated by van Hiele scholars, and connect them with important issues in relation to thinking and learning, in general, and children's thinking about geometric shapes, in particular.

## **2.2. Van Hiele Paradox A: Thinking with words *and/or* without words**

Many years ago, when my children were young, two of them were quarrelling. They asked me to be an arbiter. The first had said: "Every time we do not sleep, we are thinking." The other said: "This is not true; it may be that I am walking in the woods, but I do not think." The first: "You are thinking then, for you know it is the woods you are walking in, and to be able to know that you must have been thinking." The other: "Maybe, but for such knowledge I have not used words in my mind, so I have not been thinking."

At that time my idea was that the first child was right. You can act without thinking in words, but to do the right thing, it seems that thinking is necessary, even if without words. I told the story to H. Freudenthal and he agreed with the second girl. He said: "You can do adequate things without words, but real thinking is arguing, and that you can't do without words." I now think that he and the second daughter were right, although an opponent may maintain that it is only a question of choice.

(Van Hiele, 1986, p.9-10)

In an attempt to differentiate his model of levels of geometric thinking from Piaget's developmental hierarchy, van Hiele (1986), states that language is the only reliable route to children's thinking. In the first pages of his book 'Structure and Insight', after unfolding his philosophical dilemmas about language and thinking, van Hiele concludes with the statement that 'thinking without words is not thinking'; a statement that seems, as we shall see later on, to have been placed by many, either implicitly or explicitly, in the center of the van Hiele model.

It is, after all, reasonable that when one wishes to make a statement differentiating his stance from that of another, one will reject another's point of view and place him/herself on the opposite side. So since Piaget paid relatively little attention to language -something that was central in the critique of his theory -, van Hiele (1986), in an attempt to differentiate his position, made language a central part of his own model.

Piaget did not see the very important role of language in moving from one level to the next. It was occasionally suggested to him that children did not understand his questions. He always answered that they did understand; this could be read from their actions. But, although actions can be adequate, you cannot read from them the level at which children think. (p.5)

The paradox, now, lies in the fact that even if van Hiele declares himself confident in defending his choice concerning the absolutist importance of language, further in his book he takes a positive stance towards nonverbal thinking. A substantial part of his book is actually devoted to emphasising the value of intuition, a fact that has been eliminated by most van Hiele-based research. In the very same book in which van Hiele (1996) states that 'thinking without words is not thinking' (p.9), he also 'withdraws from the usage that gives intuition a negative connotation' (p.72) and defines intuition as the ability to 'see the solution to a problem directly, but without being able to tell'.

So here lies the first van Hiele paradox: on the one hand, van Hiele in his opening claim acknowledges verbal thinking as the only valuable way of thinking while, on the other, he supports intuition, which he defines as a way of 'thinking without words'. Van Hiele treats the aspect of 'intuition' in 'Structure and Insight' with seriousness. He analyses in detail what he calls 'the intuitive foundations of mathematics' and strongly supports that 'all rational knowledge begins with intuitive knowledge'. He explains how 'the intuitive approach begins with the presentation of a structure' (p.117) and supports the point of view that 'man is able to react directly - without the intermediary of a language - to visual structures' (p.127). As demonstrated later on, van Hiele was somehow aware of this paradox. In a paper published in 1999, he clearly admits that he was wrong in supporting that 'thinking without words is not thinking'.

Thinking without words is not thinking. In Structure and Insight (van Hiele, 1986), I expressed this point of view, and psychologists in the United States were not happy with it. They were right. (p.311)

Van Hiele's change of mind in this 1999 publication reinforced the belief that 'Structure and Insight' is characterised by paradoxes. But, moreover, as will be explained in more detail later on in this chapter, van Hiele's 1999 claim reinforced the attempt of connecting neglected aspects of the van Hiele theory with powerful ideas about thinking and learning and thus assigning van Hiele an alternative dynamic interpretation. But for the moment, let us describe the consequences that 'van Hiele paradox A' had on van-Hiele based research.

### **2.3. The consequences of 'van Hiele Paradox A' on van Hiele-based research**

The inability of van Hiele-based research to detect this paradox, van Hiele's own conflicting claims in 'Structure and Insight' concerning verbal and nonverbal thinking, his clear worries and thus a more promising, dynamic and pioneering side of the van Hiele model, all led to the reinforcement of a certain research culture in which verbal data was used to assign children to levels. Shaughnessy & Burger (1985) used children's verbal claims to conclude that the van Hiele thinking levels are adequate to describe children's geometric thinking. For example, according to the researchers, children that belong to Level 0 – visualisation, see a geometric figure as a whole and pay no attention to its components. What characterises children that belong to this stage are their (verbal) descriptions, which are 'purely visual'. 'If asked why he or she called a figure a rectangle, a student may reply, "Because it looks like a rectangle. It is like a window or a door"' (Shaughnessy & Burger, 1985, p.420). So for Shaughnessy & Burger (1985), the natural consequence is to conclude that these children see a figure as a whole and know nothing of the shape's components, since they are unable to verbalise any understanding of properties.

This type of research interpretation had deep consequences on the portrayal of children's understandings. Making lists of children's misconceptions led to a continuous restriction and limitation of what children know and are capable of. For example, Burger (1985) presents one misconception as a natural consequence of another.

Many children think of shapes as a whole without explicit reference to their components. Hence, such shapes as squares and rectangles are considered to be completely different (p.53).

Presumably, the 'logic' behind this interpretation is that since children see shapes as a whole, and squares and rectangles look different in the eyes of the children, then they are totally different. So the interpretation of their verbalisations is that children not only pay no attention to a shape's components but, also, that they can see no similarities between shape categories. This is indeed a very neat and tidy point of view; but does it do justice to children's minds? How can this interpretation explain why children can recognise a number of different rectangles (that look differently) as such?

Patronis (1985) expresses exactly the same point of view as Burger (1985) as an extension of the van Hiele theory. As stated by Patronis (1985), according to the van Hieles, at first, children do not realise a shape's properties. What they realise, at the beginning, is the whole of a shape.

Patronis (1985) goes on to state that consequently, it is **impossible** for children to see a square as a special kind of rhombus or rectangle. This interpretation not only assigns a hierarchical nature to the van Hiele model, but also presents the van Hiele system as a developmental age-related model. The first level of the model is assigned by Patronis (1985) to young children and is characterised by what children are unable of understanding. Whether the van Hiele model was indeed of a hierarchical nature is an issue I will refer to later on (§2.6.1). All these conclusions in relation to what children know - or better, do not know - about shapes were a consequence of their verbalisations and thus the acceptance of language as the only route to children's minds.

Clements et al (1999) analysed children's verbalisations in response to the interviewer's open-ended questions (e.g. 'How did you know that was [or was not] a rectangle?'). But Clements et al (1999) were led to the conclusion that 'there is a need for research using manipulatives and construction tasks'. This methodological remark is of great importance. Language is not the only way of expressing mathematical knowledge and, therefore, in order to get to children's representations, there is a need to provide multiple ways of expression.

Even though Clements et al (1999) recognised the need for research using other means of expression, thus opening new windows to children's understandings, the idea that children, at the first level of geometric thinking, view shapes as wholes and see no relationship between different shapes or between the properties of shapes dominates in many later publications by Clement and his colleagues. Clements & Sarama (2000) insist on describing the visual level as the level where children identify shapes based on their appearance and use justifications such as 'a rectangle is a rectangle because it looks like a door.' This idea of children seeing shapes as wholes is reiterated by Clements et al (2001) in a study where children were asked to compose different figures with the use of geometric shapes. In this study, the researchers created a hypothesised learning trajectory consisting of six levels/steps. Each level is characterised by the children's actions-on-objects in their attempt to compose geometric figures. The researchers state that children belonging to the first two steps of their trajectory 'view shapes only as wholes and see no geometric relationship between shapes or between parts of shapes' (p.275).

Van Hiele scholars not only based their research on the belief that 'thinking without words is not thinking', but also ignored all together van Hiele's extensive reference to intuition and his conviction that 'all rational thinking begins with intuitive knowledge'. Even though van Hiele's well-formed theory on intuition could be an important contribution to the wide existing literature on the role of intuition in mathematical and scientific thinking and learning (diSessa, 2000; Fischbein,

1987), it has been totally ignored. In an extensive review of literature on 'Geometry and Space', Clements & Battista (1992) make no reference to the aspect of intuition in the van Hiele theory. What is interesting is that, in the aforementioned paper, there is extensive reference to intuition within the context of Constructivism and Piaget and Inhelder's work.

## **2.4. Van Hiele paradox B: Subjectivity and/or objectivity**

The second paradox I have detected in 'Structure and Insight' is also connected with van Hiele's claim that 'thinking without words is not thinking'. This claim constitutes in itself a paradox. Van Hiele somehow admits that there is such a thing as thinking without words but he finds it difficult to label it as thinking. So how can the sentence: 'thinking without words is not thinking', gain some sense? As already stated in Chapter 1, it can gain sense only if what is before the 'not' is different, in some way, to what comes after it. So we need the missing word between 'not' and 'thinking'.

Thinking without words is not \_\_\_\_ thinking!

One way of completing the sentence is by borrowing from Freudenthal, as van Hiele (1986) did at some point, the word 'real'. So we could accept that ...

... thinking without words is not real thinking.

If we were, though, to accept this claim, then we would be talking about the existence of at least two ways of thinking, of which only one is acceptable, correct or of any significance in the process of learning. We would thus be taking a positive stance towards those that only assign an objective nature to mathematical knowledge. We would also be fitting van Hiele in the culture of 'misconception theory', which overemphasises the importance of making a list of all that children think or know and which, compared to 'real' knowledge, is wrong.

The paradox within van Hiele's theory is identified in the fact that, on the one hand, van Hiele accepts Freudenthal's statement that 'thinking without words is not real thinking', but on the other, he maintains a very positive stance towards the existence of a more subjective side to mathematics. Once more, I will refer to how van Hiele attempts to differentiate himself from Piaget. As stated by van Hiele (1986): 'Piaget was not aware that those [theoretical] concepts are only human constructions, which, in the course of time, may change'. And continues: '... development, with some theory as a result always must be understood as a learning process

influenced by people of that period' (p.6). Thus, van Hiele exhibits a stance towards mathematical knowledge similar to that of those who belong to the science of ethnomathematics.

Making a bridge between anthropologists and historians of culture and mathematicians is an important step towards recognising that different modes of thoughts may lead to different forms of mathematics; this is the field that we may call ethnomathematics. (D'Ambrosio, 1985, p.44)

It is indeed a paradox how the same person who accepts that there is such a thing as 'real' knowledge and 'real' thinking' can make such statements.

The conclusion 'Every square is a rhombus' is not a result of maturation, it is the result of a learning process. An intelligent person need not conclude that every square is a rhombus; this is only submission to a traditional choice. In some Greek philosophies, a square could not be a rhombus, for it had some properties a rhombus could not have. (Van Hiele, 1986, p.50)

If it were not for the claim in favour of the existence of 'real' thinking, van Hiele's position towards mathematical knowledge would have been very clear. In his meticulous philosophical discussion on objectivity and subjectivity, van Hiele (1986) takes the same stance as the one expressed by Noss and Hoyles (1996).

..... is mathematics an object of study in its own right, a cultural form, or is it a tool for understanding the mechanisms of social and scientific life? Now it is clear that a simple answer to this question is that both are true: they are two sides of a coin. But this begs question of the relationship between the two views, the relative weight which could and should be assigned to them. From a pedagogical point of view, it is the balance between these two aspects which structures how meanings are ascribed by the learner to any mathematical activity – what must be stressed and what can safely be ignored (p.13).

Similarly, van Hiele (1986) accepts this dual nature of mathematics. He states that mathematics is 'a domain of supposedly objective judgments' since they are 'discussable', 'testable' and 'accessible to a great number of people'. But, at the same time, he accepts that there are limits to this objectivity; there is no objectivity in time and space.

Some judgments are discussable and testable within a large, private group. Within the group, these judgments have great objectivity. Still there is the possibility that other people outside the group are not willing to accept the presumptions of the group and therefore do not agree with the judgment. Norms that have been valid for centuries now appear not to suffice in newer applications. Statements that were unassailable a century ago are now considered out of date and incorrect (p.218).

Most importantly though, van Hiele (1986) pursues this discussion and declares that for the practice of teaching mathematics, the 'better' mathematics has its foundation in not yet mathematical knowledge' (p.219). Van Hiele therefore accepts that from a pedagogical point of view, it would be more fruitful if mathematics were treated as a subjective field, a lesson we get from the subject's phylogeny. As I stated ten years ago:

A simple glance at the history of the subject proves that mathematics is the product of a historical and social process and that it is a subject that develops. What we take today as granted was not always so, and will probably not be taken for granted in the future. (Papademetri, 1996, p.45).

These philosophical and epistemological queries by van Hiele are not the only evidence that he was in favour of the existence and importance of acknowledging the subjective side of mathematical knowledge. It has been stressed in reviews of van Hiele's work (Clements & Battista, 1992; Hoffer, 1983), that there are two important aspects of the van Hiele model; the levels of thinking and the phases of learning. In differentiating himself from Piaget, van Hiele states that Piaget's stages are age-related whereas his level system is linked with a learning process. As declared by van Hiele himself, 'this difference is of great importance' (p.56).

The 'levels of thinking' and the 'phases of learning' are not two autonomous aspects of the model but are indistinguishably related, something that has been acknowledged by Clements & Battista (1992) and Pegg & Davey (1998) in their theoretical analysis of the van Hiele theory. Pegg & Davey (1998) see many commonalities between the van Hiele theory and Vygotsky. 'Van Hiele offered no developmental timetable for growth through the levels. In particular, he questioned the notions of growth being linked with biological maturation. Instead in ways that have much in common with Vygotsky, he saw development in terms of student's confrontation with the cultural environment, their own exploration, and their reaction to a guided learning process' (p.112). Similarly, Clements and Battista (1992) recognise that 'the phases of instruction are inextricably connected with the levels of thinking, and potentially more important for education; therefore, it is surprising and unfortunate that little research other than the van Hieles' has examined the phases directly. [...]. Additional studies are solely needed, especially given unresolved questions and concerns regarding the phases' (p.434). This still remains a gap, a misleading oversight of van Hiele-based research.

The second van Hiele paradox is, therefore, based on the fact that, on the one hand, van Hiele accepts the existence of 'real thinking', thus maintaining an objective stance towards the nature of



mathematical meaning, while on the other, he clearly declares his support towards the dual nature of mathematics by identifying, within mathematical knowledge, both a subjective and an objective side. Moreover, he reinforces his belief towards the subjective side of mathematics by connecting his thinking levels with, and presenting them as a result of, a specific teaching process, admitting that different learning cultures and philosophies might lead to different, equally acceptable mathematics. The subject of the next section is the stance of van Hiele-based research towards this paradox.

## **2.5. The consequences of ‘van Hiele Paradox B’ on van Hiele-based research**

As in the case of ‘van Hiele paradox A’, van Hiele-based research failed to detect van Hiele’s conflicting claims in relation to subjectivity and objectivity. Once again, based on van Hiele’s opening claim on the acceptance of ‘real thinking’, van Hiele has often been interpreted as supporting the ‘misconception theory’. One such example of a research study interpreting van Hiele within the framework of ‘misconception theory’ is the study – project by Burger et al (1981).

As described in a series of papers treating data collected from this project (Burger, 1985; Burger & Shaughnessy, 1986; Shaughnessy & Burger, 1985), the main aim of the study was to determine children’s understanding of common plane geometric figures. In doing so, children were involved in activities with triangles and quadrilaterals (Burger, 1985). Shaughnessy & Burger (1985), tried to interpret the initial data from the study – project in an effort to investigate how ‘elementary and secondary school students think about geometric concepts’ (p.419). The theoretical base of their investigation was the ‘theory of levels of geometric thinking’ set up by the van Hieles. One of the outcomes of this study was that ‘the van Hiele levels 0, 1, 2 are very useful in describing students’ reasoning processes in geometry’ (p.425). One of the most important outcomes of Burger’s (1985) analysis of the same data was that ‘at the beginning of a geometry course, many high school students exhibited the same geometric misconceptions as did younger children’ (p.53). So we have here an example of a project, which has van Hiele at the core of its theoretical background and aims at understanding children’s geometric thinking by locating children’s misconceptions, in an effort to classify them into a level system.

What is important is that these studies have a very constrained view of the van Hiele model. They have not taken into consideration the fact that, within van Hiele’s claims on subjectivity and objectivity, lies a paradox. Van Hiele-based research ignored van Hiele’s support towards subjectivity as well as the role that van Hiele attributed to the ‘learning phases’ described in his

theory. According to Hoffer (1983), the 'thinking levels cover only one of the three main aspects of the van Hiele model; the other two are 'insight' and 'phases of learning'. Thus, it is implied here that research eliminated two of van Hiele's main aspects. But the problem does not lie simply in the elimination of the 'phases of learning' as an important aspect of the van Hiele model, but in the elimination of the strict relationship between the 'levels of thinking' and the 'phases of learning'.

This omission has not only eliminated an important aspect of the van Hiele theory but misleadingly led researchers to the assessment of the thinking levels as an independent component, thus diminishing the van Hiele model to a developmental hierarchy (similar to that of Piaget). This led to continuous paradoxes within van Hiele-based research. The fact that many studies were driven to the conclusion that the 'assignment to levels did not seem to be strictly related to age or to grade category' (Burger & Shaughnessy, 1986) did not seem to trouble the researchers towards questioning their methodology. So there was a continuous attempt to evaluate the van Hiele level system by researchers (Burger & Shaughnessy, 1986; Gutierrez et al, 1991, Gutierrez & Jaime, 1998 Mayberry, 1983;) as an independent aspect to the 'setting' in which it occurred. As Noss & Hoyles (1996) state, 'the acknowledgment that setting is intimately bound up with performance on mathematical tasks precedes the widespread acceptance of the situated view of cognition' (p.30).

Through the methodology followed by these researchers that (perhaps not theoretically but practically) treated the van Hiele model as a developmental hierarchy, children's thinking was characterised by a sense of sameness and uniformity. As acknowledged by Pegg & Davey (1998), researchers such as Burger and Shaughnessy (1986), Fuys et al (1988) and Gutierrez et al (1991), failed to 'incorporate the notion of individual differences into the van Hiele model' even though this was their original intention. Pegg & Davey's (1998) methodological suggestion that 'what is needed is a closer examination of student understanding with an eye to the seeking and documenting of diversity' is based on their conviction that 'there is clearly great variability in the ways students learn, the structures on which students build their understandings, the roles teachers play, and the techniques teachers use' (p.115).

As a result of neglecting 'van Hiele paradox B', van Hiele has been mistakenly, but most importantly fruitlessly and misleadingly, been placed by van Hiele-based research on the wrong side of the discussion concerning the nature of mathematical knowledge. Alternatively, I aspire to open new epistemological and methodological doors for van Hiele in relation to children's

understanding of shapes by emphasising important and dynamic parts of the van Hiele model that have, up till now, been eliminated and ignored.

## **2.6. Alternative interpretations of van Hiele**

In the previous sections of this chapter, where the two paradoxes identified within the van Hiele theory were described, van Hiele has been presented as a traveller on an intellectual journey. We saw him unfolding his theory while thinking and questioning epistemological issues. His conflicting claims and personal doubts were evidence that he was still travelling at the time. His 1999 published paper provides evidence of how this journey was coming to an end.

If nonverbal thinking does not belong to real thinking, then even if we are awake, we do not think most of the time. Nonverbal thinking is of special importance; all rational thinking has its roots in nonverbal thinking, and many decisions are made with only that kind of thought (p.311).

After this change of heart, which was after all anticipated following the conflicts identified within his theory, van Hiele somehow resolves the paradoxes identified in 'Structure and Insight'. By rejecting his original claim that 'thinking without words is not thinking' he clears up the mist that was covering his conviction towards the existence of a more subjective side of mathematics as well as his support towards the existence and power of nonverbal thinking. By clearing things up, van Hiele allows us to proceed with assigning to his theory alternative and more dynamic interpretations. First, in the light of the previous discussion and van Hiele's rejection of his original claim, I will attempt to address a question that has troubled researchers: the question of whether the van Hiele model is of a hierarchical nature. By assigning a new perspective to the van Hiele level system, I will proceed with a new analysis of the first level of the van Hiele model: the visual level, by connecting it with the more general literature on the theme of 'visualisation'.

### **2.6.1. Is the van Hiele model of geometric thinking of a hierarchical nature? Is this a crucial dilemma or a pseudo-dilemma?**

In an effort to re-visit the van Hiele theory within a different dimension, it is important to address this question. The question of whether the van Hiele model is of a hierarchical nature. This is not a new question within van Hiele literature. Clements & Battista (1992) also raised a similar issue, in a different and quite interesting manner: 'Do the (van Hiele) levels form a hierarchy?' After a search in existing literature for an answer to this question, Clements & Battista (1992) concluded that 'the levels appear to be hierarchical, although there remains a need to submit this hypothesis

to rigorous tests' (p.429). In more recent publications though, Clements and Sarama (2000) still describe the van Hiele model as a hierarchical level system.

Children at different levels think about shapes in different ways, and they construe such words as square with different meanings. To the pre-recognition thinker, square may mean only a prototypical, horizontal square. To the visual thinker, squares might mean a variety of shapes that 'look like a perfect box' no matter which way they are rotated. To a descriptive thinker, a square should be a closed figure with four equal sides and four right angles. But even to this child, the square has no relationship to the class of rectangles, as it does for thinkers at higher levels. (p.482)

As commented earlier, the question posed by Clements & Battista (1992) in relation to the hierarchical nature of the van Hiele model was formulated in an interesting manner. Isn't it interesting to wonder whether a level system may or may not form some kind of a hierarchy? What is the meaning of the word 'hierarchy'?

Hierarchy: the organisation of a system in higher and lower ranks.  
Longman Dictionary of Contemporary English

Based on any definition of the word ('hierarchy'), it is paradoxical to wonder whether any level system may or may not form a hierarchy. That is why in the question driving this section's discussion the word 'level' was not used. The question is whether the van Hiele model (and not whether the van Hiele level system) is of a hierarchical nature. Thus, in order to question the hierarchical nature of the van Hiele model, one must first question the van Hiele model as a level system.

Of course, van Hiele himself in 'Structure and Insight' talked about a series of levels of geometric thinking and presented his model as a level system. But as I explained earlier in this chapter, 'Structure and Insight' was characterised by a number of paradoxes which allowed van Hiele-based research to interpret van Hiele in a certain way and eliminate other important aspects of the van Hiele theory. So even though van Hiele himself talks about the existence of levels, one may wonder whether his theory is of a hierarchical nature.

In a previous quote we saw how Clements and Sarama (2000) describe the van Hiele level as a lower form of thinking compared to higher levels of thinking, characterise each level by what children can(not) do and explain that, as we go up the level system, children know more and are capable of more. This is by no doubt an approach which describes the van Hiele model as a hierarchy. The continuous use of the word 'level' does not leave any room for doubt that the van Hiele model is of a hierarchical nature. The only way to assign another dimension to the van Hiele

model (that is not hierarchical) is to manage to move away from the idea of levels. So if we remove from claims made by van Hiele scholars any reference to levels, what will we be left with? Let's see!

According to Clements and Sarama (2000)

*'Children at different levels think about shapes in different ways, and they construe such words as square with different meanings.'* (p.482)

If we remove any reference to the levels we are left with the following claim:

*'Children think about shapes in different ways, and they construe such words as square with different meanings.'*

Moving away from the assumption of a level system makes a big difference. It allows us to think not about higher and lower forms of thinking, but different forms of thinking and meaning. The idea that children may think about shapes in different ways and assign different meanings to a specific geometric concept is the way one could think about geometric understanding within the sphere of epistemological pluralism; the idea of 'accepting the validity of multiple ways of knowing and thinking' (Turtle & Papert, 1992; Turtle & Papert, 1993;). Van Hiele himself allows this turn away from the existence of levels with his claims in his 1999 publication, to which I have referred on numerous occasions.

Nonverbal thinking is of special importance. [...]. In my levels of geometric thinking the '**lowest**' is the visual level, which begins with nonverbal thinking (p.311).

This statement guides us towards a clearer answer to the important question concerning the hierarchical nature of the levels. In clearing things up in the 1999 publication, van Hiele takes a positive stance towards nonverbal thinking which he connects with the first level; the visual level. The answer to the question in relation to the nature of the thinking levels seems to be clearer now that van Hiele takes a more clearly positive stance towards nonverbal thinking. What is mostly interesting in the 1999 published paper is that, in referring to the visual level, van Hiele uses the word 'lowest' cautiously, by putting it in quotation marks. This cautiousness allows the assignment of a new perspective to van Hiele's model of thinking levels. It allows us to treat the levels not as lower and higher stages of a hierarchy, but as diverse but equally important ways of thinking. Van Hiele seems as if he is no longer referring to his model as a level system, but as a model which accepts the existence of equally valuable but diverse ways of thinking and expressing understanding.

This new perspective of the van Hiele level system, which supports the existence not of hierarchical levels but of diverse modes of understanding and thinking, is more compatible with some of his hidden convictions and can better explain the common findings of most van Hiele-based research (Burger & Shaughnessy, 1986; Fuys et al, 1988; Gutierrez et al, 1991). Van Hiele-based research has concluded that the different thinking levels are not related to age or maturation, that children may move backwards or forwards between levels and that children might simultaneously belong to different levels. So if we substitute the word 'levels' with the phrase 'diverse modes of understanding', based on the findings of van Hiele-based studies, we can support the point of view that children may think in different ways independently of their age and may think in different ways simultaneously.

This new perspective allows us to place van Hiele among those who propose the existence of several ways of thinking, which opens new doors to understanding 'understanding'. It allows us to provide a dynamic alternative interpretation of the van Hiele theory, which shares the same belief as Noss & Hoyles (1996) in the existence of 'diverse kinds of mathematics', the same belief as Papert and Turkle in the existence of 'epistemological pluralism' (Papert, 1986; Turkle & Papert, 1992; Turkle & Papert, 1993) and the same belief as diSessa (2000) in the existence of ways of knowing 'beyond the stereotypes of knowledge we have culturally institutionalised in school and even in our common sense'. In the light of this interpretation, we aim to re-conceive the visual level; the first level of the van Hiele model.

### **2.6.2. Visual 'level' versus visual thinking**

Within van Hiele-based research, there is widespread acceptance of the existence of a visual level which has been placed at the bottom of the van Hiele model. Van Hiele (1986) himself refers to this level as the base level. But the characterisation 'base' has been interpreted in a different manner than what might have been intended by van Hiele.

In the studies (Burger, 1985; Shaughnessy & Burger, 1985; Burger & Shaughnessy, 1986; Clements et al, 1999; Clements & Sarama, 2000; Clements et al, 2001; Clements & Battista, 1992) which support the existence of a visual level as the level where children see shapes as a whole and pay no attention to shape properties, emphasis is given to the use of similes by children. The use of similes has been identified as the most common verbal proof that children see a shape as a whole. As stressed by Potari et al (2003), children spontaneously use similes to describe geometric shapes. This is apparent in the aforementioned studies. When asked to say

something about a shape, the answer they give is frequently a simile: 'it looks like a door/kite/roof'. This may be the case, but to what extent can these similes be the reflection of what children know about shapes? And moreover, does the use of similes exclude structural understandings, the understanding of a shape's structure?

A simile by itself can easily lead to the conclusion that children have very primitive ways of conceiving geometric shapes and see shapes as a whole, but when combined with other means of expressions they might turn out to be carriers of more advanced knowledge. In the study by Potari et al (2003), there was an attempt to investigate 'the role of similes in exploring the ways children conceive geometric shapes'. Besides the conclusion that 'similes are a natural way of describing geometric shapes in all age groups', the study also led to the finding that 'similes in certain cases imply a primitive way of conceiving the geometric shape while in other cases a more advanced one'. The examples given in the study, which present similes as carriers of advanced thinking, are very interesting. In the examples, the similes by themselves cannot easily be taken as proof of advanced thinking, but with the coexistence of other ways of descriptions and other means of expressions, things are different. There is, for example, the case of a seven-year-old boy trying to justify the use of the simile 'it looks like a sword' in his attempt to describe the shape kite.

The boy placed his hand along the smallest diagonal of the kite and showed the two isosceles triangles. Then showing the triangle with the smaller sides he said 'if we do this (the dissection), that (the triangle) [while rotating the shape] is what will hold it (the sword) at the back. (p.6).

This example shows that the similes used by the children are not simply the proof that children see shapes holistically. When children are asked to justify the use of a simile, they expose advanced knowledge. In the example above, the use of gestures in combination with the use of manipulatives helped towards the exposition of such advanced thinking. Another example in the study by Potari et al (2003) is that of Angeliki, an eleven-year-old, who gave the following description of a kite: 'it has four angles. Two of them are equal and the others as well. Two of them are small, the others are big. It looks like a spaceship.' If Angeliki had chosen to use only the simile in her attempt to describe the shape, should the conclusion be that she had no knowledge of the shape's properties?

If children are simply asked to say why a rectangle is a rectangle they have the choice of using any means to answer. Apparently, more often than not, they use similes. This is compatible with

van Hiele's (1999) description of the visual level as the level where 'figures are judged by appearance'. The question is whether judging a figure by appearance excludes structural understanding? The use of similes should not be taken as the only proof of what children know about a shape. We should wonder whether similes can be carriers of advanced understanding and whether they can be based on the understanding of properties. It is interesting to think about what a simile is and how it is used in everyday language. I especially like the following definition.

Simile an expression making a comparison in the imagination between 2 things, using the words like or as: 'As *white as snow*' is a simile.

Longman Dictionary of Contemporary English

By using a simile, one attributes a characteristic of one thing to another. In the simile 'as white as snow' the property of the colour of snow is attributed to another thing. If you simply say 'this is like snow' it is not clear which property of snow you are attributing to the object. The use of a simile can be interpreted in a number of ways: this is as white as snow; this is as cold as snow. Similarly, as supported by the data from the study conducted by Potari et al (2003), by using similes children assign specific components of objects to shapes. Similarly to the use of similes in everyday language, when children use similes to describe shapes, they attribute properties of physical and everyday objects to specific shapes. But frequently, the use of the simile in itself cannot be adequate proof of this advanced understanding. So in order for children to expose advanced structural understanding, they should be asked to justify their answers and be encouraged to use, equipped with, and allowed multiple means of expressions (use of gestures, manipulatives, constructions, etc).

Lehrer et al (1998), in a three-year longitudinal investigation, asked thirty seven children (grades 1-3) to compare and contrast triads of shapes. For each triad, children had to determine which two were most alike and justify their choice. The fact that their findings provided evidence of the existence of variation among appearance-based reasoning (van Hiele visual level) led them to defy the 'visual' level of development. In analysing their findings they state that

... children's reasoning most often involved the visual appearance of figures as suggested by van Hiele. This appearance-based reasoning, however, encompassed many distinctions. Sometimes children compared figures to prototypes of other figures ('It's squarish') or to prototypes of real-world objects ('It looks like a ramp'). At other times, children focused on the size of the figure (it's skinny') or to attributes resulted from properties of angles (it's pointy' or 'it's slanty'). Some children regarded figures as malleable objects that could be pushed or pulled to transform them into other figures. .... Some children for example decided that a chevron and a triangle were alike because 'if you pull the bottom [of the chevron] down, you can make it into this [the triangle].' Children mentally animated an action on a side of one



figure, often accompanied by appropriate gestures analogous to a 'drag' operation with a computer mouse, to morph it into another figure. .... Consequently, although we might describe children's reasoning as primarily 'visual', children's justifications involved many distinctions about form that appeared to involve several different types of mental operations, ranging from detection of features like fat or thin, to comparison to prototypical forms, to the action-based embodiment of pushing or pulling on one form to transform it into another. These distinctions appear to defy description by a single, 'visual' level of development (p.141-142).

Similarly Clement's et al (1999) support the view that children base their decisions on visual features, but are at the same time capable of recognising parts and properties of shapes. Nevertheless, we find that within research cultures which aim at listing misconceptions and/or at categorising children into level systems, important, rich understandings which children express about the properties of shapes are ignored and children are judged mostly in relation to what seems wrong in their replies. It is interesting how the same data can be interpreted in different ways within different frameworks. Whereas Lehrer et al (1998) interpret children's answers like 'it's pointy' as evidence that children focus on attributes that result from a shape's properties, Burger (1985), uses the same utterance ('it's pointy') as evidence that 'many children rely on imprecise qualities to identify shapes' (p.53).

In a study carried out by Clements et al (2001), the purpose of which was 'to chart the mathematical actions-on-objects young children use to compose geometric shapes', an instrument was designed to assess the levels of a hypothesised trajectory concerning the composition of geometric shapes. The hypothesised trajectory consisted of 6 levels. Each level's description included what a child can and cannot do when the child belongs to that level. In assessing the trajectory, children were asked to use pattern blocks to fill in simple frames. For further discussion, it is important to include the following example from the study where Mary, a four-year-old, is trying to combine pattern blocks to fill a specific simple frame of a human body.

Mary, a pre-school (4.3 years) child, exhibited actions that typified the behaviours of our early level of composition: the Piece Assembler. When Mary began working on the above described puzzle man she tried correctly to place a trapezoid in the foot. The trapezoid was 180° opposite the orientation needed to fill the frame and Mary, through her minor rotations in each direction, was unable to arrive at the requisite orientation and thus rejected the piece. Moments later, she uses the trapezoid to fill the arm, this time successfully rotating the shape to match the frame (p.277).

According to the researchers, 'findings generally supported the hypothesis that children demonstrate the various levels of thinking when given tasks involving the composition and decomposition of 2-D figures, and that older children, and those with previous experience in

geometry, tend to evince higher levels of thinking' (p.277). What characterises the previous study? According to the researchers, little Mary 'tried (correctly) to place a trapezoid in the foot' of the puzzle man but 'was unable to arrive at the requisite orientation and thus rejected the piece. [...] Mary returned to the legs and concatenated 4 squares to (incorrectly) cover one leg, and two to cover the other before deciding to move to another item. Using squares inappropriately shows that Mary was not attending to angle, behaviour typical of all of the children categorised as Piece Assemblers.'

A Piece Assembler, according to Clements & al (2001) 'can concatenate shapes to form pictures. In free-form "make a picture" tasks, for example, each shape used represents a unique role, or function in the picture. Can fill simple frames using trial and error. Uses turns or flips to do so, but again by trial and error; cannot use motions to see shapes from different perspectives. Thus, [children at this step], view shapes as wholes and see no geometric relationship between shapes or between parts of shapes' (p.275). A Piece Assembler reminds us of the van Hiele visual level.

What is it that makes Mary's anecdote and its interpretation by Clements et al (2001) interesting? The attempt to fit the findings to the theoretical background that originated the research in the first place led to a certain interpretation that seems to overemphasise some parts of children's responses (which seem inappropriate and wrong) and eliminate others (which were correct). Trying to fit Mary into a certain level led the researchers to focus on what Mary did 'incorrectly' or 'inappropriately', paying little attention to what Mary did 'correctly'. Children are hardly ever correct by accident. In the previous example, Mary correctly chose a trapezoid even though she failed to place it the right way to complete a certain frame. What understanding was hidden behind Mary's correct choice is equally important to her final choice of rejecting the shape. In addition, no attention was given to the setting in which the activity took place. What motivation did Mary have to try harder and build on her original and correct intuition, which led her to choose the right shape (a trapezoid) and try to fit it in the frame? The study by Clements et al (2001) took into no consideration of the fact that 'every cognitive act must be viewed as a specific response to a specific set of circumstances' (Resnick, 1991).

This section of Chapter 2 has described, up to this point, how the interpretation of van Hiele's visual level led researchers towards eliminating important and rich structural understandings about shapes and assigning a more 'primitive' interpretation to the visual level. Next, we shall see how by carefully studying van Hiele one can interpret the visual level in a more promising way.

At the beginning of this chapter, we referred to the paradoxes, which were identified within van Hiele's theory. It is important to view these paradoxes within their epoch. The era when van Hiele published 'Structure and Insight' (1986) was highly influenced by two main mathematics education research cultures: 'misconception theory' and 'hierarchical systems described by levels of understanding'. According to Noss & Hoyles (1996), the second culture was a result of the first one.

The progression within mathematics education research gradually allowed the expression of more pioneering views. As Papert (1993) states, school curricula designers might want us to believe that intellectual activity in general, and mathematical development specifically, progress step-by-step from one clearly stated and well confirmed truth to the next. According to Hoyles (1996), hierarchies in mathematics education are largely artefacts of research methodology.

Van Hiele's theory was trapped between eras and this is probably why it was characterised by paradoxes. Highly influenced by their era, the van Hieles were originally concerned with what they saw as the difficulties which children had with geometry. Their theory started evolving in the 1950's until in 1986 Pieter van Hiele attempted a meticulous analysis of his and his wife's theory (Dina van Hiele-Geldof), which included the 'levels' of thinking. Even though he proposed a rather revolutionary level system, in the sense that he introduced a level system that did not have a developmental character, the existence of levels within his theory clashed with his pioneering views on nonverbal thinking and the dual nature of mathematics (subjectivity and objectivity). To conclude this chapter, we shall unfold van Hiele's theory on visual structures, which gives a new perspective to the visual level and allows us to connect van Hiele with the broader literature on visualisation.

Hershkowitz (1989) identifies van Hiele as an example of theory dealing with the relationship between visualisation and geometry learning and states that 'according to van Hiele's theory, [...], visualisation is the first level and a necessary one in the hierarchy of geometry thought'. Even though this statement assigns a hierarchical nature to the van Hiele level system, it points towards an interesting and perhaps unique remark that connects van Hiele's visual 'level' with the broader use of the notion of 'visualisation'. This is a step away from the more simplistic interpretation of the visual level as the level where children's descriptions are 'purely visual', in the sense that children see shapes as wholes paying no attention to structure.

According to Bishop (1989), in his review of research on visualisation, 'the notion of 'visualisation' interacts in the research with the ideas of imagery, spatial ability, diagrams and intuition'. Similarly, Hershkowitz (1989) shares the point of view that 'there is a consensus among math educators and researchers that visualisation, or spatial ability, is important because it enhances a global and intuitive view and understanding in many areas of mathematics'.

Within the body of literature on 'visualisation' different terms are used for the definition of the object which constitutes the product of the process of *visualisation*: *concept image* (Vinner, 1983), *visual theorem* (Davis, 1993), *image* (English, 1997; Wheatley & Cobb, 1990), *mental picture* (Davis et al, 1991), *mental representation* (Davis & Maher, 1997), *visual image* (Cruz et al, 2000; Presmeg, 1986), *visual representation* (Hershkowitz et al, 1996) and *image schemata* (Wheatley, 1997). Apart from these definitions, terms like *imagery* (Battista et al, 1991; Clements, 1981; Presmeg, 1997), *visual thinking* (Hershkowitz et al, 1992) and *visual reasoning* (Wheatley, 1997) are also used in parallel with or instead of the term visualisation. Despite the wide range of existing terms within this literature, there is widespread agreement that visualisation presents a powerful and dynamic beginning to the complex abstractions of mathematics. Parallel to this agreement, a number of researchers emphasise that there is at the same time another side of visualisation which narrows a concept's image (Hershkowitz, 1989) and creates 'visual perception limitations' (Warren & English, 1995). This second side of visualisation is more compatible with the idea that children have misconceptions which interfere with learning.

This 'dual nature' of visualisation reflects the dilemma we are faced with when dealing with children's existing knowledge and leads to the following practical question: What is more productive for pedagogy? Is it more productive to identify children's misconceptions or treat children's existing knowledge as a rich resource on which to build mathematical meaning? Smith et al (1993) make a very important claim; that 'many of the assertions of misconceptions research are inconsistent with constructivism'. Consequently Smith et al (1993) suggest that 'if we take constructivism seriously, we must either reconsider the solely mistaken character of misconceptions or look for other ideas to serve as productive resources for students learning'. Since, as suggested by diSessa (1988), children's existing knowledge is the only resource we have to build scientific understanding in children's minds, it is important to have a more positive stance towards children's intuitions. Thus, within the boundaries of this thesis we shall refer to visualisation having in mind only the more positive side of the term, the side that supports the point of view that visualisation presents a powerful and dynamic beginning to the complex abstractions of mathematics.

The connection of 'visualisation' with intuition within the related literature is parallel to the connection of visualisation with non-verbal thinking and thinking with images. Clements (1981) describes visual imagery as 'a mode of thinking which is nonverbal, involving internal representations which could best be described as images of a largely spatial, and often visual character'.

Another widely accepted idea within literature on visualisation is that 'the origin of images is in experiences' (Wheatley, 1997). We find similar statements in Davis et al (1991), Davis & Maher (1997), Hershkowitz et al (1992) and Tall & Vinner (1981). Within these studies, there is an effort to provide and search for activities and ways of developing and reinforcing visual thinking in the mathematics classroom.

What is the connection between this literature on visualisation and van Hiele? Whether there is a connection or not depends on how one interprets van Hiele. In the following quote, Wheatley (1997) makes a distinction between van Hiele's visual level and the powerful description of images.

When an individual looks at a drawing of a geometric figure such as a rectangle and then the drawing is no longer present, the person may be able to represent a previously constructed image of a rectangle. The nature of a person's mathematical schemes and prior experiences with 'rectangle' will greatly influence the nature of the image formed. For some it may simply be an undifferentiated whole or what van Hiele (1986) called level 1 activity. On the other hand, the concept of rectangle may be sufficiently rich that the image includes sides, right angles, and parallelism (p.282).

Whereas images may be rich and include structural understanding, van Hiele's visual level excludes the understanding of a shape's properties. Such a claim takes into no consideration the van Hiele paradoxes presented in this chapter. Here, the intention is to explain in more detail how eliminated claims in van Hiele's effort to analyse his theory in 'Structure and Insight' support the same beliefs as the rest of the literature on visualisation. The shared beliefs within literature on visualisation are summarised in the following list:

1. Visualisation is at the basis of mathematical thinking since it provides rich intuitive understandings which can support the learning of mathematics in a dynamic manner
2. Visualisation is highly connected with nonverbal thinking; thinking in mental pictures rather than words.
3. Visualisation originates from personal experiences.

Based on this list, can van Hiele's theory be interpreted as a theory of visualisation? Even though van Hiele (1986) supports that recognising an isosceles triangle by its property: ('two sides equal') is quite different to recognising an isosceles triangle because of its gestalt (p.75), he does not diminish the importance of the second route to recognising a shape.

It is true that at the base level there is a language, but the use of this language is limited to the indication of configurations that have been made clear by observation. At the first level you can say: 'This is a rhombus'.

It is an important phenomenon that without further explanation (how would such an explanation be possible!) another person can say: 'And this also is one.' But let us not mistakenly presume that such phenomena are restricted to geometry. This is the kind of phenomena all human communication is built on. You show your baby a little hairy beast and you say: 'This is a dog.' Afterward the child sees a big Alsatian and he says: 'Dog.' [...].

This phenomenon underlies a great part of human knowledge, and in the chapters of structure I have given many examples of it. It is one of the important principles of the Gestalt (structured) psychology, but it seems to have been forgotten. When Skemp (1971) considers this phenomenon he speaks of abstraction and supposes that the child makes distinctions between various properties and thereby decides the application of names. This is an important issue: it is relevant to the origin of human thinking (p.49-50).

In this quote, van Hiele (1986) traces the origin of human thinking in nonverbal understandings originated by observations, which allow children to distinguish properties. The point of view that observations are highly important for intuitive visual thinking is also stressed by the literature on visualisation. In his analysis of visual theorems, Davis (1993) supports Gauss's stance that 'mathematics is a science of the eyes'. Observations create strong experiences, which play an important role in visualisation. Van Hiele (1986) defines the 'ability to see the solution to a problem directly without being able to tell' as intuition and supports the point of view that intuition is cultivated through looking closely at visual structures.

In psychology and even more in philosophy one often distinguishes only two methods of reaching conclusions: those that are derived from discursive thinking and the others, which are brought about unconsciously. This distinction blocks the way to seeing things as they really are. It throws everything that differs from discursive thinking, including reading visual structures and drawing conclusions on prejudicial grounds, into one pile and gives it the name 'intuition'. The difference between these two ways of coming to a conclusion is enormous: the decision on the grounds of a visual structure may be just as reliable as a decision on the grounds of discursive thinking; in both cases one comes to a correct decision as long as no surprises occur. [...] One who acts from a "faultless intuition" probably concludes on the grounds of visual structures. Because often it is difficult to express these structures in words, conclusions sometimes seem mere guesses. [.....]. I join Mallinckrodt (1959), who translates *intueri* as 'looking at something, viewing closely'. In this way I withdraw from the usage that gives 'intuition a negative connotation'.

A decision based on an intuition, that is on a 'close consideration' without the aid of discursive thinking, may be correct. If we have access to a strong structure, the sureness of the decision may be quite justified. (p. 72)

Van Hiele's detailed analysis of his theory is full with such claims in favour of visual structures, intuitive and non-verbal thinking which he connects in similar ways to the literature on visualisation: 'the intuitive approach begins with the presentation of a structure' (p.117), 'all rational knowledge begins with intuitive knowledge' (p.125), 'man is able to react directly – without the intermediary of a language – to a visual structure, (p.127), 'the construction of visual structures in our minds very often does not need a language' (p.16).

Besides references from 'Structure and Insight', we also have van Hiele's claims in the 1999 published paper where he resolved the 'van Hiele paradoxes'. In this paper, van Hiele talks about visual (nonverbal thinking) as the basis of all rational thinking and provides, in the same way as 'visualisation' researchers provide, a variety of activity examples which aim at 'enriching children's store of visual structures' and 'enabling children to build a rich background in visual and descriptive thinking involving various shapes and their properties'. He explains how by solving puzzles, matching and fitting games the child can nourish visual structural understandings of shapes, which are important for the construction of mathematical ideas.

According to van Hiele (1999), for children, geometry begins with playful activities, which allow them to 'integrate what they have learned into what they already know'. Moving beyond his belief that 'thinking without words is not thinking', enabled van Hiele to look into learning and understanding in a more open-minded way. So we now have before us a van Hiele who emphasises the importance of exploration and of working visually in order to develop a more structural knowledge of shapes. Most importantly, van Hiele points out that children's engagement into such exploratory, visual, playful activities 'gives teachers a chance to observe [...] children, and to assess informally how they think and talk about shapes'. This remark is important not only for teachers but for researchers as well.

Van Hiele's positive attitude towards intuition, nonverbal thinking and the point of view that in the process of learning children 'integrate what they have learned into what they already know' may allow us to connect van Hiele with powerful and constructive ideas; Like the idea of the 'modularisation of knowledge'. According to Papert, (1993) 'when knowledge can be broken into 'mind-size bites' it is more communicable, more assimilable, more simply constructable'. This idea can also be recognised in Noss & Hoyles' (1996) key concern of 'mathematical abstraction

as a process, which builds upon layers of intuitions and meanings' (p.105). Similarly, diSessa (1988) proposes the fruitful idea of 'knowledge in pieces'; the idea that 'intuition' is a 'fragmented collection of ideas' and that these ideas 'are the only material we have to develop scientific understanding in our student's heads' (p.51).

Besides the general ascertainment that intuition plays an important role in the development of understanding, diSessa (2000) goes beyond the tendency to use the word intuition for understandings we cannot really describe or determine by providing a detailed description of intuitive knowledge. According to diSessa (2000), intuition is fragmented, problematically related to language, characterised by a lack of systematicity and commitment but at the same time rich, flexible, diverse and generative, frequently effective and sometimes even correct. Some of these attributes have been pinpointed by van Hiele himself. As we saw earlier, according to van Hiele (1986), a decision based on an intuition (which 'probably concludes on the grounds of visual structures') might be correct and often difficult to be expressed in words.

So in contradiction to the idea of the existence of a visual level as a lower form of thinking which excludes structural understandings, we have the idea of visual structures which, as we have seen, is compatible with the idea of visual thinking. Whereas the visual level is characterised by what children know (or do not know), the idea of visual structures allows us to think not about what children know but the nature of what they know and the ways they use to express their understandings.

## **2.7. Van Hiele-based research: Epistemological and methodological gaps**

The aim of this chapter was to highlight aspects of the van Hiele theory which have been largely ignored and were identified as an extension of what we described as the 'van Hiele paradoxes', in an effort to assign a new alternative dynamic perspective to the van Hiele theory. I explained how studies that were based on the van Hiele model of geometric thinking were mostly influenced, implicitly or explicitly, by van Hiele's claim that 'thinking without words is not (real) thinking'. As a result, important aspects of the van Hiele theory have been lost. In conclusion, it is considered that most van Hiele-based research ...



1. .... has tended to ignore van Hiele's extensive reference to intuitive thinking and the importance of nonverbal thinking
2. ... overlooked van Hiele's doubts concerning the hierarchical nature of his model and his tendency towards acknowledging that there are no 'lower' and 'higher' forms of understanding; but rather equally important but diverse modes of understanding and expressing understanding
3. ... eliminated the importance that van Hiele assigns to acknowledging the subjective side of mathematical knowledge for pedagogical purposes, and
4. ... managed to dissociate the 'levels of thinking' from the 'phases of learning' diminishing, as a result, van Hiele's model to a developmental hierarchy.

Consequently, substantial pieces of research concerning geometric thinking exhibit the following weaknesses from a methodological point of view:

1. There is a tendency to evaluate children mainly through their verbal ability. Verbal data has been used as the means for assigning children to levels.
2. The emphasis has been on children's 'wrong responses', 'misconceptions'. As a result, there has been a downplaying of children's important, rich, intuitive understandings in the process of fitting them into level systems.
3. There has been an evaluation of the cognitive act with no reference to important aspects of the setting (e.g. child-adult interaction, activity) in which the cognitive act takes place. As an extension to this remark, young children were assessed in settings which were designed without taking into consideration the particularities of their age.

A particular implication of these orientations is that they led to a restricted view of what children know about shapes. This restricted view has tended to degrade children's structural understandings. In the remaining chapters of this thesis, I will describe a study that was designed based on these methodological remarks in an effort to reveal, describe and analyse children's structural understandings of shapes.

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## CHAPTER 3

### RESEARCH METHODOLOGY

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#### 3.1. Aim of the study

As stated in the previous chapters, the main aim of this study is to describe and analyse young children's structural understandings of shapes through an investigation of squares. As illustrated in Figure 2 the following Research Questions (RQ) emerge from the above main aim:

RQ1 What do children know about shapes?

RQ2 How is this knowledge expressed?

RQ3 How is this knowledge used in the process of constructing shapes?

The piloting procedure (Chapters 4) and the process of data analysis (Chapter 5) allowed the transformation of the aforementioned research questions into the following four Operational Research Aims (Figure 2, ORA):

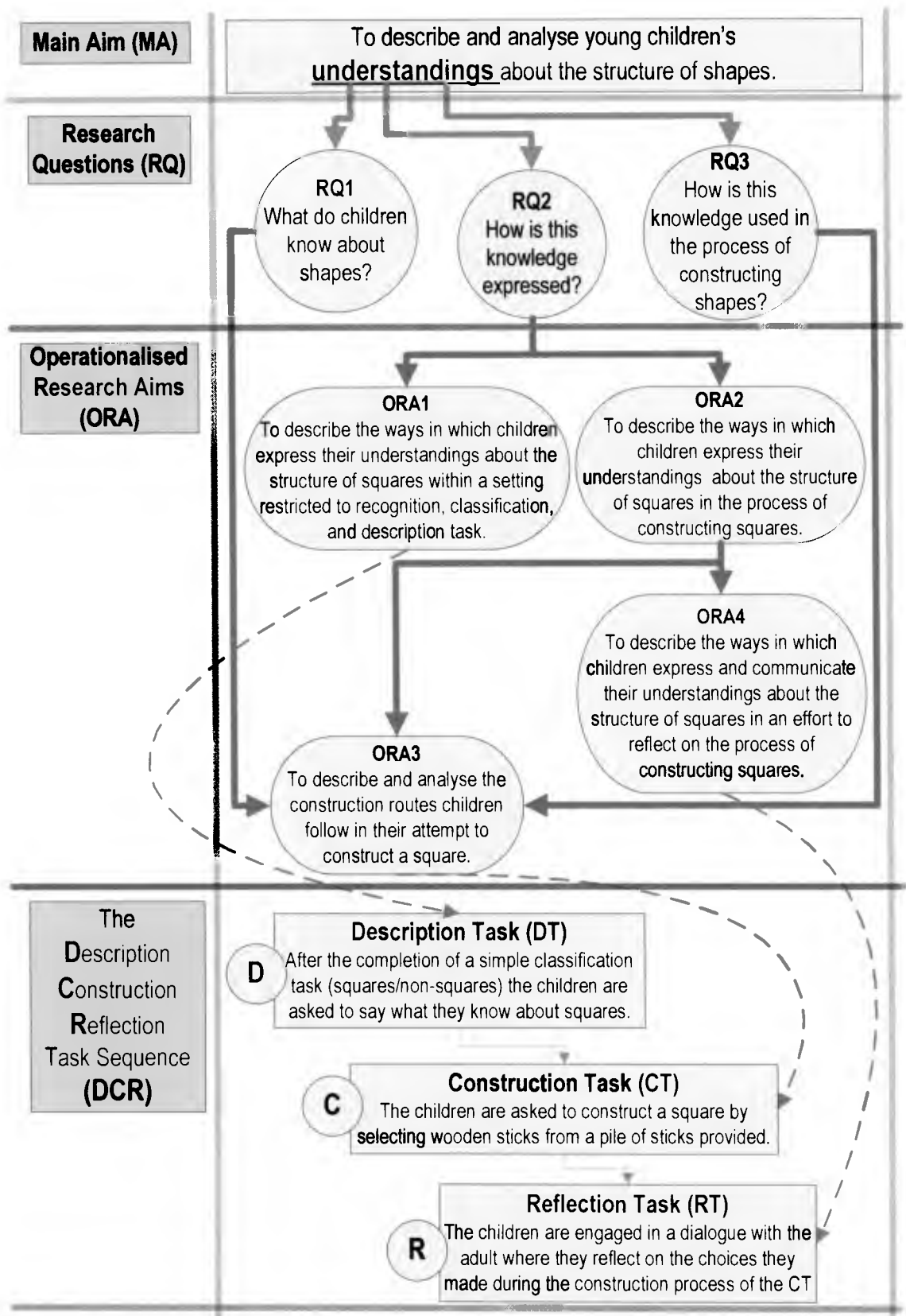
ORA1: To describe the ways in which children express their understandings about the structure of squares within a setting restricted to recognition, classification and description tasks.

ORA2: To describe the ways in which children express their understandings about the structure of squares in the process of constructing squares.

ORA3: To describe and analyse the construction routes children follow in their attempt to construct a square.

ORA4: To describe the ways in which children express and communicate their understandings about the structure of squares in an effort to reflect on the process of constructing squares.

Figure 2: Overview of the aims and design of the study



By following the arrows one can see how each part of the task sequence is connected to each ORA and how each ORA re-addresses the original RQs of the study. For example, by investigating the data collected through the CT we will be able 'to describe and analyse the construction routes children follow in their attempt to construct a square' (ORA3). By following the arrows connecting the RQs with the ORAs, it is made clear that through the investigation of the children's construction routes during the CT (ORA3) we will be able to address all of the study's RQs (RQ1/RQ2/RQ3). ORA3 and ORA4 were the product of breaking down ORA2. Thus, ORA2 will not be addressed directly but indirectly through ORA3 and ORA4.

The study's RQs (Figure 2) were addressed through data collected using task-based interviews (the choice of this source of data is the subject of §3.3). Through a piloting procedure that was conducted before the main study, and which will be described in detail in Chapter 4, a three-phase task sequence was designed for the task-based interviews (Figure 2, The Description, Construction, Reflection Task Sequence).

### **3.2. Overview of the methodology for data collection**

In view of addressing the main aim and research questions formulated in the previous section, a research study was designed and conducted, where 52 five-year-olds were involved individually in task-based interviews within their school settings. The interviews were conducted by a group of thirteen preschool student teachers in two public schools in Nicosia, Cyprus. I had trained the student teachers for this purpose myself as part of a teacher-training course they were attending at the University of Cyprus. Using these student teachers during the process of data collection allowed (a) a far greater sample than would have been possible otherwise, and (b) the implementation of naturalistic elements in the setting in which the task-based interviews took place (this issue will be addressed in §3.3).

#### **3.2.1. Setting of the task-based interviews**

The setting for the task-based interviews, which was the method chosen for data collection, is described in Figure 3. As shown in Figure 3, the focus of attention of the setting was on the '*child's understandings of shapes*'. This, however was investigated in relation to the following two components:

- The '*task sequence*' involved and
- The adult present in the setting ('*student teacher - interviewer*')

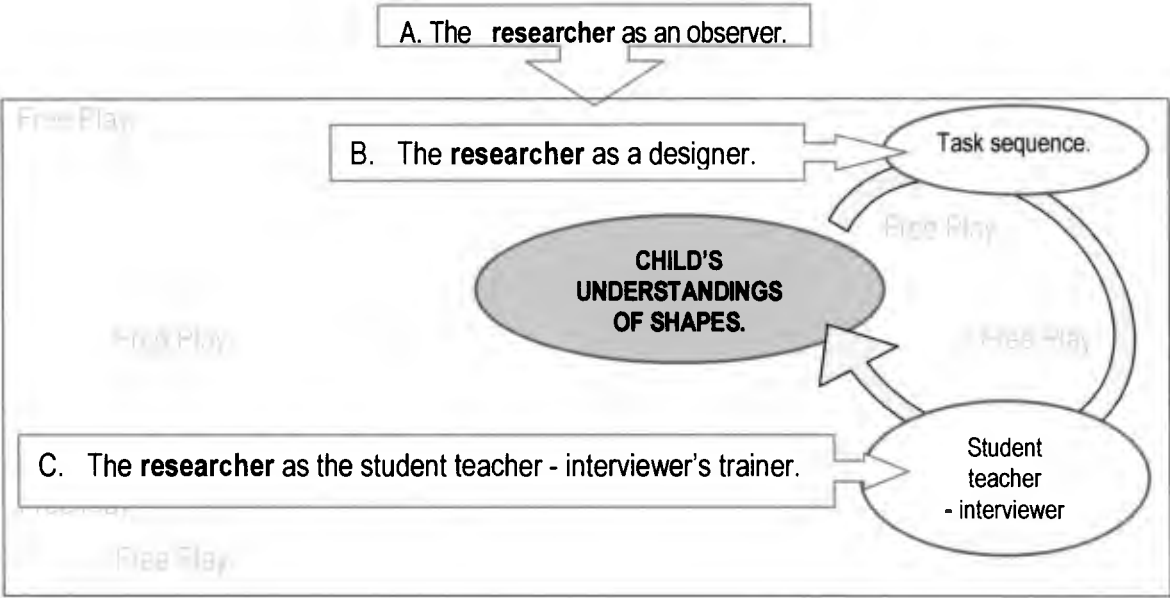
Within this setting arrangement I was able to play a triple role, as illustrated again in Figure 3:

- A. '*The researcher as an observer*',
- B. '*The researcher as a designer*' and
- C. '*The researcher as the student teacher - interviewer's trainer*'.

Furthermore, an important element of the setting was the choice of time and place: all interviews were conducted in the children's own school as part of their everyday involvement in '*free play*'

activities. In the following sections I will provide a detailed description of each element of the setting for the interviews, as these were conducted in the main study.

**Figure 3: Description of the setting for the task-based interviews**



The blue ellipses illustrate the three main elements of the setting whereas the three red arrows indicate the triple role of the researcher.

### 3.2.2. The task sequence

The task sequence used for the interviews which were conducted for the main study was designed through an iterative piloting procedure completed in three cycles. The piloting process constitutes an important part of the study and is the subject of the next chapter.

A simplified description of the task sequence used in the task-based interviews is described in Figure 2 (p.39). The Task Sequence (TS) used in the study, consisted of three main activities: The Description Task (DT), the Construction Task (CT) and the Reflection Task (RT). During the DT, the children were asked to say what they knew about squares after the completion of a simple classification task. In the CT that followed, the children were asked to construct a square by selecting wooden sticks from a pile of sticks provided. Finally, during the RT the children were engaged in a process of reflecting with the adult in order to justify the choices they had made in their attempt to construct a square. More information about the task sequence will be provided further on (§4.2).

### **3.2.3. The role of the researcher**

As illustrated in Figure 3, I (as the researcher) was part of the setting of the task-based interviews by undertaking three roles: the role of the observer, the role of the task designer and the role of the interviewers' trainer. All of the interviews were conducted by trained student teachers and were videotaped. Even though I was not physically part of the setting, I was an integral part of and was influencing and controlling the setting through the activities involved, the training of the adults conducting the interviews and the interview script tool used by the interviewers, which were the products of the piloting procedure.

### **3.2.4. Participants**

The fifty-two children involved in the study were attending pre-primary programmes in two Cypriot public schools. The public schools which took part in the study were the schools in which the student teachers (interviewers) were 'working' as pre-service teachers, as part of a teaching course they were attending at the University of Cyprus. Each student teacher interviewed four children randomly selected by me from the class's list of pupils. The children's parents had given their consent for the participation of their children in the study before the data collection took place. The youngest child involved in the study was four years and ten months old and the oldest was six years and eight months old. Twenty-five of the children were girls and twenty-seven were boys.

The thirteen student teachers involved in the study were studying at the Department of Education of the University of Cyprus, in the four-year programme of studies for preschool teachers. The main study was conducted during Spring Term 2003. The student teachers were in their third year of studies at the time. During this third year of studies, student teachers at the University of Cyprus follow a School Experience course, of a duration of one Academic Term (thirteen weeks). During the academic years 2001-2003, I had full responsibility for this course in relation to preschool education student teachers. As part of the course, the student teachers were placed, in couples, in public preschool classrooms and were asked (a) to engage themselves in specific activities (child/classroom observations, planning, conducting and reflecting on learning situations, etc.) and (b) to become active members of the school and classroom community (Papademetri, 2003).

The student teachers' effort to deal with the everyday 'problems' of school life and make the most of their presence in schools was supported by the University's pedagogical community. During the course, the student teachers attended regular three-hour meetings (discussion groups and workshops) once a week with the teacher-educator in charge, in this case myself. Within this framework, the thirteen student teachers were trained and supported to conduct the task-based interviews as an activity for cultivating communication skills. The training programme that was used for preparing the thirteen student teachers involved in the study was one of the three products of the piloting procedure (Chapter 4).

As mentioned above, these thirteen student teachers were placed, in couples, in a pre-primary classroom (the number of student teachers being odd, three of them were placed together in one classroom). The student teachers were placed in schools during the third week of the course, after attending some preliminary meetings with me at the University<sup>5</sup>. During the 9<sup>th</sup> and 10<sup>th</sup> week, they attended two three-hour meetings during which they were trained for the interviews. During the last three weeks of the course (weeks 11-13), they conducted the interviews. Thus, the student teachers were trained for the interviews after seven weeks of attending regular pedagogical meetings and conducted them at least after seven weeks of presence in the school.

As part of the training programme, the student teachers were given specific guidelines in relation to the interviewing procedure. The guidelines were provided both in writing (Appendix A) and orally and concerned their role, the time and place of the interviews, the task sequences, the videotaping procedure, etc. The student teachers were also provided with all materials needed for the tasks as well as an interview script tool that was the third product of the piloting procedure. A detailed description and analysis of the interview script tool is provided in §4.3.

While one student teacher was conducting the interviews the other completed an observation form (Appendix B). The student teacher undertaking the role of the observer had to transcribe the child's actions. The observation forms were later used as a starting point for transcribing the interviews. The student teachers had already worked with their pair on other activities and were accustomed to working as a team. All interviews were videotaped. Each couple submitted videotapes of and completed observation forms for each interview.

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<sup>5</sup> The School Experience course is conducted by the University of Cyprus with the consent and full support of the Ministry of Education and Culture. The University of Cyprus has an agreement with the Ministry of Education and Culture in relation to the teacher training course, based on which I had free access to schools. The research was conducted by the student teachers as one of their course assignments and the schools' headmasters as well as the classroom teachers were informed in relation to the research.

### **3.2.5. Time and place of the interviews**

Based on the guidelines that were given to the student teachers, all interviews were conducted in the children's own school as part of their everyday involvement in free play activities. Free play is a crucial part of any early childhood education programme. As stated in the Cyprus National Curriculum (2000), an important characteristic of free play is that children are freely engaged in controlled activities.

Within free play, different children engage in different activities in different ways, alone, with teachers or with other children, in different parts of the classroom or school. The children choose for themselves where to play or, when the teacher finds this more appropriate, are assisted by their teachers to 'work' on a specific activity. Given the many variations of play to which the children involved in the study were already accustomed within free play settings, the task-based interviews were considered by them as yet another usual activity. For practical reasons, the interviews were not conducted in the main playroom, in order to control the noise level, but this did not pose any problems since the children were accustomed to 'play' in other parts of the school within their free play time.

The student teachers were also guided to choose places away from noise and movement that could disturb the children and interfere with the quality of the videotaping, to use the floor if possible (e.g. closed gym with carpet) and to prepare the chosen area in order to make the children feel welcome and comfortable.

In the remaining part of this chapter we will analyse the theoretical framework behind this research design.

### **3.3. Theoretical framework of the research design**

As was described in the overview of this study's methodology in the previous section, the data was collected through task-based interviews. In this section, I will describe the theoretical framework which (a) led to the choice of task-based interviews for data collection and (b) determined the design of the interview's setting. In relation to the latter, I will address some important issues arising from the effort to serve two different research domains: the domain of mathematics education research and the domain of early childhood education research. Thus, after sketching the demands identified by the fact that this study was a study about 'early



childhood mathematics education', two important issues will be addressed: the theoretical framework which led towards (a) using a construction task as the core of the task sequence and (b) using student teachers for data collection.

### 3.3.1. Why task-based interviews?

I would like to define 'task-based interviews' as the method in which data is collected through interviewing the subjects under investigation in relation to their understandings while engaged in specific tasks designed for this purpose. Thus, this research method has its origin in the methodology of the 'teaching experiment', (Cobb & Steffe, 1983; Steffe et al, 1983). As stated by Cobb & Steffe (1983), 'the constructivist teaching experiment is used in formulating explanations of children's mathematical behaviour. Essentially, a teaching experiment consists of a series of teaching episodes and individual interviews ...'. According to Kilpatrick (1987), the term 'teaching interview' seems to be a more proper term for the methodology described as teaching experiment:

Steffe and his colleagues (1983) have pioneered an extension of Piaget's clinical interview in which a child is set a mathematical task, the response is analysed in terms of a model of the child's understanding of the task constructed from an interpretation of that and other responses, additional tasks are given to test the model, and instruction is provided by the interviewer in an effort to develop the child's conceptual structures and to model that development as it occurs. The term teaching experiment is often used to describe such an interview ... Teaching interview seems a more appropriate term for what Steffe and his colleagues do. (p.17)

Even though von Glasersfeld (1983) considered 'teaching interviews' as a methodological consequence of radical constructivism, a paradigm defined by the conviction that there is a 'need to abandon our search for objective truth', this study supports the position that 'teaching interviews' complement both the objective and subjective side of mathematics; a position also supported by Cobb (1990) in his analysis of Steffe and his colleagues' work.

Steffe and his colleagues did not assume that whole numbers have a mind-independent existence when they investigated the mathematical worlds of first and second graders. An openness to the possibility that young children's arithmetical knowledge could be knowledge-in-action was appropriate given their purposes. Nonetheless, they necessarily had to make realist assumptions while interacting with children. Interviews and teaching sessions were conducted against the background of an assumed shared spatiotemporal reality of physical objects. (p.205)

In an interview, the interviewer's effort is to find out the interviewees' feelings, beliefs, understandings. There is, therefore, a search for meaning in the interviewee. In a 'teaching interview' we have, parallel to this effort, the design of tasks, which aim at teaching. Hence the assumption of a teaching object. The powerful idea about 'teaching interviews' is that one can ensure 'openness' towards children's understandings while accepting, at the same time, the 'spatiotemporal reality' of mathematical knowledge. There might be no objectivity in time and space, but there is objectivity in a specific moment and place. Designing tasks or lesson plans is unavoidably based on an assumed reality. This assumed reality is, to a certain extent, shared with the subjects under investigation during the teaching interviews.

Even though the term 'teaching interviews' has been supported because of the sense of complementarity that it carries, it is thought that the choice of the word 'teaching' brings to mind, more than would be actually desirable, the old-fashioned images of schooling where the teacher, as the authority, would transfer (real) knowledge into children's heads. This is why I chose the term 'task-based interviews' in order to describe the method used in this study.

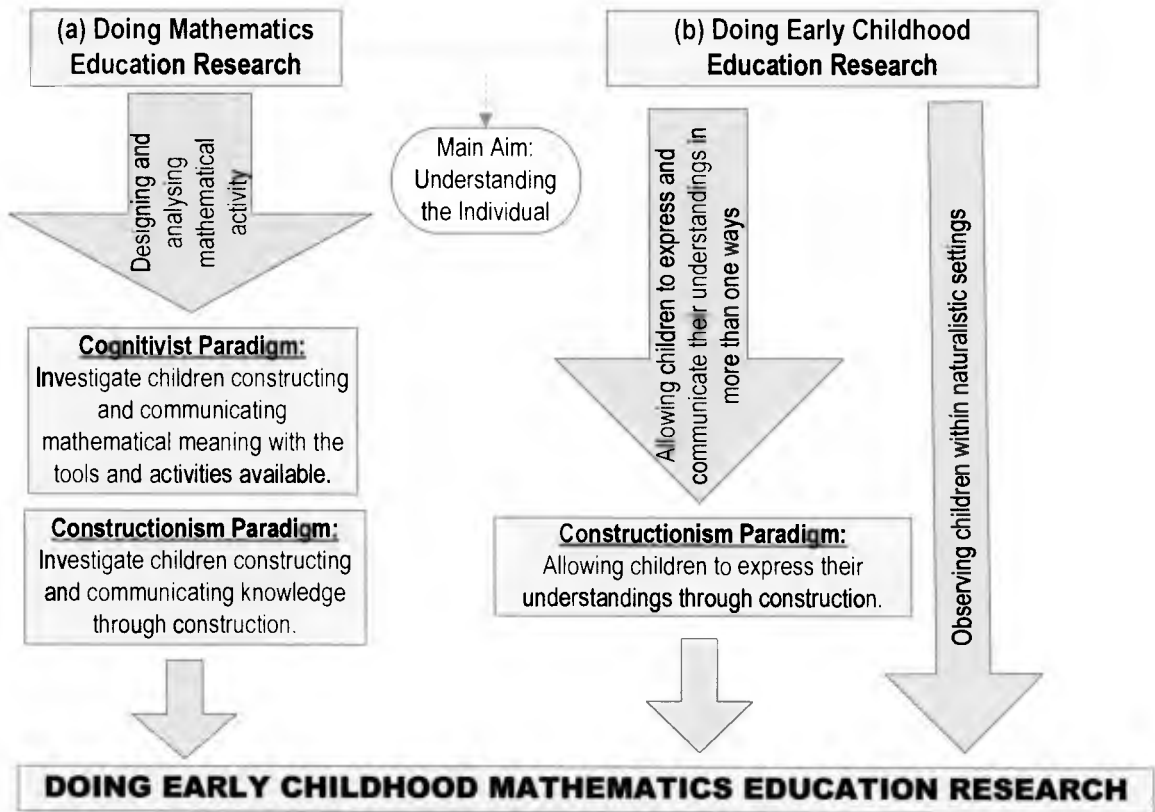
As stated by Goldin (2000), 'in comparison with conventional paper-and-pencil test-based methods, task-based interviews make it possible to focus research attention more directly on the subjects' processes of addressing mathematical tasks, rather than just on the patterns of correct and incorrect answers in the results they produce' (p.520). This definition of 'task-based interviews' excludes the use of task-based interviews as a form of oral test-based method. Whilst the use of task-based interviews as a source of data ensures the perspective of openness towards children's understandings, it provides, at the same time, the researcher with a sense of control on the setting in which the cognitive act will take place. As stated by Goldin (2000), 'the value of task-based interviews ... lies in the fact that they provide a structured mathematical environment that, to some extent, can be controlled' (p.520).

To conclude, the choice of task-based interviews supports this study's interest in processes rather than evaluating answers, but most importantly, it serves the purpose of achieving complementarities in relation to the nature of mathematical knowledge. The acceptance of both a subjective and objective side to mathematics is at the basis of this study's epistemology and, consequently, its methodology construction. The theoretical framework behind the design of the setting for the task-based interviews is the subject of the following section.

3.3.2. Doing Early Childhood Mathematics Education Research

In the process of designing the details for the setting of the interviews there were certain issues that had to be considered. These emerged from the fact that the study was concerned with doing both mathematics education and early childhood education research. Thus, I had to think about what doing early childhood mathematics education research required. A graphical description of the theoretical framework behind the choices in relation to designing a study within the domain of early childhood mathematics education research is provided in Figure 4.

Figure 4: Methodological requirements from the perspective of Early Childhood Mathematics Education Research



The big arrows contain the methodological requirements of each domain (mathematics education research and early childhood education research) and in some cases connect each domain with specific research paradigms.

As illustrated in Figure 4, 'doing early childhood mathematics education research' embraces two research domains: (a) the domain of 'mathematics education research' and (b) the domain of 'early childhood education research'. From both perspectives, the 'main purpose' is to 'understand the individual'.

The emphasis from the perspective of mathematics education research, as described in Figure 4(a), was on 'designing and analysing mathematical activity'. This led to the 'cognitivist paradigm', which is characterised by an attempt to 'investigate children constructing and

*communicating mathematical meaning with the tools and activities available*'. This, in its turn, led as shown in Figure 4 to the '**constructionism paradigm**', which allows us to '*investigate children constructing and communicating mathematical meaning while ... constructing*'.

On the other hand, the perspective of early childhood education research (Figure 4(b)) required the identification of ways of '*allowing children to communicate and express their understandings in more than one ways*' which led, once again, to the '**constructionism paradigm**', which '*allows children to express their understandings through construction*'. In addition, there was the requirement of '*observing children within their natural learning settings*' which was an important issue from the perspective of early childhood education research.

Thus, this theoretical framework required the merging of all these requirements (Figure 4) which I will now proceed with justifying, in more detail, before explaining how they were integrated into this study's research design.

### **3.3.3. The role of construction in the methodology design**

According to the way 'task-based interviews' has been defined above, this method allows the collection of data through interviewing the subjects under investigation while engaged in specific tasks designed for this purpose. From this point of view, the choice of 'task-based interviews' emphasises the importance of designing mathematical activity. As a starting point, this study values Hoyles' (2001) insistence on the conviction 'that studies in mathematics education should involve some discussion of mathematical activity, however this is defined'.

The conviction that designing mathematical activity is important is based on the belief that 'activity, concept and culture are interdependent' (Brown et al, 1989) and that 'treating knowledge as an integral, self-sufficient substance, theoretically independent of the situations in which it is learned and used' assumes 'a separation between knowing and doing', a separation between 'knowing what' and 'knowing how'. Thus, it is based on Vygotsky's powerful idea that 'the activity in which knowledge is developed is an integral part of what is learned' (Brown et al, 1989).

Noss & Hoyles (1996), within the framework of computer-based research emphasise the 'need to focus on tools and settings' as well as 'on the ways in which the understanding of mathematical ideas is mediated by the tools available for its expression.' (p.50). A main outcome of computer-based research is that the use of 'new tools' can lead to 'new cultures', which open the way to the

expression of 'new understandings'. Similarly, in this study the interest is focused on the investigation of the ways in which specific tools and activities can bring into mathematical learning a certain culture, which will allow access to the expression of 'new' understandings. Sfard & Linchevski (1994) state that 'the massive use of computer graphics in teaching function will reverse the 'natural' order of learning so that the structural approach to algebra will become accessible even to young children'. I hope that likewise, by equipping young children with specific tools to think with, this study will reverse the widespread idea that young children have no knowledge of the structural formulation of shapes and see shapes as wholes. Rather, I aspire to sketch young children's rich intuitive understandings of shape properties and components despite their inability to express these intuitions in words.

Thus, within this epistemological framework, the choice of task-based interviews as the method for data collection acquires a specific meaning. Knowledge expressed by the subjects during the interviews is not considered as independent of the tasks used. The choice, or better, the design of the activities, are mutually dependent of the knowledge expressed. Consequently, a substantial part of the research design relates to designing activities that will allow children to construct and express structural understandings of shapes. This led to the cognitivist-constructionism perspective and consequently to the use of construction tasks.

### ***The use of construction tasks in investigating mathematical knowledge***

As this study pays great attention to activity design and tool mediation, it goes beyond the cognitive assumption that knowledge is constructed in children's heads. Furthermore, it shares the same assumption as the cognitivist research paradigm, i.e. that the knowledge constructed in children's heads is highly connected with the tools at hand. According to Hoyles (2001), cognitivist research is 'concerned with students expressing their mathematics with the tools available' (p.275).

In addition, this study shares the constructionism principle that learning takes place in situations where learners are allowed to build and reflect on their own models (Kafai, 2006) and that with the appropriate tools available learners can 'build things and ideas simultaneously' (Noss & Hoyles, 2006). According to Papert (1991), 'the simplest definition of constructionism evokes the idea of learning-by-making' (p.6) and builds on the powerful idea of thinking-as-constructing. Besides the fact that constructionism builds on the constructivist assumption of meaning-actively-



constructed by the individual, it suggests that actual, physical construction can lead children to new understandings.

Constructionism is both a theory of learning and a strategy for education. It builds on the “constructivist” theories of Jean Piaget, asserting that knowledge is not simply transmitted from teacher to student, but actively constructed by the mind of the learner. Children don't get ideas; they make ideas. Moreover, constructionism suggests that learners are particularly likely to make new ideas when they are actively engaged in making some kind of external artifact – be it a robot, a poem, a sand castle, or a computer programme – which they can reflect upon and share with others. Thus, constructionism involves two intertwined types of construction: the construction of knowledge in the context of building personally meaningful artifacts. (Kafai & Resnick, 1996, p.1)

Within a constructionist framework, Kafai & Resnick (1996) emphasise the importance of children ‘reflecting upon their thinking and sharing their understandings with others’. Similarly, Noss & Hoyles (1996) refer to the importance of focusing on ‘the ways in which the understanding of mathematical ideas is mediated by the tools available for its expression’. So both constructionists and cognitivists raise the same point of children communicating and expressing mathematics. Enabling children to express and communicate their mathematics is a very important issue for this study, for two reasons; first of all, because communicating is considered as an integral part of thinking, and secondly because communicating was an issue that had to be dealt with delicately because of the subjects’ age.

Communicating is strictly connected with language. But how is language defined? As an extension of Vygotsky's theory, Rogoff (1990) expressed the idea of cultural tools. As stated by Shepardson (1999), ‘cultural tools for Rogoff consist of the technologies used to solve problems, the skills and procedures for using the technologies, and the language used for thinking, organising reality, and structuring activity.’ (p.624).

The construction task integrated in the task sequence that was designed for the purposes of this study acted as such a cultural tool, not only because it was used by the children as an object-to-think-with, to borrow Papert's (1993) expression, but also because construction was the language used by the children to express and communicate their thinking-in-change, to borrow an expression by Hoyles (2001).

This study builds on the perspective that communicating is an integral part of thinking. The constructionist culture allows children to think in new ways, by allowing them to communicate in new ways. Communicating is not simply expressing thinking, but rather reflecting on and

formulating understanding. Thus, in designing activities, I were looking to provide children with what Noss & Hoyles (1996) define as an 'autoexpressive' language. An autoexpressive language acts both as a thinking tool and an expressive tool.

But besides this support of construction from the perspective of mathematics education research, construction might turn out to be a very powerful research tool also from the perspective of early childhood education research.

### ***The use of construction in interviewing young children***

There is, therefore, an interest in investigating the potential roads that constructionism can open for research and education into the world of the mind. But not any mind. I am referring, specifically, to the mind of the very young child. The attempt to investigate constructionism is not new and it has been well achieved by others. The contribution of this study relates mostly to the age of the subjects. As was explained in Chapter 2, research in young children and shapes often lacks experience with, and understanding of, the individuality of the subjects. As a result, it leads to misleading conclusions, which underestimate the grandeur of the young mind. Besides my long-standing concern to carry out meaningful research within the domain of mathematics education, I always had to be alert of the fact that this research was about young children.

In Chapter 2 we saw how verbal data was used to evaluate young children's understandings, neglecting the fact that there are special issues to be considered in relation to young children's ability to communicate. Any attempt to do research with young children has to respect the fact that children think differently to adults.

One cannot overlook Piaget's main contribution to understanding children's thinking, which is that children think differently to adults. " Piaget's fame rested upon his demonstration that children have, as it were, a logic of their own and that their logic is as well formed as adult logic, but different ..." (Bruner, 1990, p.ix). But Piaget's conviction that children think differently to adults was incompatible with his methodology. As Bruner (1990) states: 'he did not extend that profound insight into the determination of how children were in fact viewing the tasks that he gave them'. And here lies the important contribution of others. We have, for example, Margaret Donaldson's main principle that the outcome of a cognitive act is strictly connected to the factors of the context in which it takes place. By changing some of the variables that formed Piaget's tasks, she proved that children's thinking is not as limited as Piaget had demonstrated in his theory. But as pointed

out by Donaldson (1978) herself, even though this brings children's thinking closer to that of adults, we should not reject Piaget's original insight; that children think differently.

It appears, then, that the theories about the growth of language and thinking which have been most influential over recent years are, in important respects, ill-founded. This does *not* mean that these theories are wrong in their entirety. Nor should we conclude that, because children turn out to be in some respects closer to adults than has been supposed, they are really just like them after all. It may simply be that we have to look for the differences elsewhere. (p.59)

The question is: 'Where should we look for these differences?' To answer this question, we should look into how the word 'language' has been defined within theories concerning language and thought. Language, implicitly or explicitly, implies the use of words. But is this the only context that can be given to 'language'?

**lan-guage** **1** the system of human expression by means of words **2** a particular system of words, as used by a people or nation **3** any system of signs, movements, etc., used to express meanings or feelings  
Longman Dictionary of Contemporary English,

Even though Vygotsky (1962) states that 'to understand another's speech it is not sufficient to understand his words ...we must understand his thought' (p.51), there is still a tendency of restricting language to speech, to a verbal communication system. What other routes are there to thought besides words? Within the computer age and Papert's revolutionary import of the programming language and the idea of 'thinking in images' (Papert, 1986), the discussion about language and thinking is elevated to a different level and context. Until before 'new technologies' opened new roads to thinking and understanding, children's use of words was analysed in order to unravel what children know or do not know, what children can or cannot do. But what the computer revolution showed was that language (the process of communicating and expressing) is not only about words. Words are not the only carriers of meaning and knowledge.

So now it might be the right time to return to the question: 'Where should we look for the differences between children and adults?' One difference might lie in the way meaning is expressed and communicated. As Loris Malaguzzi states 'the child has a hundred languages and a hundred hundred more. But they steal ninety-nine the school and the culture. Separate the head from the body. They tell the child: To think without hands to do without head, to listen and not to speak. To understand without joy ...' (poem translated by Lella Gardini, in Edwards et al, 1998).



So school, culture, curricula and, to a large extent, research tradition have stolen from children all but one language. Expressing and communicating with words was, for far too long, more or less the only way to children's minds. From the beginning of this thesis, in the Introduction Chapter, I described how my long and deep experience with young children has made it clear to me that children can do, and thus know, more than they can say. This point was actually the motivation for this study; my strong desire to find ways of allowing young children to show exactly what they are capable of, what they know and how they think. So in order to allow children to unravel their rich understandings, we have to equip them with multiple means of expression.

Constructionism provides a culture in which multiple ways of expression are made available so that multiple modes of understanding can be expressed. If school and culture have managed to separate the mind from the body, constructionism might be able to bring the two back together. Thus, besides the fact that constructionism has been a major contribution to mathematics education, for early childhood education (and research) it may solve a practical problem. It might be a way out of the verbal evaluation, which led to underestimating the ability of young children.

### **3.3.4. The role of the student teachers for data collection**

Besides this major issue of allowing children multiple ways of communicating, there was another issue to be considered in relation to the fact that this study was about young children. Interviewing children, young children to be more precise, is not an easy task. There are many issues to be considered. As stated by David (1992), 'interviewing children has been seen as a very flawed research method' (quoted in Brooker, 2001, p.164). And it is indeed a very flawed research method, and a very risky one in the hands of researchers who are not familiar with young children. The risk lies not in the limitations young children may have, as might have been implied, but rather in the interviewers' 'lack of understanding the subjects under investigation'.

Grieve and Hughes (1990) stress the importance of observing children in natural surroundings and support Margaret Donaldson's view that experimental evidence should be supplemented with naturalistic observations. Similarly, Brooker (2001) states that 'well-known studies of young children's language (Tizard and Hughes, 1984; Wells, 1985) confirm the commonsense observation of anyone working with the very young, that children's utterances are better in every way (longer, clearer, more complex, more thoughtful) when the children are in a familiar environment, with familiar adults' (p.164-165). The choice of the word commonsense as an

epithet for this conviction could not be more successful. For those who have had the opportunity to work and observe children on a daily basis this is commonsense. That is why it is a striking experience for people with this kind of familiarity with young children to come face to face with research concerning young children, when such research ignores commonsense knowledge of how young children think and act. It is precisely through this familiarity and understanding of young children that this study aspires to contribute something worthwhile to mathematics education research and practice.

The 'commonsense' knowledge (for people familiar with the nature of young children) that it is more likely to penetrate into children's minds if you investigate them in a familiar environment, with familiar adults, was something that could not be ignored. So this led to a search for a research design that would allow the integration of a more naturalistic approach within the cognitivist-constructionist paradigm supported earlier. The following section aims to clarify the outcome of this search.

### **3.4. Interviewing children within naturalistic settings**

So based on the requirements of this study, as these were described in Figure 4 and analysed in §3.3.3 and §3.3.4, there was a need for findings ways of interviewing children within naturalistic settings. The practical problem that needed to be solved was mostly related to the researcher's role. In contrast to a more naturalistic method of data collection, where the researcher observes children learning in classroom settings, the cognitivistic researcher's role is rather different. S/he has a dual control of the setting s/he studies (normally within the setting of task-based interviews); first, as the designer of the activities involved and tools used and second, as the interviewer or, in other words, as the participant observer. The researcher is not simply observing others while constructing and sharing meaning, but as a researcher is physically part of the setting and somehow influences what is 'learned'. The claim here is that these two roles are not contradictory to each other, and one is not necessarily better than the other. Actually, for this study there was a need to search for ways of combining the two.

A similar approach is found in Cobb et al (1993). Cobb et al (1993) describe a year-long teaching experiment in one second-grade classroom where data was collected by videotapings of every mathematics lesson for an entire year. Even though the initial orientation of the study was that of a cognitive constructivist approach and the original intention was to 'analyse individual children's learning as they participated in classroom mathematical activity', this changed in the process as

the researchers developed, as they themselves claim, a 'growing awareness of the inefficiency of their original orientation'. This growing awareness concerned the idea that mathematics is a social activity as well as an individual constructive activity. For the study described in this thesis, this awareness was connected to the age of the subjects, which demanded investigating individual activity within naturalistic settings.

The powerful idea borrowed from the study by Cobb and his colleagues is that of the creation of 'a small pedagogical community'. In the study designed by Cobb et al (1993), the classroom teacher conducted all lessons that were videotaped and analysed. The classroom teacher became a member of the research staff and joined regular project meetings once a week. There was, therefore, the creation of a 'small pedagogical community that engaged in joint pedagogical problem solving'. For the purposes of the study designed for this thesis, there was no need to create a pedagogical community since, as a teacher educator, I was already an active member of such a society. As a teacher educator in charge of the teacher training programme for pre-service preschool teachers at the University of Cyprus, I was mainly concerned with creating precisely such a prosperous pedagogical community, which aimed at 'joint pedagogical problem solving'. Thus, I had the advantage of making use of this community to achieve a better research design.

To be more precise, rather than conducting the task-based interviews myself, I prepared my student teachers within the University's pedagogical community to do so. One of the advantages of this decision was that the student teachers were doing their practice experience in preschools where they were placed as pre-service teachers and were considered as members of the school staff. Therefore, by the time they had to conduct the interviews, they were familiar with the school culture but, most importantly, the children were familiar with the student teachers whom they considered their own teachers and were used to 'playing'<sup>6</sup> with. The preparation of the pre-service teachers was carried out as part of their usual training during a teacher-training course, within the university's pedagogical community. This arrangement allowed the investigation of children while interacting with familiar adults within familiar settings.

In looking for the right research design, I realised that my role as a researcher was, in more than one way, identical to my role as a teacher educator. On a daily basis, my 'job' as a teacher educator was to conduct classroom observations in preschools where my student teachers were placed for their practice experience. The observations aimed at providing student teachers with feedback and helping them improve and develop their teaching skills, mainly in relation to

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<sup>6</sup> Any activity within early childhood education is treated as a form of play.

communicating effectively with children towards the construction of knowledge. The student teachers would take different courses on preschool pedagogy with me before and during their presence in preschools. In addition, there were supportive discussion groups organised on a weekly basis concerning pedagogical issues. Thus, in most cases while being the outside observer of 'lessons' I was, to some extent, influencing the setting in which learning occurred. As an outside observer, I was able to observe with an open mind the construction and communication of meaning while it was actually happening and at the same time I could recognise my influence on the setting.

Thus, in designing my study I had the opportunity to make use of my capacity as teacher educator. Instead of conducting the task-based interviews myself, I 'trained' my student teachers to do so as part of their everyday routine of interacting with children in preschool settings in the context of a teacher training course. The student teachers were placed in schools for a period of eleven weeks. During this time they became active members of a specific class. This meant that they acted as just another teacher for the children. By the time they had to take part in the interviews (between the 9<sup>th</sup> and 11<sup>th</sup> week of their attendance in their school) the children were familiar, comfortable and used to being involved in play, tasks and other activities with them. The interviews were therefore just another interesting thing to do with one of their teachers.

In integrating naturalistic elements into research about cognition, Moschovich & Brenner (2000) supported Cole's (1978) conviction that 'the analysis of any behaviour should begin with a descriptive analysis of at least one real world scene' (quoted in Moschovich & Brenner, 2000, p.466). Moschovich & Brenner (2000) build on Cole's statement and argue that the descriptive analysis of the real world scene will 'inform the design of an experiment that preserves some aspects of the real-world setting while modifying others' (p.466). So designing interviews should be based on classroom observations. In this study, this was achieved on two levels.

First, given my experience of classroom observations as a teacher educator, I was able to build the setting of the interviews and train my student teachers in relation to the broader culture of the schools involved in the study, with which I was very familiar. The schools involved in the study were the public schools used for the student teachers' practice experience. All public schools in Cyprus follow the same curriculum without major variations. The many years of classroom observations in different public schools have made me familiar with the community culture involved. The interviews and training of the students were consequently based on that familiarity. Moreover, the two schools involved in the study were schools with which I had a long and regular

cooperation and were more than willing to welcome pre-service teachers. Furthermore, the nine weeks that the student teachers had spent in the specific school and culture before conducting the interviews, provided them with a sense of familiarity with the school community and culture and the individuality of the children.

As was described in Figure 4 (p.47), the age of the subjects demanded observing children within their natural learning settings. The rationale behind this requirement is that children's understandings are better investigated in familiar environments, while the children are interacting with familiar adults (Brooker, 2001; Donaldson, 1978; Grieve & Hughes, 1990; Tizard and Hughes, 1984; Wells, 1985). The use of the student teachers brought the setting of the interviews closer to the children's natural learning setting. The children were investigated while interacting with familiar adults. The second naturalistic element that was integrated into the setting of the task-based interviews was related to the familiarity of the environment.

In relation to the above, besides the use of the student teachers and the physical setting (the children's school) we also had familiarity in relation to the choice of time for conducting the interviews within the school programme. The interviews were conducted during free play. Free play is a crucial part of early childhood education programmes since it is fully in line with the basic pedagogical principles and philosophy of early childhood education. As was explained earlier, during free play children are freely engaged in controlled activities. The teacher's role is to provide a rich learning environment where the child interacts freely with adults, other children and the objects available. Thus, in free play different children play different activities in different ways, alone, with teachers or with other children, in different parts of the classroom or school. So free play guarantees a balanced co-existence of a controlled learning setting with a sense of free activation where children are constructing and communicating meaning.

Brooker (2001) points out the benefit of conducting interviews with young children within play areas. Similarly, we stress the benefits of interviewing children during free play. For children who are involved in free play on a daily basis during specific hours of their school programme, free play is interpreted as the time when there are a lot of things to do and a lot of ways to play; some activities are old and some are new and there is the possibility of playing alone, with other children or with a teacher. Thus, given the many possibilities to which children are accustomed in free play environments, the subjects of this study were interviewed within familiar settings.

### **3.5. Concluding remark**

In this chapter we described the research methodology in relation to the setting in which the data was collected and supported the methodological choices by sketching the study's theoretical framework. The following chapter provides a thorough description of how the task sequence, the interview script tool and the training programme that were used in the main study were designed through the piloting procedure. A detailed description of the final products of the piloting procedure (task sequence, interview script tool, training programme) will also be provided in the next chapter.

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# CHAPTER 4

## Designing through Piloting

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### 4.1. Introduction: The piloting procedure

In the previous chapter we presented the main characteristics of this study's research design and theoretical framework. The subject of this chapter is to describe the piloting procedure and the products designed through it.




The empirical part of this study was organised in a series of pilot studies, which led to the design of the main study that was used for the collection of the final data for analysis. The fact that this study involved designing mathematical activity was the main purpose for conducting a series of piloting phases rather than one pilot study. Furthermore, there was a need for careful management in relation to training the student teachers in conducting the interviews. This element of the study was another reason for conducting a series of pilot studies. The piloting was not only about trying out and rehearsing the methodology, but mainly about designing the right interview setting that would allow addressing the main aim: to investigate young children's structural understandings of shapes through an investigation of squares.

The setting of the task-based interviews, as this was described in the previous chapter, consisted of four aspects: the child, the activities, the adult, and the adult-child interaction. The aspect 'child' was the aspect under investigation whereas the other three aspects were to be controlled by the products of the piloting procedure. Thus, conducting a series of pilot studies aimed at designing the following:

- A 'good' task sequence for creating the desired setting,
- A 'helpful' interview script tool for facilitating the child-adult interaction and
- A 'successful' training programme for preparing the (student teacher) interviewers for the interviews.

During the piloting procedure there was a continuous effort to design, try out and re-design the task sequence, the interview script and the training programme. The piloting procedure was completed in three phases. Table 1 provides a list of each phase with its main activities and purposes in relation to the three desired outcomes. As shown in this Table, the process of designing, trying out and re-designing the task sequence involved pilot phases 1-2. The interview script and training programme were the objects of pilot phases 2-3.

**Table 1: Overview of the piloting procedure**

	Task sequence	Interview script	Training programme
<b>Phase I (Autumn, 2001)</b> → Researcher 'playing' with children in playrooms	Trying out Preliminary task sequence  Designing Task sequence		
<b>Phase II (Autumn, 2001)</b> → Researcher conducting task-based interviews	Trying out Task sequence	Designing Interview script	Designing Training programme
<b>Phase III (Autumn, 2002)</b> → Researcher training student teachers → Student teachers conducting task-based interviews		Trying out Interview script  Improving Interview script	Trying out Training programme  Improving Training programme
<b>Main study (Spring 2003)</b> → Researcher training student teachers → Student teachers conducting task-based interviews			

In the first phase of the piloting procedure (Autumn, 2001), I conducted semi-structured, free play style interviews with young children, where a preliminary task sequence was tried out. The outcome of Phase I was the design of the task sequence that was used in the main study. This sequence was tried out in Phase II (Autumn, 2001), where once more I conducted task-based interviews myself. No changes were considered necessary in relation to the activities. Therefore, based on this task sequence, an interview script was prepared and a training programme was planned, with the purpose of being tried out in Phase III. In Phase III (Autumn, 2002), a group of eighteen student teachers was trained to conduct interviews and proceeded with doing so. Each student teacher conducted two interviews. This final phase of the piloting procedure allowed me to try out once more the effectiveness of the task sequence, but most importantly to try out and finally improve the interview script and training programme.

I did not determine myself the number of student teachers involved in the final phase of the piloting procedure and the main study. The student teachers were involved in the study as part of a School Experience Course they were following at the time. Thus, the number of student



teachers involved was the actual number of student teachers following the specific course. In the piloting procedure (Phase III), the number of student teachers was greater than the number of student teachers involved in the main study, but the student teachers involved in the main study conducted more interviews each (Table 2)

**Table 2: Number of interviews conducted during each phase of the research**

Research phase	Number of student teachers involved	Number of interviews conducted by each student teacher or the researcher	Number of interviews conducted during the phase
I	0 (interviews conducted by the researcher)	9	9
II	0 (interviews conducted by the researcher)	6	6
III	18	2	36
Main Study	13	4	52

The remainder of this chapter provides (a) a detailed description of the story behind designing, trying out and re-designing the task sequence and the final product of this process, (b) an analysis of the interview script designed through the piloting procedure to guide the student teachers through the interviews and (c) a description of the training programme as this was designed in the course of the piloting process to prepare the student teachers for the interviews.

#### 4.2. Designing the task sequence: The Description, Construction, Reflection (DCR) framework

As was emphasised in the previous chapter, this study pays great attention to designing mathematical activities. The role of the researcher is transformed into that of a director, ‘directing’ the setting in which the learning act will take place. The core of this setting is the activities, which aim at forming a dynamic microworld within which interesting learning situations will occur. More specifically, this study is about designing and investigating mathematical meaning making within constructionist cultures, an issue I have already extensively referred to in the previous chapter.

To meet the research aim as described in the beginning of Chapter 3, the search for a task sequence focused on designing construction tasks that would allow children to express structural understandings about squares. We will proceed with the description of how the task sequence evolved throughout the piloting phases.

In the first phase of the piloting procedure; I conducted semi-structured, free play style interactions with young children in preschool playrooms. The interaction with the children was

considered semi-structured, since it was based on a preliminary task sequence but there was no preparation in relation to my interaction with the children during the tasks. I spent five days in preschools where, during free play within the school's playroom, I created a 'corner'<sup>7</sup> where children could come on their own free will to play with me. The children's teacher introduced and 'advertised' the new corner in an effort to make the children feel more comfortable with the unknown adult and also make the new corner seem an interesting choice of play. All interactions were videotaped. The videotapes along with the researcher's fieldnotes were used for feedback. The task sequence and materials used in this phase were as shown in Table 3.

**Table 3: Overview of the preliminary task sequence that was used in Phase I of the piloting procedure**

Phase I / Preliminary Task Sequence		
Task Sequence	Material used	Description of task
Classification task (squares/non-squares)	Cutout squares and other shapes (with variety in relation to color and size)	The children were asked whether they recognised the squares and whether they knew the shapes' name and finally they were asked to classify the shapes into two groups (squares/non-squares).
Defining squares in words		The children were asked to say (in words) what they knew about squares.
Construction task	Wooden sticks: there was an effort of achieving variety in relation to length while including many sets of 4-6 equal sticks.	The children were asked to choose sticks from the pile of sticks provided in order to construct a square. After the construction of a square the children were engaged in a discussion with the researcher where they had to explain their choices in relation to the sticks they used and their spatial arrangement.

The aim of this pilot phase was to initiate the process of clarifying and elaborating the potential of construction tasks in relation to investigating children's understandings of squares. Nonetheless, the unstructured free play style interaction involved in this phase started with a classification task (Table 3). This was considered essential since the completion of the classification task ensured that the children had mental representations of the shape under investigation before proceeding with the construction activities. The children were presented with a number of cutout squares and a variety of other shapes and were at first asked to say what they see. Being flexible to their replies, I would try and see if the children could recognise the squares, name them and distinguish them from other shapes. Thus, the children had to classify all shapes into two groups: squares/non squares. If a child had difficulty in doing so, I would proceed with scaffolding interaction to help the child complete the classification task.

After the procedure of recognising and classifying the cutout shapes, the children were asked to say what they knew about squares (Table 3, Defining squares). Some of the questions used for

<sup>7</sup> During free play, playrooms are organised into a variety of 'corners' (art corner, dough corner, costume corner, alphabet corner, block corner, library corner, pretend play corner, music corner, science corner, maths corner, etc.) from which children can choose or may be encouraged by their teachers to go to in order to 'play'.

this discussion were borrowed from other studies (Shaughnessy & Burger; 1985, Clements et al; 1999), which tried to evaluate children's understandings of shapes through their utterances. The questions used were the following: 'How did you know these were squares?' 'Why do you say these are squares?' The aim of including this discussion was to be able to compare children's verbal responses (and thus other studies' findings) with the children's exposition of knowledge during and after the construction task.

After this discussion, the children were asked to choose sticks from a pile of wooden sticks provided in order to construct a square while having in front of them the squares from the starting task (Table 3, Construction Task). Again, the researcher was flexible in relation to the interaction with the children and interfering with their efforts. The children were left uninfluenced to complete their attempt and the interventions were mainly supportive. When the child would complete the construction, the interview would be related to whether the construction was successful or not. Either way, the interview would proceed with scaffolding intervention. In those situations where a child's construction was not a square (three out of the nine children that participated in this phase of the study failed to construct a square in their first attempt) the child was asked whether it looked like a square, or like the squares from the classification task and, while being flexible to the child's responses, I would try and facilitate the child towards constructing a square. Following the successful construction of the square (after the first or the second attempt), the child was asked to explain the choices made in relation to the sticks used and the way they were placed. Again, I was flexible in relation to the scaffolding involved.

Many clarifications are needed in relation to the scaffolding procedure and the supportive intervention on behalf of the interviewer, but this will be done later on. For the moment, it is important to say that as a qualified preschool teacher and given my experience as a teacher educator, I was capable of interacting with the children in phases I and II in an appropriate manner. This behaviour was later on transcribed in order to prepare an interview script. More details will be provided in the section concerning the preparation of the interview script through the piloting procedure (§4.3).

The use of the videotapes and fieldnotes led to important remarks in relation to the task sequence. First of all, the starting classification task was considered essential to ensure that the children could recognise the shape under investigation. Furthermore, the children used the cutout shapes from the classification task (sometimes after being encouraged to do so and sometimes by their own initiative) throughout the rest of the task sequence in the following ways: (a) for

feedback during their attempt to construct the shape; during the construction process they would look at the group of squares to see whether their construction looked like the cutout shapes or not, (b) as manipulatives to justify their choices during the construction task; for example in order to justify why they used those sticks instead of others they would take a cutout shape and point at its sides to support their choices.

The discussion that followed the classification task where children were asked to say what they knew about squares was successful, as it allowed comparison (a) with other studies that used verbal data for analysis and (b) with the children's exposition of structural understanding during the construction tasks. At this point, children would indeed (as shown by other studies) give answers (mostly similes) that did not explicitly exhibit structural understandings. For example, to the question why they say a square is a square they gave answers like 'because it looks like a window, a house, a toast'.

As expected, the construction task provided the children with a tool to use for thinking and constructing meaning and a way of expressing structural knowledge about the shape. To be more precise, the construction task provided evidence of the fact that children possess specific structural knowledge about the shape as well as evidence of how they use this knowledge in the process of learning. Children would start their construction by making some 'correct' choices while discovering new knowledge in the process. For example, in their effort to construct a square, three children tried initially to construct a quadrilateral with four right angles only to realise in the process that the opposite sticks had to be equal.

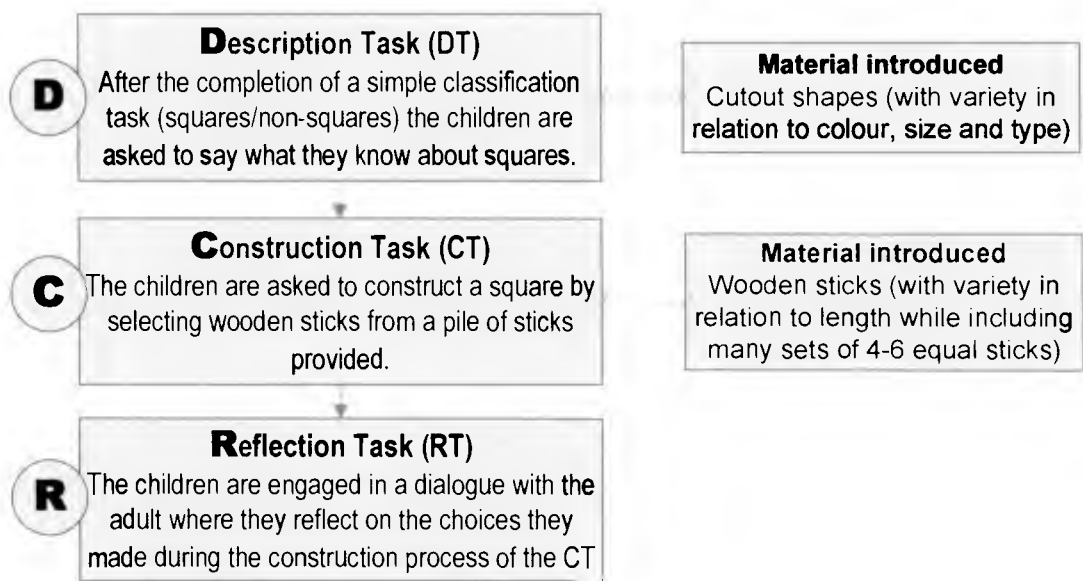
During the construction task, the children exhibited a much more sophisticated way of expressing their understandings about the shape than observed in the discussion prior to the construction task. Even though in most cases there were no clear signs of structural understandings during the short discussion prior to the construction tasks, at this point the children exhibited rich structural knowledge of the shape. The exposition of structural understanding was done in multiple ways: with the use of language (conventional and unconventional), with the use of manipulatives (cutout shapes and sticks), with the use of gestures.

It was therefore considered that the task sequence used in Phase I could lead to the collection of the data needed to allow us to address the main aim and answer the set of research questions as these were formulated in Chapter 3. Even though no essential changes were required in relation to the task sequence, as a result of pilot Phase I, I proceeded with re-defining and re-structuring

the preliminary task sequence. As will be explained later on in Chapter 5, this procedure evolved parallel to the process of transforming the research questions into a set of operational research aims; a process that was essential during the data analysis procedure.

Figure 5 presents an overview of the task sequence, as this was structured and defined after the completion of the first phase of the piloting procedure. In re-defining the task sequence, it was decided that the interviews would develop through a three-phase framework. In Phase A involving a Description Task (DT), the children would be involved in classification and recognition activities. In Phase B, the children would be involved in a Construction Task (CT) where they would be given wooden sticks of various lengths in order to construct a square. Finally, in Phase C defined as the Reflection Task (RT), the children would be involved in a process of reflecting with the interviewer on the construction process of Phase B.

Figure 5: Overview of the final version of the task sequence: The Description, Construction, Reflection (DCR) Framework



This Description-Construction-Reflection (DCR) framework on which the re-structuring of the task sequence was based, was considered an effective way to address the original research questions. The connection between the research questions and the three phases of the task sequence was originally introduced in Chapter 3 (Figure 2, p.39). More clarifications in relation to this relationship will be provided in Chapter 5 where I will describe how the DCR framework evolved parallel to the process of formulating the study’s operationalised aims.

In the second phase of the piloting process, I proceeded with trying out the DCR framework, which was the outcome of the starting phase of that process. I returned to the classroom I had

conducted pilot Phase I for five days and again created a corner during free play where children could come and play with me. Again, all interviews were videotaped and the videotapes along with the interview fieldnotes were used for feedback.

#### **4.3. Designing the interview script**

The aim of preparing an interview script was to support the student teachers' interaction with the children during the interviews and prepare them to be able to react in 'an appropriate manner' (as this will be defined further on) to all of the children's possible responses and behaviours.

By the end of Phase II of the piloting process, a preliminary version of the interview script was prepared. As I have already explained in the first two phases of the piloting procedure, I interacted with children in playrooms in order to design the task sequence. In the interactions involved in these two phases no interview script were used. As a qualified preschool teacher, I was able to interact with the children in an 'appropriate' manner; meaning that I was able to make good judgment on when and how to intervene, how to explain to the children what they had to do and support them through the interviews. In order to prepare the preliminary script at the end of Phase II of the piloting procedure, the interaction with the children involved in Phases I & II was transcribed. In view of obtaining a complete portrait of all possibilities of interaction, this preliminary script was continuously enriched. Thus, it was constantly transformed throughout the piloting procedure.

The interview script was tried out in Phase III of the piloting process by a group of eighteen student teachers. The fact that eighteen student teachers conducted two interviews each in phase III of the piloting procedure allowed the observation of thirty-six videotaped interviews. This contributed towards (a) having a better picture of the range of different reactions by the children during the interviews and (b) pinpointing specific parts of the interview where a number of student teachers did not interact 'appropriately' with the children. These observations allowed the improvement of the interview script. In order to avoid this 'undesired' behaviour, specific notes were added to the final version of the interview script to remind the interviewers of the kind of interaction they should use. Following these additions, the interview script was considered adequate to be used for the main study.

The final version of the interview script (Appendix C) portrays the role of the interviewer during the interaction with the children. The interview script involves the following four types of

interaction: instructive, inquiring, supportive and scaffolding. Definitions and examples of each of these types of interaction are provided below<sup>8</sup>.

**Instructive interaction** refers to the type of interaction which involves direct instructions. Instructive interaction occurs mainly at the beginning of an activity and is characterised by an explanation of what is expected of the children. In other words, the interviewer's role is instructive when specific instructions concerning the aim of an activity are given to the child.

For example:

- *'Now I would like you to separate all these shapes into two groups. Put all the squares in one group and all the shapes that are not squares in another group.'* (Appendix C).
- *'Now we will put aside the group of shapes which are not squares and leave all the squares here in front of us so that we can see them. Can you choose some of these sticks and make a square?'* (Appendix C).

**Inquiring interaction** refers to the type of interaction which involves questioning the children in order to allow them to say what they know or to explain their choices during the activities. Inquiring interaction occurs mainly at the end of an activity and the interviewer's role is to allow the children to express their understandings, and to support and explain their choices during the construction tasks.

For example:

- *'Why did you take these sticks in order to make your square?'* (Appendix C).
- *'How did you know all these were squares?'* (Appendix C).

**Supportive interaction** refers to the type of interaction which provides support to children during their effort in completing a task. There are two sub-categories of supportive interaction within the interview script.

- **Attention (re-) focusing:** where the interviewer for example repeats an instruction by highlighting specific elements, which the child seems to have paid no attention to or which might be helpful for the child's attempt to complete a task.

For example:

- *'You must be careful so that the square you make is nice like the ones in front of you'* (Appendix C).

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<sup>8</sup> A similar analysis of the interaction between interviewer and children can be found in Jones (1998).

- **Encouragement:** where the interviewer encourages and stimulates the child to continue and put effort into the activity.

For example:

- *'Well done. Go on!'* (Appendix C).

**Scaffolding** refers to the kind of interaction involved in those parts of the interview where the interviewer needs to react in relation to the child's response to an instruction or inquiry. The interview script provided the interviewer with specific reactions to each possible response of the child throughout the interview. The script provided guidance by instructing the interviewer what to do in each case: *'In case a child ....., then you should .....*'. There are two sub-categories of scaffolding within the interview script:

- **Supportive scaffolding:** where the interviewer provides support throughout a task when necessary, or when a child seems to be following an unfruitful path towards completing a task, or when the child's attempt at a task is not successful.

For example:

- *'In case a child constructs a shape that is not a square then you should ask the child if s/he likes his/her construction and to look carefully at the squares from the classification task. Ask the child to choose other sticks or change some of the sticks used in his/her construction. ('Would you like ...). Encourage and support the child to be careful and think about his/her choices in order to make a **nice** square like the ones in front of him/her. If the interaction is not successful and you feel you have tried everything, construct a square yourself ('would you like it if I built a square for you?). Then you ask the child to try again.'* (Appendix C).

- **Inquiring scaffolding:** which aims at engaging children in a discussion that will allow them to unfold and cultivate their ability to express and communicate mathematical meaning.

For example:

- *'In case a child answers that s/he selected these four specific sticks instead of others because they are small/big then you should ask: 'How many big/small sticks do we need to make a square?' ...'. 'Do you mean that with four small/big sticks I can make a square?... Let's try.' Pick up four sticks that are relatively small or relatively big but that are not equal, describing at the same time what you are doing in a manner that you are re-voicing the child's answer. 'I took four small/big sticks. Can I make a square with these sticks?' If the answer is yes then ask the child to try. If the answer is no then ask the child 'why?' (Appendix C).*



Whereas instructive and inquiry interaction was the part of the interaction which had to be followed as given with no intervention and flexibility on the part of the interviewers, supportive interaction and scaffolding required differentiation and flexibility and the interviewer had to react in relation to the child's responses. Scaffolding interaction is expressed in the interview script in the form of 'in case a child ..... then you should .....'. The piloting procedure helped in order to complete the interview script with 'all' of the children's possible responses and thus the proper interview reaction for each case. I had the opportunity to observe a relatively large amount of children throughout the three phases of the piloting procedure, a process which provided me with a relatively full picture of possible reactions by the children.

The final phase of the piloting procedure (Phase III) in which student teachers were trained to conduct interviews helped in order to point out specific, common 'mistakes' made by the student teachers during the interviews. So the piloting procedure also helped in adding specific notes to the interview script, which were considered helpful for the interviewers in order to avoid these common mistakes. The notes that were added to the final versions of the interview script concerned the following observed common 'mistakes':

- (a) There were specific parts of the interviews where the interviewer's role was simply to pose a given inquiry question to the child and accept any answer given with no further interaction. After receiving an answer from the children, many interviewers would ask further questions in order to guide the children towards answers that they thought were satisfactory. So the note '*Accept all answers as correct*' (Appendix C) was added to the interview script in relation to these parts of the interview.
- (b) In some cases, the interviewers would 'slightly' change the formulation of an instruction. This 'slight' change, however, had a major impact on the picture that the child created in relation to the aims of an activity. To be more precise, in the construction task the given instruction in the interview script was formulated in such a manner as to ask the children to construct shapes that would look like the ones in front of them. Some interviewers would pick up one specific square and ask the child to 'construct a square that would be the same as this one'. As a result, the children were trying to construct a square that had the same size as the one the interviewer was holding. A note was therefore added to the interview script to remind the interviewers not to use the expression '*make the same square as this one*' but rather '*make a square that looks like these squares*' (Appendix C).

- (c) Some confusion was observed in relation to the parts of the interview where scaffolding was required. These parts of the interview script included all possible reactions along with the right response on the part of the interviewers. It was observed that some interviewers would try and go through all the possible responses (sometimes in the order given in the interview script) somehow independently to each child's response. Thus, a note was added to remind the interviewer that *'in the directions given for scaffolding you will need to move backwards and forwards according to the child's answers as the discussion develops'* (Appendix C).

Special attention to these 'mistakes' was obviously given, as we shall see later on, during the training of the student teacher/interviewers.

#### **4.4. Designing the student teacher's training programme**

The final product of the piloting procedure was the training programme that would prepare the student teachers for conducting the interviews in the main study effectively. The group of student teachers involved in the pilot study (Phase III) was - as were the student teachers involved in the main study - studying at the Department of Education, University of Cyprus, in the four-year programme of studies for Preschool Teachers. Pilot Study Phase III was conducted during Autumn Term 2002. The student teachers were all in their third year of studies at the time of their enrolment in the study, just like the group of student teachers involved in the main study. Within the framework of a teacher-training (School Experience) course they were attending at the time, they were trained and supported to conduct the task-based interviews. I was the teacher educator responsible for this teacher-training course.

Based on the way the School Experience Course was organised, the student teachers were placed into schools during the third week of the course after attending some preliminary meetings with me at the University. Following their placement in schools, the student teachers had to spend most of their time, on a weekly basis in the school, being actively involved in the school's activities. Parallel to this, they were attending, on a weekly basis, three-hour classes at the University with me. During the 9<sup>th</sup> and 10<sup>th</sup> week, they attended two three-hour meetings during which they were trained for the interviews. They conducted the interviews during the last three weeks of the course (weeks 11-13). This was the procedure followed in Phase III of the piloting procedure as well as in the main study.

With the completion of piloting phases I and II (Autumn Term, 2001), and the design of the task sequence, a preliminary version of the training programme was prepared. This training programme was implemented in Phase III as part of the School Experience Course of Autumn Term, 2002. Pilot study III provided evidence that, in general, the training programme was successful. The involvement of the student teachers provided interesting data and did not interfere with the data's ability to give answers to the research questions. Nonetheless, the feedback from phase III led to some improvements in relation to the training programme. These improvements had to do (a) with the general guidelines provided to the student teachers (b) with the material used in the interviews and (c) with the interview script.

In relation to the general guidelines provided to the student teachers after pilot phase III, improvements were considered essential due to some technical problems that were pinpointed in relation to the aspect of videotaping, the completion of the observation sheet and the use of the interview script. In relation to videotaping, in many cases the student teachers would position the camera in a way that did not allow clear vision of the children's actions and choices. Furthermore, in some cases the children were working in areas outside the camera's field of view. Thus providing specific guidelines in relation to the way they had to position the camera and practical ways of restricting the area in which the children would 'play' (with the use of visual limits) in relation to the camera's field of view, were considered essential.

As far as the completion of the observation sheet was concerned, most observers in pilot phase III did not provide detailed information and were unable to keep up with the interview. Thus, the training programme was enriched with the addition of guidelines of (a) how to provide detailed information and (b) how to use drawings rather than verbal narrations to transcribe the children's actions in order to save time.

To conclude, on the improvements concerning the general guidelines, it was observed that in some cases the student teachers did not use the interview script effectively. During the interviews, they would spend a lot of time looking at the interview script to see how to proceed, which led to long pauses of perplexity, puzzlement and confusion. It was therefore considered necessary to emphasise in the training programme to the student teachers that they had to familiarise themselves with the interview script well before conducting the interviews, in order to avoid, as much as possible, looking at it during the interviews. They were asked to try out the procedure with one child before conducting all four interviews that were to be used for the study.

Based on these changes, an improved handout was prepared for the student teachers in relation to general guidelines (Appendix A).

In addition, in phase III it was observed that even though the student teachers were given exact guidelines on how to prepare the material needed for the activities, it was not exactly as it should be. It was therefore decided that the student teachers were not to prepare the material themselves. The material needed was to be prepared and provided by me.

Finally, the observation after the completion of phase III that student teachers made some common mistakes led (a) to the need for adding specific notes to the interview script to alert them in order to avoid these mistakes and (b) to pointing out the need for paying special attention to these mistakes during the training programme. All the comments that were added to the interview script after phase III of the piloting procedure were explained in §4.3.

Thus, the third outcome of the piloting procedure was the training programme described in Table 4. The training programme is described as part of the teacher-training course. The training programme of Table 4 was the one used in the main study. Based on this training programme, the student teachers in the main study were placed in couples in preschool classrooms during the third week of the course. During the 6<sup>th</sup> week, they were given a take-home activity (Appendix E) and were asked to read a set of papers concerning research on Children and Shapes. They were also given a set of questions to think about while reading the papers. The student teachers were told that this activity would prepare them for the meeting of the 9<sup>th</sup> week with their educator.

During the 9<sup>th</sup> week, the student teachers attended the first three-hour meeting concerning their involvement in the research project. After a discussion based on the take-home activity that was given to them during the 6<sup>th</sup> week of the course, they were presented with the research project. During the literature discussion as well as the presentation of the research project, emphasis was given to the need for serious consideration of the fact that there is a need for careful management when doing research with young children. They were presented with the way they would be involved in the study and were given general guidelines both in writing (Appendix A) and orally on how their involvement would be successful. Guidelines were provided in relation to the role of the interviewer, the role of the observer, the time and place of the interviews, the activities and material needed, the videotaping procedure, the use of the interview script, etc. They were also given the interview script and were told to read it carefully before coming to the next session.

**Table 4: The training program that was used for preparing the student teachers for the interviews**

Term week	Meeting Subject	Activity
3		1 <sup>st</sup> week of attendance in preschool settings.
6		4 <sup>th</sup> week of attendance in preschool settings Take-home activity: Literature readings on research concerning Children and Shapes.
9	Research project: 'Children and shapes' Discussion: <ul style="list-style-type: none"><li>• Literature review</li><li>• Project presentation</li><li>• Activity analysis</li></ul>	7 <sup>th</sup> week of attendance in preschool settings
10	Research project: 'Children and shapes' Workshop activities <ul style="list-style-type: none"><li>• Discussing videotaped incidents</li><li>• Role play</li></ul>	8 <sup>th</sup> week of attendance in preschool settings
11-13		9 <sup>th</sup> -11 <sup>th</sup> week of attendance in preschool settings. Conducting task-based interviews.

During the 10<sup>th</sup> week, the student teachers attended a second three-hour meeting, which was a workshop that aimed at training them to conduct the interviews. The workshop included the following activities:

- Discussing videotaped incidents: the student teachers were given the opportunity to watch videotaped incidents from Phases I and II of the piloting procedure. In the discussion that followed, the student teachers had to comment on the children's behaviour reflecting on the discussion of the previous meeting. They also had the opportunity to see me in the role of the interviewer and comment on the way I had conducted the interviews.
- Role play: the student teachers were asked to work in groups of three and take in turns the role of the interviewer, the interviewee child and the observer in an attempt to go through the interview script. They were asked to do so three times until they all had the opportunity to play all three roles. During this activity I would observe and transcribe some comments that I felt needed to be addressed after the completion of the activity.

**4.5. Concluding remark**

In this Chapter I have described the piloting procedure that led to the design of the task sequence, the interview script and the student teachers' training programme that were used in the main study. In the following chapter, I will describe the data analysis process which started evolving parallel to this piloting procedure.

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# CHAPTER 5

## Data Analysis

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### 5.1. Introduction

Chapter 3 provided a description of the methodology that was used for data collection, whereas Chapter 4 illustrated the piloting procedure through which the task sequence, the interview script and the interviewers' training programme were designed. Chapter 5 aims to describe the last part of the methodology; that of data analysis. Data analysis involved two processes: first, a process of reformulating the research questions in view of making them more investigable and second, the process of developing a coding scheme. The coding scheme was developed in two stages. A preliminary coding scheme was developed with the use of the data collected from the piloting procedure (first stage). This was then revised with the use of the data collected from the main study (second stage). The process of reformulating the research aims was evolving parallel to the first stage of developing a coding scheme. This chapter is organised into two main parts corresponding to the two stages of the process of developing a coding scheme.

### 5.2. Developing a coding scheme - First stage

The three phases of the piloting procedure provided three types of data. Since all interviews conducted during the piloting procedure were videotaped, the first type of data was in the form of videotaped interviews. The second type of data was in the form of fieldnotes. This type of data was gathered through the two phases of the piloting procedure (I, II) during which I conducted the interviews personally. These fieldnotes were my own notes while conducting these interviews. Lastly, the final phase of the pilot study (Phase III), in which interviews were conducted by student teachers, provided data in the form of observation forms completed by trained student teachers undertaking the role of the observer. By studying and processing all of the above data, I was able to reformulate the research questions and develop a preliminary coding scheme.

In Chapter 3, we explained that the main aim of the study was analysed into three research questions which, in turn, were further detailed into four operational research aims (Figure 2, p.39).

This final stage of transforming the research questions into operational research aims is closely connected to the process of designing a coding scheme. In the previous chapter, we described the process followed in order to design a task sequence that would address the research aims. As already explained in Chapter 4, the relationship between reformulating the research questions into operational research aims and designing the task sequence was a two-way relationship. The task sequence was obviously designed to address the research questions of the study, but through the process of designing the task sequence and studying the data collected from its application during the pilot study, the research questions were reformulated and refocused. This was the process that generated the operational research aims of the study. This same process also led to a simultaneous process of categorising the data. This categorisation process constituted the basis for the coding scheme that was eventually used for analysing the data of the main study.

What became apparent from the careful examination of the data collected from the pilot study was that the task sequence consisted of three phases, each of which could address specific aspects of the main research aim. The result of this acknowledgement was the re-structuring of the preliminary task sequence used in Phase 1 of the piloting procedure. As was described in Chapter 4, the preliminary task sequence was redefined in the form of a three-phase task sequence (the DCR Task Sequence) consisting of a Description Task (DT), a Construction Task (CT) and a Reflection Task (RT). Each task constituted a main category of data.

Figure 6 describes the operational research aims as these were formulated through the piloting procedure and the corresponding interview tasks/data categories. Going through the data collected from the pilot study, I developed and gradually completed, for each of these categories, a preliminary list of sub-codes, which was later enriched and revised with the use of the data collected from the main study.

It was obvious from the first two phases of the piloting procedure that the data was capable of providing information in relation to the children's understandings as these were expressed in two different settings. First of all, with the use of the data collected from the DT one could '*describe the ways in which children expressed themselves about squares within a setting restricted to classification, recognition and description tasks*' (Figure 6, ORA1). Second, through the children's involvement in the CT as well as in the RT, there was the possibility '*to describe the ways in which children express their understandings about the structure of squares in a process of constructing squares*' (Figure 6, ORA2).

Figure 6: Main categories of data and corresponding operational research aims

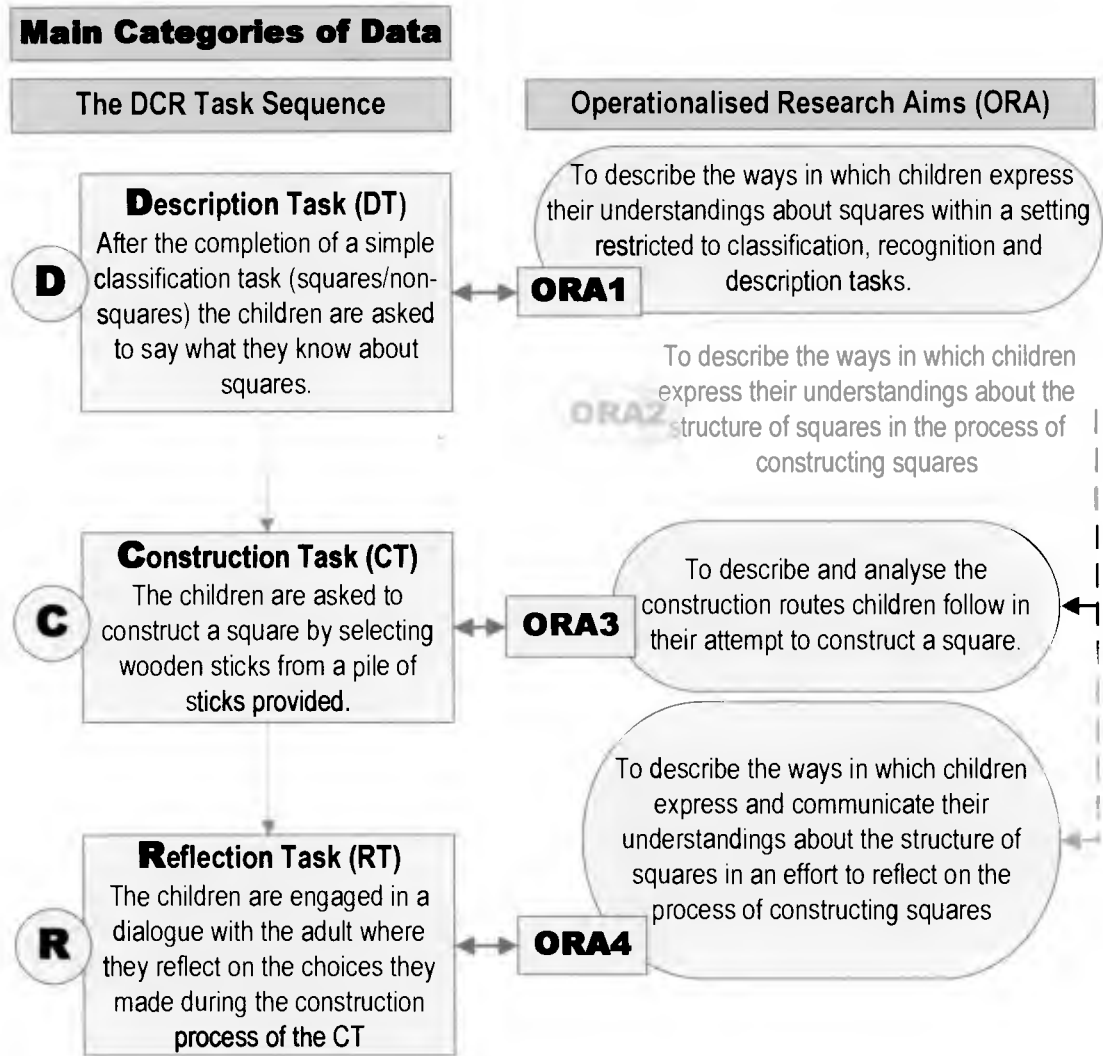


Figure 6 illustrates the way in which the three phases of the DCR Task Sequence form a separate main category of data and address the study's ORAs. ORA3 and ORA4 were the product of breaking down ORA2. Thus ORA2 is not directly connected to any main category of data but is indirectly connected to the CT and the RT.

Through their involvement in the CT, the children were given the opportunity to express their understandings through construction, while through the RT, they were given the opportunity to express their understandings through a process of reflecting on the construction process of the CT. Thus, ORA2 (Figure 6) was divided into two ORAs corresponding to the two final tasks of the DCR Task Sequence. The CT provided data that would allow us 'to describe and analyse the construction routes the children followed in their attempt to construct a square' (Figure 6, ORA3), whereas the RT provided data which gave us the possibility 'to describe the ways in which children express and communicate their understandings about the structure of squares in an effort to reflect on the process of constructing squares' (Figure 6, ORA4).



This simultaneous process of formulating the ORAs of the study and re-structuring the task sequence led to the three ORAs (Figure 6, ORA1/OR3/OR4) corresponding to the three phases of the DCR Task Sequence (Figure 6, DT/CT/RT). At the same time, the three phases of the task sequence represent the study's main categories of data (Figure 6).

In this chapter of the thesis I have, up to now, described how the piloting procedure assisted the process of reformulating the original research questions in view of making them more investigable and, consequently, how it allowed the development of the main categories of data. As was stated at the beginning of this chapter, by going through the data collected from the pilot study, I developed and gradually completed, for each of these categories, a preliminary list of sub-categories of data. This preliminary list started evolving from the beginning of the pilot study and was completed through a more systematic procedure with the use of the data collected from the final phase of the piloting procedure (Phase III).

Phase III of the piloting procedure provided a large amount of data from thirty-six interviews conducted by trained student teachers. This data was in the form of videotapes and observation forms completed by trained student teachers undertaking the role of the observer during the interviews. I went through all of the observation forms while watching the videotaped interviews, in an effort to identify themes and patterns, and thus develop sub-categories of data. The observation forms (Appendix B) that the student teachers were asked to complete during the interviews were structured in such a way as to allow me, later on, to take notes during this initial stage of open coding. A wide margin was left blank on the right-hand side of the form for 'comments', that the student teachers were specifically instructed not to fill in.

During the process of open coding, by going through the videotapes, I complemented the observation forms - where it was considered necessary - and took notes in an effort to identify and name interesting phenomena and patterns in relation to the categories of data. The units for this analysis varied according to the nature of the category. In some cases, the unit of analysis was the entire task sequence, a whole activity or a process, while in other cases it was a word or a sentence, a gesture, a choice or a specific action. This procedure led to the development of a list of sub-categories for each main category of data. Table 5 describes all of the sub-categories that emerged from this procedure for each main category of data and the corresponding units of analysis.

**Table 5: Preliminary sub-categories of data and corresponding units of analysis**

Main categories		Sub-categories	Unit of analysis
1	Description Task (DT)	Manipulatives	A specific choice and/or use of manipulatives
		Gestures	A certain gesture (hands, head, movement, fingers)
		Conventional/Unconventional language, Simile, Self-evident answer	A word, sentence or phrase
2	Construction Task (CT)	Strategies 1,2,3,.....	The square construction process
3	Reflection Task (RT)	Manipulatives	A specific choice and/or use of manipulatives
		Gestures	A certain gesture (hands, head, movement, fingers)
		Conventional/Unconventional language, Simile, Self-evident answer	A word, sentence or phrase

A number of sub-categories were identified in relation to the ways of expressing knowledge about squares during the DT (Main Category 1, Table 5). The children would use '*manipulatives*' (cutout shapes), '*gestures*' (use of hands, fingers or body to sketch a shape's component or for 'drawing' the shape), '*conventional language*' (use of formal language like 'equal sides', 'right angles') or '*unconventional language*' (use of alternative, inventive, exceptional, informal, everyday language) to express their understandings. The use of '*simile*' ('it looks like a house') and '*self-evident answers*' ('I know it is a square because I see it') were also identified as sub-classes of the unconventional language sub-category. The same sub-categories were used for the third main category which concerned the ways of expressing knowledge about squares through the RT (Main Category 1, Table 5). At this point, using the same sub-categories for these two main categories seemed a practical strategy that would allow me to compare how children expressed themselves about squares before and after the CT.

In relation to the second main category of data (Table 5), which concerned the construction routes followed by the children during the CT, different patterns were identified among the children. Based on these patterns and commonalities among the children's effort to construct the shape, I was able to sketch different strategies. The strategies identified during this stage of developing a coding scheme are described in detail further on. The sub-categories we have described based on the information of Table 5 were then transformed into codes. Table 6 provides a list of all of the codes that were developed after the piloting procedure and a description for each such code.

**Table 6: Description of the preliminary coding system**

<b>ORA1 DESCRIPTION TASK (DT): The child...</b>	
Manipulatives <b>DT-MANIP</b>	...uses manipulatives (wooden sticks, cutout shapes).
Gestures <b>DT-GEST</b>	...uses gestures (hands, fingers, body).
Conventional Language <b>DT-CONVLANG</b>	...uses formal mathematical language (e.g. equal sides, right angles).
Unconventional Language <b>DT-UNCONVLANG</b>	...uses unconventional (alternative, original, inventive, creative, exceptional, different, informal, everyday) language.
Simile <b>DT-SIM</b>	...uses simile (e.g. like a door, like a window).
Self-evident answer <b>DT-SELFEV</b>	...uses answers like 'because it looks like a square' or 'because of its shape' or 'because it looks like one'.
<b>ORA3 CONSTRUCTION TASK (CT): The child...</b>	
Square Construction Strategy 1 <b>CT-S1</b>	...selects four equal sticks and constructs a square.
Square Construction Strategy 2 <b>CT-S2</b>	...selects three equal sticks and constructs an open shape with right angles and then completes the construction.
Square Construction Strategy 3 <b>CT-S3</b>	...uses two equal sticks to construct a right angle and then completes the construction.
Square Construction Strategy 4 <b>CT-S4</b>	...selects two equal sticks and places them parallel to each other and then completes the construction.
Square Construction Strategy 5 <b>CT-S5</b>	...uses two unequal sticks to construct a right angle and then completes the construction.
Square Construction Strategy 6 <b>CT-S6</b>	...randomly selects unequal sticks and places them in such a way as to construct a quadrilateral with four equal sticks.
Square Construction Strategy 7 <b>CT-S7</b>	...randomly selects unequal sticks and places them in such a way as to construct a quadrilateral which resembles a square but has no right angles.
<b>ORA4 REFLECTION TASK (RT): The child</b>	
Manipulatives <b>RT-MANIP</b>	...uses manipulatives (wooden sticks, cutout shapes).
Gestures <b>RT-GEST</b>	...uses gestures (hands, fingers, body).
Conventional Language <b>RT-CONVLANG</b>	...uses formal mathematical language (e.g. equal sides, right angles).
Unconventional Language <b>RT-UNCONVLANG</b>	...uses unconventional (alternative, original, inventive, creative, exceptional, different, informal, everyday) language.
Simile <b>RT-SIM</b>	...uses simile (e.g. like a door, like a window).
Self-evident answer <b>RT-SELFEV</b>	...uses answers like 'because it looks like a square' or 'because of its shape' or 'because it looks like one'.

In relation to the square construction routes followed by the children, seven strategies were identified during this initial phase of open coding. They are described in more detail in Table 6. Even though identifying commonalities and patterns among the children allowed the identification and description of these seven strategies, I could realize, at this point, that this analysis was not

adequate to describe and analyse the children's routes to construct a square. This realization was the result of the observation that even though there were commonalities among the children, these did not involve the children's entire construction route.

As one can see in the descriptions of Table 6, in most cases a strategy does not portray the entire construction route followed by the children but the part of the process which was common among a number of children. Thus, in these cases children would continue their attempt in order to complete their construction by experimenting with different sticks and their spatial arrangement. As far as this experimentation is concerned, there was certain variability among the children that followed the same strategy.

For example, among the children that began their attempt to construct a square by using two equal sticks to construct a right angle (Table 6, CT/S2) there was a variation in relation to the way in which they completed their construction. Some children would proceed by trying out different sticks for the third side of the shape until they found one that was equal to the other two and then experiment with different sticks in order to close the shape. Other children would randomly add a third stick to their construction (which was unequal to the other two) and then experiment in different ways in order to close the shape. Thus, children following the same strategy could follow different routes in order to complete their construction and therefore end up with different shapes at the end of their construction attempt. At this stage of developing a coding scheme, no codes were identified in relation to this variability. This was something to be resolved in the final phase of developing a coding scheme after a careful examination of the data collected from the main study, and will be described in a following section.

### **5.3. Developing a coding scheme - Second stage**

During the second stage of developing a coding scheme, the preliminary coding scheme was revised based on a careful examination of the data collected from the main study. Through this process, all the sub-categories for each main category of data were changed into a more advanced format. In the following sections of this chapter, I will describe in detail the final sub-categories for each main category of data and how these were developed. First of all though, I will describe how the data collected from the main study was prepared for the analysis.

### **5.3.1. Preparing data for analysis**

After completing the process described in the previous section, there was a need for transforming the raw data collected from the main study into a more manageable and convenient form. All interviews were videotaped by the student teachers and all of the videotapes were handed over to me as the researcher. In addition, the student teachers also handed over an observation form for each interview. The observation form (Appendix B) completed by student teachers undertaking the role of the observer during the interviews conducted for the main study was structured in such a way as to be used for transcribing the interviews at a later stage. The student teachers were instructed to complete the observation form only in relation to the child's verbal and non-verbal behaviour. Columns were also provided for the interviewers' behaviour comments (and coding), which the student teacher - observers were instructed to leave blank.

After carefully watching the videotapes, I transformed these observation forms into interview transcript forms by filling in the column relating to the interviewer's behaviour that was left blank by the student teachers and by adding more details to the columns in relation to the child's verbal and non-verbal behaviour, which were already completed by the student teachers undertaking the role of the observer. Some transcriptions required more additions and alterations than others. In addition to the above, I was taking notes in the 'comments' column in relation to interesting stories within the data. A serial number was added on the top of the right hand-side of the cover of each transcript. The numbering started from 1.

After the data was prepared for analysis, I applied the preliminary coding scheme (Table 6, p.79). This resulted in a process of revising the sub-categories for each main category. The following sections provide a detailed description of this revising procedure.

### **5.3.2. Data analysis of the Description Task (DT)**

The data of the main study reconfirmed the original decision that no coding scheme was needed for the children's involvement in the classification tasks, since all of the children completed the task with no difficulty. Now the result of applying the preliminary coding scheme to the data of the DT was the formation of a more advanced system of categorising the children's answers. First of all, it became apparent upon carefully examining the data from the main study that it was essential not only to describe the ways in which understandings were expressed by the children but also to define a system which would allow the distinction between expression means that

implicitly included structural elements from other means of expressions. Secondly, I came to realise that the sub-categories I had identified in the preliminary coding scheme (Table 6, p.79) were not covering all of the children's answers. Thus, the preliminary categories were revised. This led to the coding scheme described in Table 7. For each sub-category of data, Table 7 provides the corresponding code and a description.

As shown in Table 7, the following six categories were categorised as Structural Answers:

- conventional language (use of verbal reasoning such as '*...because it has four sides/four angles*'),
- conventional language/manipulatives (use of verbal reasoning such as '*...because it has four sides/four angles*' and correspondingly shows parts of a cutout square),
- unconventional language (use of verbal reasoning such as '*...because it has four lines/four edges*'),
- unconventional language/manipulatives (use of verbal reasoning such as '*...because it has four lines/four edges*' and correspondingly shows parts of a cutout square),
- manipulatives (the child says something like '*... because it is like this*' and shows parts of a cutout square, or slides his/her fingers along the shape's perimeter), and
- construction (the child says something like '*... because it is like this*' and constructs parts of the shape such as a right angle with the use of hands or 'draws' a square with a finger in the air or on the table/floor).

On the other hand, we had a number of categories which did not implicitly include the expression of structural understandings. As shown in Table 7, there were nine such categories:

- simile ('*...because it looks like a house/window*'),
- self-evident answers ('*...because I see it is a square*'),
- self-evident/manipulatives (the child says something like '*...because it is like this*' and shows a cutout square),
- self-evident/class inclusion (the child uses a self-evident answer which is connected with the objective of the classification task ('*...they belong to the same group*'),
- simile/construction ('*...because we use it to make a house*'),
- colour/size ('*...there are different colours/sizes*'),
- teacher ('*...the teacher told me*'),
- no answer (the child says '*...I don't know*' or does not reply to an interviewer's question) and
- other (answers that did not fit into any of the other eight categories).

**Table 7: Codes and description of the children's categories of responses during the Description Task (DT)**

Description Task - DT <sup>9</sup>		
Sub-Categories	Code	Description: The child...
(Structural) Conventional Language	(STR) <b>CONVLANG</b>	...uses conventional language to describe the shape's structure (use of words like angles, sides, etc).
(Structural) Conventional Language- Manipulatives	(STR) <b>CONVLANG/ MAN</b>	...uses conventional language to describe the shape's structure (use of words like angles, sides, etc) in combination with the use of cutout squares.
(Structural) Unconventional Language	(STR) <b>UNCONVLANG</b>	...uses unconventional (alternative, original, inventive, creative, exceptional, different, informal) language to describe the shape's structure.
(Structural) Unconventional Language – Manipulatives	(STR) <b>UNCONVLANG/ MAN</b>	...uses unconventional (alternative, original, inventive, creative, exceptional, different, informal) language to describe the shape's structure in combination with the use of cutout squares.
(Structural) Manipulatives	(STR)MAN	...points to parts of a cutout square to describe the shape's structure.
(Structural) Construction	(STR)CONSTR	...'constructs' something with the use of hands or fingers to describe the shape's structure.
Simile	<b>SIM</b>	...uses a simile to describe what the shape looks like ('it looks like a window', 'a house', etc).
Self-evident answers	<b>SELF EV</b>	...gives self-evident answers ('because it looks like a square', 'because of its shape', 'because I see it is a square', etc).
Self-evident - Manipulatives	<b>SELF EV/ MAN</b>	...points to manipulatives (e.g. cutout shapes) to show what a square looks like (e.g. 'because they look like this').
Self-evident/ Class-inclusion	<b>SELF EV/ CLASSINCL</b>	...gives a self-evident answer which is highly connected to the objective of the preceding classification task ('In order to make a group!').
Simile/ Construction	<b>SIM/ CONSTR</b>	...refers to the use of the shape to construct something ('we use it to make a house', etc).
Colour/size	<b>COL/SIZ</b>	...refers to the different colours and sizes of squares.
Teacher	<b>TEACH</b>	...refers to the teacher ('the teacher told us').
No Answer	<b>NO</b>	...states that he doesn't know anything else about squares or does not reply.
Other	<b>OTHER</b>	...gives answer that does not fit into one of the other categories.

### 5.3.3. Data analysis of the Construction Task (CT)

Even though some themes and patterns in relation to the CT began emerging from the data collected from the pilot studies, the final coding scheme for this category of data was only completed after a thorough examination of the data collected from the main study. In my effort to

<sup>9</sup> Each code of the coding scheme consists of two parts separated by a dash (-). The first part of each code points out the main category of data (interview task) and is always illustrated on the first row of the relating Table of Codes. So all codes included in each table begin with the abbreviation indicated in the first row (e.g. DT). The second part of the code points out the sub-category of data and is indicated in the 'CODE' column (e.g. SIM). This format is applied in all of the Tables which provide Lists of Codes from this point onwards.

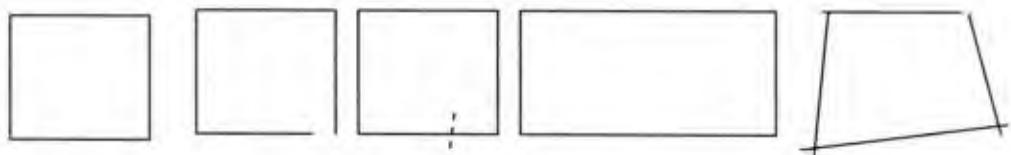
identify an analysis tool for analysing the data collected from the CT, I faced a practical problem. Even though I had identified this problem from the first stage of developing a coding scheme, it was resolved during the second stage of the process. As previously acknowledged (§5.1.), even though I could identify some themes and patterns among the children's effort to construct a square, finding some categories in relation to the complete process followed by the children turned out to be impossible.

There were some commonalities in relation to some actions in the children's attempt to construct a square, but the route each child followed was unique. The case of two or more children following exactly the same process was rare and thus, categorizing fifty-two children into 20-25 categories was not considered an effective way to categorise the data. So what could be the criterion for a meaningful categorisation? To answer this question, a very careful and repetitive examination of the nature of the task in relation to the data collected was considered essential. This thorough examination led to the conclusion that even though the original identification of strategies (Table 6, p.79) was a powerful way of analysing the data, further analysis was needed.

In the CT, the children were asked to construct a square by selecting wooden sticks from the pile of sticks provided. After the children were presented with the CT, they were left 'uninfluenced' to complete their construction. The only intervention at this point on behalf of the interviewer was of a supportive nature. A child's attempt to construct the shape was considered complete when the child would either verbally or non-verbally 'state' that s/he had finished. Some children would say 'finished' while others would stop, 'sit back' and wait for the next instruction, indicating therefore that they considered their construction complete and (in some way) successful.

Sometimes, the children's first attempt was successful (meaning that the outcome was a square) and sometimes not (meaning that even though the outcome had some properties of a square it was not a square). Figure 7 illustrates some examples of the final product of the children's first attempt to construct a square.

**Figure 7: Examples of products of the children's first attempt to construct a square.**



The slightly oblique dotted line crossing through the side of a construction indicates that that specific side is constructed with the use of two sticks.



Which of the examples in Figure 7 should be considered as a 'successful construction'? This is an interesting question. At this point, I would simply like to state that for the purposes of conducting the interviews (and not for the purposes of the analysis or for the sake of discussion) an attempt was considered successful by the interviewer only if the child had used four equal sticks to construct the shape. In case the first attempt was not 'successful', the interviewer would proceed with a scaffolding procedure in order to help the child successfully construct a square. Following the directions in the interview script (Appendix C) the interviewer would ask the child to try again. In case a child 'failed' to construct a square in this second attempt, the interviewer would construct a square with four equal sticks for the child to see and ask the child to try once again. So each child had up to three attempts in order to construct a square. In all interviews, after the successful construction of a square (or three failed attempts) the interviewer would proceed with the RT in which the child was asked to reflect on his/her choices in relation to his/her construction.

As a result of the way the CT, the RT and the interview script were designed the children went through three phases. The first phase was the **intuitive phase**, where the children had to draw on their intuition in order to begin their construction. The second phase was the **experimentation phase**, where children had to construct new knowledge on their already existing intuition through trial in order to complete their construction or in order to make other attempts until they ended up with a square. In some cases, this phase involved scaffolding intervention on behalf of the interviewer. Not all children had to go through the experimentation phase. The third phase was the **reflection phase** where the children were involved in a discussion with the interviewer and were asked to articulate their choices and actions thus expressing their newly built understandings. In the cases where the children failed to construct a square after the third attempt, the experimentation phase would extend parallel to the reflection phase.

In the previous analysis of the procedure followed by the children during the CT and the RT, we described the intuitive phase as the phase where children would draw on their intuition in order to begin their attempt to construct a square. Whereas the word 'intuition' was originally used in the process of data analysis simply as a synonym to the phrase 'existing knowledge', the thorough examination of the data collected from the main study allowed me to embark on a process of defining and describing intuitive knowledge. The examination of the data for the purposes of the analysis allowed me, at a first stage, to borrow some of the main characteristics which diSessa (2000) attributes to scientific intuition, to which I have referred in Chapter 2, and define intuitive

knowledge as the fragmented knowledge which children bring with them in a learning situation and are mostly able of expressing in ways which are contrary to what is formally acknowledged as correct. This issue of defining and describing the nature of intuitive knowledge will be re-addressed further on in Chapter 9. Based on the study's findings, I will attempt to describe in more detail the nature of intuitive knowledge as this was identified within this study's data. To be more precise, I will investigate whether this study's data can confirm the characteristics which diSessa (2000) attributes to intuitive knowledge.

In the following pages, I will describe the coding scheme that was developed for each of the children's attempt to construct a square.

### ***Data analysis of the children's first attempt to construct a square***

First of all, I will describe the coding scheme that was developed for analysing the children's first completed (not necessarily 'successful') attempt to construct a square. The importance of analysing the children's first attempt lies in the fact that this attempt includes the intuitive phase and thus allows us to focus on the children's intuitive knowledge about squares.

In their first attempt to construct the shape, the children followed different strategies that led them to a construction which was sometimes a square and sometimes not. Nevertheless, all of these strategies involved the exposition of specific structural knowledge about the shape. What became apparent from carefully studying the raw and transcribed data was that children would base their construction on a specific choice, **a foundational action** that had the attribute of stability. This specific action, which involved the choice of specific sticks and/or their spatial arrangement, was an action that remained intact until the end of their attempt and exhibited understanding of a specific property (or properties) of a square. This foundational action was the children's first action in their attempt to construct a square. The children would then proceed with other choices and actions in order to complete their construction. All of these other choices and actions during the construction attempt involved the element of experimentation and thus indicated that children were in a constructing process of building new knowledge on their original intuition (experimentation phase).

The identification of the children's foundational actions in the CT allowed the identification of nine square construction strategies among the fifty-two children that participated in the study. These nine categories are described in Table 8. They include the seven strategies that were identified in

the first stage of the process of developing a coding scheme (§ 5.1). Two additional strategies were identified within the data collected from the main study. Table 8 provides for each square construction strategy, the code that was used for coding the data and a description of the foundational action involved.

Three of the nine strategies identified did not include an experimentation phase. The foundational action in these three strategies was identical to the complete construction route followed by the children. These were S1, S6 and S9 (Table 8). S1 was the strategy, which led children directly to the successful construction of a square. Children following this strategy would select four equal sticks and use them to construct a quadrilateral with four right angles with no experimentation required. Children following S6 would randomly select four sticks and construct a four-sided shape with four right angles that resembled a square or a rectangle<sup>10</sup>. The outcome of this strategy would be shapes with gaps and or extensions. Children following S9 would select one stick at a time and construct a four-sided shape, which somehow looked like a square with its sides almost or not equal and its angles almost or not right.

All of the remaining six strategies involved the element of experimentation. Children following these strategies would start their construction attempt with a foundational action and then experiment in order to complete that attempt. Children's foundational action, when following S2 (Table 8), was to select three equal sticks and construct an open shape with right angles. Children following S3 (Table 8) would begin their construction by selecting two equal sticks and constructing a right angle. According to the description of S4 (Table 8), children, just like in S3, would start their attempt of constructing a square by selecting two equal sticks. But instead of placing the two sticks to create a right angle, like in the case of S3, they would place them parallel and aligned. In the case of S5 (Table 8), children would randomly select two sticks and create a right angle.







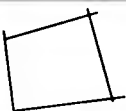
In all of the strategies described in the previous two paragraphs, the foundational action involved the selection of sticks and their spatial arrangement. On the other hand, in the case of S7 and S8 the foundational action only involved the selection of sticks. As in the case of S2, children following S7 (Table 14), would begin their construction by selecting three equal sticks. But the foundational action of S7 does not include spatial arrangement. As in the case of S3 and S4, the children following S8 (Table 8) would begin their attempt to construct a square by selecting two equal sticks. But after the selection of the equal sticks they would start experimenting.

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<sup>10</sup> Any reference to a rectangle from this point forward concerns non-square rectangles.

**Table 8: Codes and description of the strategies the children followed during their first attempt to construct a square**

Construction Task (Attempt A) – CT(A)










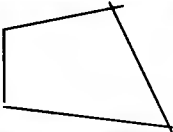

Strategies	Code	Foundational Action		
		Verbal Description The child .....	Choice of Sticks	Spatial Arrangement
1	S1	..... selects four equal sticks and places them one by one creating right angles and thus constructing a square.	4 equal sticks	
2	S2	..... selects three equal sticks and constructs an open shape with right angles.	3 equal sticks	
3	S3	..... selects two equal sticks and constructs a right angle.	2 equal sticks	
4	S4	..... selects two equal sticks and places them parallel and aligned.	2 equal sticks	
5	S5	..... randomly selects two (unequal) sticks and creates a right angle.	2 unequal sticks	
6	S6	..... randomly selects four (unequal) sticks and tries to construct a four-sided shape with four right angles.	4 unequal sticks	
7	S7	..... selects three equal sticks.	3 equal sticks	-
8	S8	..... selects two equal sticks.	2 equal sticks	-
9	S9	..... selects one stick at a time and tries to construct an irregular quadrilateral which somehow looks like a square with its sides not equal and its angles not right.	4 unequal sticks	

As I have already explained, the identification of the construction strategies could not cover the data collected from the complete construction route followed by the children. Thus, further analysis was considered essential. Besides the patterns identified within the data which allowed the identification of the nine strategies described in Table 8, patterns and commonalities were also identified in relation to the final product of the children's first attempt to construct a square. Eleven categories were identified in relation to this aspect. The products of the children's first attempt to construct a square are described in Table 9. The eleven products identified among the data were classified into three types (Table 9, Type A/Type B/Type C). The corresponding code for each product is also shown in Table 9.

As one can see from the descriptions in Table 9, all of the product categories were 'quadrilaterals', meaning that all of the children constructed a four-sided shape at the end of their first attempt to construct a square even though in some cases this construction had gaps and/or extensions. The difference between each Type (A/B/C) of category lies in the properties of the 'quadrilateral' that the children constructed. All of the products included in Type A were squares with or without some 'flaw' (e.g. gap, extension). On the other hand, all of the products included in

Type B were (non-square) rectangles with or without some ‘flaw’ (e.g. gap, extension). Finally, products under Type C were simply characterised by the property of having four sides.

**Table 9: Codes and description of the products of the children’s first attempt to construct a square**

Construction Task (Attempt A) - CT(A)				
Type	Product	Code	Verbal Description	Graphical Description
A	1	P1	Square with four equal sticks.	
	2	P2	Square with four sticks (gap).	
	3	P3	Square with four sticks (extension).	
	4	P4	Square with more than four sticks.	
B	5	P5	Rectangle with two sets of equal sticks.	
	6	P6	Rectangle with four sticks (gap(s)).	
	7	P7	Rectangle with four sticks (extension).	
	8	P8	Rectangle with more than four sticks.	
	9	P9	Rectangle with four sticks (gaps and extensions).	
C	10	P10	Irregular quadrilateral that resembles a square but has no right angles.	
	11	P11	Irregular quadrilateral with some angles right and/or some sides equal.	

The slightly oblique dotted line crossing through the side of a construction indicates that that specific side is constructed with the use of two sticks.

To be more precise, as illustrated in Table 9, Type A includes the following four categories: (a) square with four equal sticks (P1), (b) square with a gap (P2), (c) square with an extension (P3) and (d) square with more than four sticks (P4). Type B includes the following five categories: (a) rectangle with four equal sticks (P5), (b) rectangle with a gap (P6), (c) rectangle with an extension (P7), (d) rectangle with more than four sticks (P8) and (e) rectangle with gaps and extensions (P9). Finally, Type C includes the following remaining two categories: irregular quadrilateral that

resembles a square but has no right angles (P10) and irregular quadrilateral with some angles right and some sides equal (P11).

Therefore, through the analysis which I have described up to this point in relation to the data that was collected from the CT, patterns were identified in terms of the strategies that the children followed and the final product of their first attempt to construct a square. But does this analysis cover the entire construction route followed by the children? It actually does only in those cases where the children's construction route did not include an experimentation phase and consequently the foundational action was identical to the complete construction route followed by the children. As I explained when I was describing the children's construction strategies, there were three such strategies: S1, S6, S9 (Table 8, p.88).

So where does this leave us in relation to the data that was collected from the children that followed one of the other six strategies? In respect of these children, the analysis described up to now covers only the beginning (the foundational action which characterises the strategy followed) and the outcome (product) of their attempt. What happens in relation to the data in between the beginning and the end of the construction route? Upon carefully going through the data in these cases, two important observations were documented. The first observation was that children following the same strategy did not necessarily end up with the same product. The second observation was that children that did follow the same strategy and ended up with the same product did not necessarily follow the same route.

These observations exhibit the variability identified among the children's attempt to construct a square. How could this variability be dealt with? One way of dealing with it would be to form categories by corresponding strategies to products. This corresponding process gives us all the categories shown in Table 10. As one can see, this matching process gives us a very large number of categories (99 categories). Each child can fit into one of these categories according to the strategy s/he followed and the product of his/her construction attempt. Since the categories are more than the subjects of the study, we know from now that not all categories will appear in the data. Nevertheless, fitting the children into these categories will allow us, in Chapter 7, to identify patterns (e.g. which strategies led to which products) and describe the variability that existed among the children's construction routes.

**Table 10: Formation of categories by corresponding the children's strategy to the product of their first attempt to construct a square**

Construction Task (Attempt A) - CT(A)

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11
S1	S1/P1	S1/P2	S1/P3	S1/P4	S1/P5	S1/P6	S1/P7	S1/P8	S1/P9	S1/P10	S1/P11
S2	S2/P1	S2/P2	S2/P3	S2/P4	S2/P5	S2/P6	S2/P7	S2/P8	S2/P9	S2/P10	S2/P11
S3	S3/P1	S3/P2	S3/P3	S3/P4	S3/P5	S3/P6	S3/P7	S3/P8	S3/P9	S3/P10	S3/P11
S4	S4/P1	S4/P2	S4/P3	S4/P4	S4/P5	S4/P6	S4/P7	S4/P8	S4/P9	S4/P10	S4/P11
S5	S5/P1	S5/P2	S5/P3	S5/P4	S5/P5	S5/P6	S5/P7	S5/P8	S5/P9	S5/P10	S5/P11
S6	S6/P1	S6/P2	S6/P3	S6/P4	S6/P5	S6/P6	S6/P7	S6/P8	S6/P9	S6/P10	S6/P11
S7	S7/P1	S7/P2	S7/P3	S7/P4	S7/P5	S7/P6	S7/P7	S7/P8	S7/P9	S7/P10	S7/P11
S8	S8/P1	S8/P2	S8/P3	S8/P4	S8/P5	S8/P6	S8/P7	S8/P8	S8/P9	S8/P10	S8/P11
S9	S9/P1	S9/P2	S9/P3	S9/P4	S9/P5	S9/P6	S9/P7	S9/P8	S9/P9	S9/P10	S9/P11

**Data analysis for the children's second attempt to construct a square**

As explained earlier on, the coding scheme previously described was related to the children's first attempt to construct a square. A similar coding scheme was developed for those children whose first attempt was not successful and had to try again. In case the first attempt was not successful, the interviewer would follow the directions in the interview script (Appendix C) to ask the child to try again. In this second attempt three approaches were identified, as described in Table 11. The existence of these approaches is based on the fact that in this second attempt the children had in front of them the construction of their first attempt. So some children would try and fix their original construction (Table 11, A1), others would undo their original construction and try again from the beginning (Table 11, A2) while others would try to fix their original construction but would, at some point in the process, decide to undo it and start all over again (Table 11, A3).

**Table 11: Codes and description of the approaches the children followed during their second attempt to construct a square**

Construction Task (Attempt B) – CT(B)

Approaches	Codes	Description: The child...
1	A1	...tries to fix the construction that was the product of his/her first attempt.
2	A2	...undoes the construction of his/her first attempt and starts all over again.
3	A3	...tries to fix the construction that was the product of his/her first attempt but then decides to undo it and start all over again.

As described in Table 12, A1 had three sub-categories. Children following A1 would either *fix* the mistake in the product of their first attempt in order to end up with a square with four equal sticks or make a number of *adjustments* in relation to the sticks and the spatial arrangement of their original product or *improve* their original product by changing the spatial arrangement of the sticks they had originally used. All children that followed the first sub-category (*fix*) were children whose original construction was a Type A product (square with a flaw). These children ended up with a

square with four equal sticks in this second attempt. For example, a child that had constructed a square with a gap in his/her first attempt would replace the smaller stick of the original construction with one that was equal to the other three.

**Table 12: Codes and description of the sub-categories in relation to the approaches of the children's second attempt to construct a square**

Construction Task (Attempt B) – CT(B)A		
Approaches	Sub-categories/Codes	Description: The children would...
1	Fix	...fix the mistake in the product of their first attempt in order to have a square with four equal sticks.
	Adjust	...make all necessary changes (change up to three sticks and their spatial arrangement) so that their shape acquires a new form
	Improve	...improve the product of their first attempt without changing any of the sticks so that it would look more like a square
2	Strategies 1-9	...use one of the strategies in Table 8 (p.88) to start all over again.
3	Combination of the sub-categories of approaches 1 and 2.	...first attempt to fix, adjust or improve the product of their first attempt. They would then undo their original construction and apply one of the strategies in Table 8 (p.88) to start all over again.

On the contrary, only some of the children that followed the second sub-category (*adjust*) ended up with a square with four equal sticks. Children which followed the third sub-category (*improve*) did not end up with a square with four equal sticks but with a shape that was an *improved* version of their original construction. For example, a child that had constructed a quadrilateral with no equal sticks and no right angles in his/her first attempt could *improve* it by turning it into a rectangle.

When following A2, the children would use one of the Strategies described in Table 8 (p.88). Finally when using A3 they would use a combination of the sub-categories of the other two approaches. Again, this second attempt could lead to a successful or unsuccessful product. The coding scheme used for the product of this second attempt was the same as the one used for the first attempt (Table 9, p.89). I simply had to replace the phrase Attempt A with the phrase Attempt B.

### ***Data analysis for the children's third attempt to construct a square***

As explained earlier, the children could try once again if they had failed in their second attempt. The interviewer would construct a square and ask the child to try as well one last time. Again, for this third attempt I used the same coding scheme as for the second attempt by replacing the phrase Attempt B with the phrase Attempt C.



5.3.4. Data analysis of the Reflection Task (RT)

After the CT, the children were involved in a discussion with the adult, where they had to reflect on the choices they had made in order to construct a square. This discussion consisted of two parts. The first part had to do with the choice of sticks and the second with their spatial arrangement. The interviewer would begin each part of the discussion with a question and then follow the interview script (Appendix C) to interact with the child. The opening question for the first part of the discussion was: ‘Why did you take these sticks in order to make your square?’ The opening question for the second part of the discussion was: ‘How must we place these sticks in order to make a square?’ Each part of the discussion was considered a separate unit of analysis and was analysed with the use of a different coding scheme for reasons I will explain shortly. Table 13 demonstrates the codes that were used for each part of the discussion.

As explained previously in §5.2, my initial intention was to use the same coding plan used for the DT to analyse the data collected from the RT, so that I could compare the two. But the careful examination of the data collected from the main study led to a very important realisation. The nature of the discussion involved in the RT after the children’s attempt(s) to construct a square was very different to the discussion involved in the DT. A different coding scheme was therefore considered essential. I will deal with the two parts of the RT separately. For each part of the RT, I will describe the nature of the dialogues involved and the coding scheme that was developed.

Table 13: Codes and description of the two parts of the Reflection Task (RT)

Reflection Task – RT		
Part	Code	Description: The part of the discussion which began with the question:
A	A	‘Why did you take these sticks in order to make your square?’ and ended when the interviewer would proceed with posing the question: ‘How must we place these sticks in order to make a square?’
B	B	‘How must we place these sticks in order to make a square?’

The Reflection Task (Part A)

The differences identified between the data that was collected from the DT and the data collected from the RT(A) were due to two aspects. First, whereas in the DT the interviewers were instructed to accept all of the children’s answers as correct with no further scaffolding intervention, in the discussion following the construction of a square the interviewers were instructed to follow the interview script (Appendix C) until they would get a specific response by the children. In relation to the choice of sticks, the interviewers were instructed to follow the interview script until the

children would give a conventional answer (e.g. 'I chose these sticks because they were the same/equal'). Thus, whereas in the case of the DT the children expressed their understandings without further interaction with the adult, in the case of the RT(A) the children were, in most cases, involved in an extensive dialogue until they would give an answer that would 'satisfy' the interviewer. In other words, whereas in the case of the DT the children expressed their understandings about squares by purely replying to a question posed by the interviewer, in the RT(A) the dialogues involved the aspect of communication. The interviewers were trying to interact with the children in order to lead them to a specific answer and the children were trying to explain their understandings in different ways, in their effort to get through to the interviewers.

Furthermore, the discussion between the interviewer and the child within the RT was built on the preceding construction experience. With the use of the interview script (Appendix C) the children were asked to justify the choices they had made in order to construct the shape. In this sense, the transcribed dialogues were **situated**, meaning that they were characterised by a certain style and quality which was determined by the setting of the task. The setting of the task influenced the type of language that was used. All dialogues involved language that was referring (a) to the construction act and the manipulatives used, and (b) to the structure of the shape. Whereas only some categories of answers were characterised as structural answers in the DT, all of the children's answers in the dialogues of the RT were structural answers. Moreover, these answers were situated on the preceding construction act. This is why I will use the term **con-structural language** to describe the quality of the language involved in the RT.

Consequently, the data collected from the RT(A) provided, in most cases<sup>11</sup>, extensive dialogues between the interviewer and the interviewee, which contained highly interesting information. Whereas in the DT the unit of analysis was a child's response to a question, in the case of the RT(A), I decided to use the entire dialogue as the unit of analysis.

By carefully examining the data, I identified specific patterns in relation to the way these dialogues evolved. In these dialogues, the children would use different ways to express their understandings and justify why they chose the sticks they did in order to construct a square. Even though the language used was con-structural language, there was a difference among the dialogues in relation to the communication route that was followed. As explained earlier, the interviewer was trying to interact with the child so that the child would give a conventional answer.

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<sup>11</sup> There were cases where the child would reply to the opening question with a conventional answer. In these cases, no scaffolding was required and thus the dialogues were short.

Table 14 provides a description of what was considered conventional and unconventional answer during this phase of the interviews.

**Table 14: Definition of conventional and unconventional answers given by the children during the first part of the Reflection Task (RT) regarding the choice of sticks for the construction of a square**

Reflection Task (Part A) – RT(A)		
Type of answer	Code	Description
Conventional	CONV	'I used these sticks because they were the same/equal.'
Unconventional	UNCONV	All other answers or responses were characterised as unconventional answers.

A communication route is characterised by the alternation of conventional and unconventional answers. One would expect that two communication routes would have been identified among the data. The first communication route would be the one starting with unconventional answers and ending up with a conventional answer and the second would be the one where children would give a conventional answer from the beginning. The fact is that we identified six different communication routes among the fifty-two children. Table 15 provides all the relevant information

**Table 15: Codes and descriptions of the communication routes followed by the children in the first part of the Reflection Task (RT) regarding the choice of sticks**

Reflection Task (Part A) – RT(A)	
Communication Routes/ Codes	Description: The child...
CONV	...gives a conventional answer from the beginning.
UNCONV→CONV	...gives an unconventional answer at the beginning. Thus the interviewer proceeds with the scaffolding intervention until the child gives a conventional answer.
CONV→UNCONV→CONV	...gives a conventional answer from the beginning but for some reason the interviewer proceeds with scaffolding intervention during which the child gives unconventional answers before repeating the conventional answer.
UNCONV→CONV→UNCONV→CONV	...gives an unconventional answer at the beginning. Thus the interviewer proceeds with the scaffolding intervention. At some point of the dialogue, the child gives a conventional answer. Nevertheless, the interviewer proceeds with the scaffolding interaction. As a result, the child gives unconventional answers again. At the end, the child gives a conventional answer and the interviewer ends the interaction procedure.
NO REPLY→CONV	...does not reply to the interviewer's question but after the scaffolding intervention the child gives a conventional answer.
NO REPLY→UNCONV→CONV	...does not reply to the interviewer's opening question. Once the scaffolding intervention begins the child starts giving unconventional answers. At some point, the child gives a conventional answer and the interviewer ends the dialogue.

The first communication route (Table 15, CONV) was the shortest since the child would immediately give the conventional answer and thus the interviewer would move on to the second part of the dialogue, which had to do with the spatial arrangement of the sticks. The second communication route (Table 15, UNCONV→CONV) was the route where the children would begin the dialogue by giving unconventional answers. After interacting with the adult, they would

give a conventional answer and thus the interviewer would move on to the second part of the discussion.

The two communication routes described in the previous paragraph were the anticipated ones. Surprisingly, four additional routes were identified. Two additional routes were identified (Table 15, CONV→UNCONV→CONV, UNCONV→CONV→ UNCONV→CONV) due to the fact that, in some cases, the interviewers would proceed with the scaffolding interaction even though a child had given a conventional answer.

Finally, we had the case of children who did not reply to the interviewers' opening question. In some of these cases, the children would give conventional answers immediately after the interviewers would start the scaffolding procedure (Table 15, NO REPLY→CONV) whereas in other cases the children would give unconventional answers at the beginning of the scaffolding procedure before ending with a conventional answer (Table 15, NO REPLY→UNCONV→CONV).

***The Reflection Task (Part B)***

The second part of the RT (RT(B)) concerned the spatial arrangement of sticks and was the final part of the interviews. Because of the way the interview script (Appendix C) was designed, this final part of the interviews was completed in four steps. Each step was analysed in a different way. Table 16 provides a description of each step. Each such step was assigned a different code.

As one can see in Table 16, in the discussion that concerned the spatial arrangement of the sticks, the interviewers were instructed towards a different scaffolding intervention than the one of RT(A). Initially, the children were asked to respond to the interviewers' opening question - 'How must we place these sticks [a set of four equal sticks] in order to make a square?' (Table 16, Step 1). In the case of RT(A), the interviewer was instructed to interact with the child until s/he would give a conventional answer. In relation to the spatial arrangement of sticks, it was considered 'needless'<sup>12</sup> to try and lead the children towards a conventional answer (e.g. 'We have to construct right/90° angles'). The interviewers were instructed to accept all of the children's answers to the opening question ('How must we place these sticks in order to make a square?') as correct and then proceed according to the interview script (Appendix C).

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<sup>12</sup> The vocabulary of five-year-olds does not normally include expressions like 'right/90° angles'.

**Table 16: Codes and description of the four steps of the second part of the Reflection Task (RT) regarding the spatial arrangement of sticks for the construction of a square**

Reflection Task (Part B)- RT(B)		
Step	Code	Description: The interviewer...
1	ST1	...asks the child 'How must we place these sticks [a set of four equal sticks] in order to make a square?' and accepts all answers as correct.
2	ST2	...constructs a rhombus with the set of four equal sticks and asks the child 'Is this a square?'. If the child's answer is 'No' the interviewer proceeds with Step 3. If the answer is other than 'No' the interviewer follows the interview script until she gets a negative answer from the child.
3	ST3	...asks the child 'Why [is this not a square]?'
4	ST4	(in case during step 3 the child does not transform the rhombus into a square) ...asks the child to do so.

According to the interview script (Appendix C) the interviewers were instructed to take four equal sticks and construct a rhombus (Table 16, Step 2). Then they would ask the children whether they believed this was a square or not ('Is this a square?'). If the child's answer was 'No' the interviewer would proceed to Step 3. If the answer was 'Yes' the interviewer would follow the interview script until the child would give a negative answer.

In Step 3 (Table 16), the interviewers were instructed to ask the child: 'Why [is this not a square]?'. Again the children's responses were accepted with no further interaction. As we will see shortly, some of the children would transform the rhombus into a square in order to justify why the shape constructed by the interviewer (rhombus) was not a square. In those cases where the children did not use this type of justification, the interviewer was instructed to ask the child to transform the rhombus into a square (Table 16, Step 4).

In Table 17 we have a description of the coding scheme that was developed for analysing Step 1 (ST1). Seven categories were identified among the children's responses to the interviewer's question (How must we place these sticks [a set of four equal sticks] in order to make a square?). Four of these categories involved some kind of construction. Some children constructed something with the use of sticks or verbally described a construction process which involved sticks (Table 17, Construction (Sticks)), and some children constructed something with sticks and at the same time made use of cutout squares or previous square constructions (Table 17, Construction (Sticks)/Manipulatives). Some children constructed something with their fingers (Table 17, Construction (Fingers)) while others constructed something with their fingers and, at the same time, made use of cutout squares or previous square constructions (Table 17, Construction (Fingers)/Manipulatives).

**Table 17: Sub-categories of the children's responses to the question: 'How must we place these sticks [four equal sticks] in order to make a square?'**

Reflection Task (Part B-Step 3) – RT (B-ST1)		
Sub-Categories	Codes	Description: The child...
Construction (sticks)	CONSTR (Sticks)	...constructs something with the use of sticks or verbally describes a construction process which involves sticks.
Construction (sticks)/ Manipulatives	CONSTR (Sticks)/MAN	...constructs something with the use of sticks and at the same time makes use of manipulatives (cutout squares or previous square constructions).
Construction (fingers)	CONSTR (Finger)	...constructs something with the use of his/her fingers.
Construction (fingers)/ Manipulatives	CONSTR (Finger) /MAN	...constructs something with the use of his/her fingers and at the same time makes use of manipulatives (cutout squares or previous square constructions).
Unconventional Language	UNCONVLANG	...gives a verbal answer which is different to a conventional answer ('right/90° angles').
Self Evidence /Manipulatives	SELFEV/MAN	...points to manipulatives (e.g. cutout shapes, previous square constructions) to show what a square looks like (e.g. 'because they look like this').
No reply	NOREPLY	...does not reply.

Three more categories were identified. Some children used unconventional language (Table 17, Unconventional Language), some children used self evident answers by pointing towards cutout squares or previous square constructions (Table 17, Self evidence/Manipulatives) and others did not reply at all to the interviewer's question (Table 17, No Reply).

Since Step 2 (Table 16) of this part of the interview involved a closed question ('Is this [rhombus] a square?'), only three categories were identified among the children's responses (Table 18). Some children gave a positive answer (Table 18, YES), others gave a negative answer (Table 18, NO) while others did not reply (Table 18, NOREPLY).

**Table 18: Sub-categories of the children's responses to the question: 'Is this [rhombus] a square?'**

Reflection Task (Part B-Step 2) – RT(B-ST2)		
Sub-Categories	Code	Description: The child...
Yes	YES	...answers 'yes'.
No	NO	...answers 'no'
No reply	NOREPLY	...does not reply to the question 'Is this a square?'

Moving on to Step 3 (Table 16) of the RT(B), ten categories were identified among the subject's responses to the interviewer's question -'Why [is this not a square]?'. Table 19 provides a list of these categories.

In two of the categories included in Table 19 the children interfered with the rhombus the interviewer had constructed to explain why it was not a square. To be more precise, some children changed the position of all of the sticks of the rhombus in order to 'fix' it and make it a

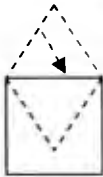
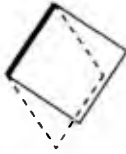
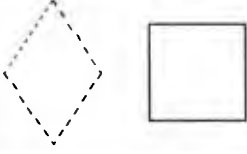
square (Table 19, Fix) while some children transformed the rhombus into a square by changing the position of only the three sticks parallel to using some kind of verbal unconventional justification (Table 19, Unconventional Language/Transform). Two more categories involved the use of verbal unconventional justification. A number of children simply used unconventional language (Table 19, Unconventional Language) whereas another group of children used unconventional language in combination with pointing towards cutout squares or previous square constructions (Table 19, Unconventional Language/Manipulatives). Two of the categories in Table 19 involved some kind of construction. Some children used sticks to construct something (Table 19, Construction(Sticks)) while others used their fingers (Table 19, Construction(Fingers))

**Table 19: Sub-categories of the children’s responses to the question: ‘Why is this [rhombus] not a square?’**

Reflection Task (Part B-Step 1) – RT(B-ST3)		
Sub-Categories	Codes	Description: The child...
Fix	FIX	...changes the position of all of the sticks of the rhombus so that s/he ends up with a square that has a prototypical orientation.
Construction (sticks)	CONSTR(Sticks)	...constructs something with the use of sticks.
Construction (fingers)	CONSTR(Finger)	...'constructs' something with the use of his/her fingers.
Unconventional Language	UNCONVLANG	...gives a verbal answer which is different to a conventional answer ('right/90° angles').
Unconventional Language/Transforms	TRANSF	...keeps one of the sticks of the rhombus in its original position and changes the position of the other three sticks. The child ends up with a square that has no prototypical orientation. Parallel to transforming the shape, the child gives a verbal answer which is different to a conventional answer ('right/90° angles').
Unconventional Language/Manipulatives	UNCONVLANG/MAN	...gives a verbal answer which is different to a conventional answer ('right/90° angles'). Parallel to that, the child points to manipulatives (e.g. cutout shapes, previous constructions) to support his/her answer.
Self evidence	SELF EV	...gives self-evident answers ('Because it's not a square.', 'Because it's a different shape')
Self Evidence/Manipulatives	SELF EV/MAN	...points to manipulatives (e.g. cutout shapes, previous constructions) to show what a square looks like (e.g. 'because they look like this').
Simile	SIM	...constructs something with the use of sticks.
No reply	NOREPLY	...does not reply.

Finally three categories were identified in relation to the procedure the children followed in order to transform the rhombus into a square (Table 16, Step 4). Some children ‘fixed’ the rhombus by changing the position of all of the sticks so that they would end up with a horizontal/vertical orientation (Table 20, Fix). Other children transformed the rhombus into a square by changing the position of only three of the sticks (Table 20, Transform). Finally, some children reconstructed the shape, meaning that they took the sticks of the rhombus in their hands and used them to construct a square right from the start (Table 20, Re-construct).

**Table 20: Sub-categories of the children's attempt to transform the rhombus into a square**

Reflection Task (Part B-Step 4) – RT(B-ST4)			
Sub-Categories	Codes	Description: The child...	Visual Description
Fix	FIX	...changes the position of all of the sticks so that s/he ends up with a square that has a prototypical orientation.	
Transform	TRANSF	...keeps one of the sticks in its original position and changes the position of the other three sticks. The child ends up with a square that has no prototypical orientation.	
Re-construct	RE-CONSTR	...takes the sticks which the interviewer used to construct the rhombus in his/her hands. Then s/he uses the sticks to construct a square.	

### 5.3.5. Issues of reliability

During the second stage of developing a coding scheme, a qualified Early Childhood teacher actively involved in qualitative research with a PhD in the field of Early Childhood Education spent some time to support the process. To be more precise, she viewed parts of videotaped interviews from the main study and used the same observation form (Appendix B) used by the student teachers to transcribe the children's actions. Then, we met, viewed the same interviews together and compared her transcripts with mine. Special attention was given in this session to the construction routes followed by the children during the CT. This was done at the early stages of the data analysis process. At a second stage, we met and spent some time discussing and practicing the coding system I had developed. Then she coded eight transcripts which were used for a reliability check. There was a 91% rate of agreement between her coding scheme and mine.

### 5.4. Concluding remark

In the three following chapters, the findings of this study are presented based on the data analysis described in this chapter. To be more precise, Chapters 6, 7 and 8 treat the findings in relation to the children's involvement in the Description Task (DT), the Construction Task (CT) and finally the Reflection Task (RT) respectively.



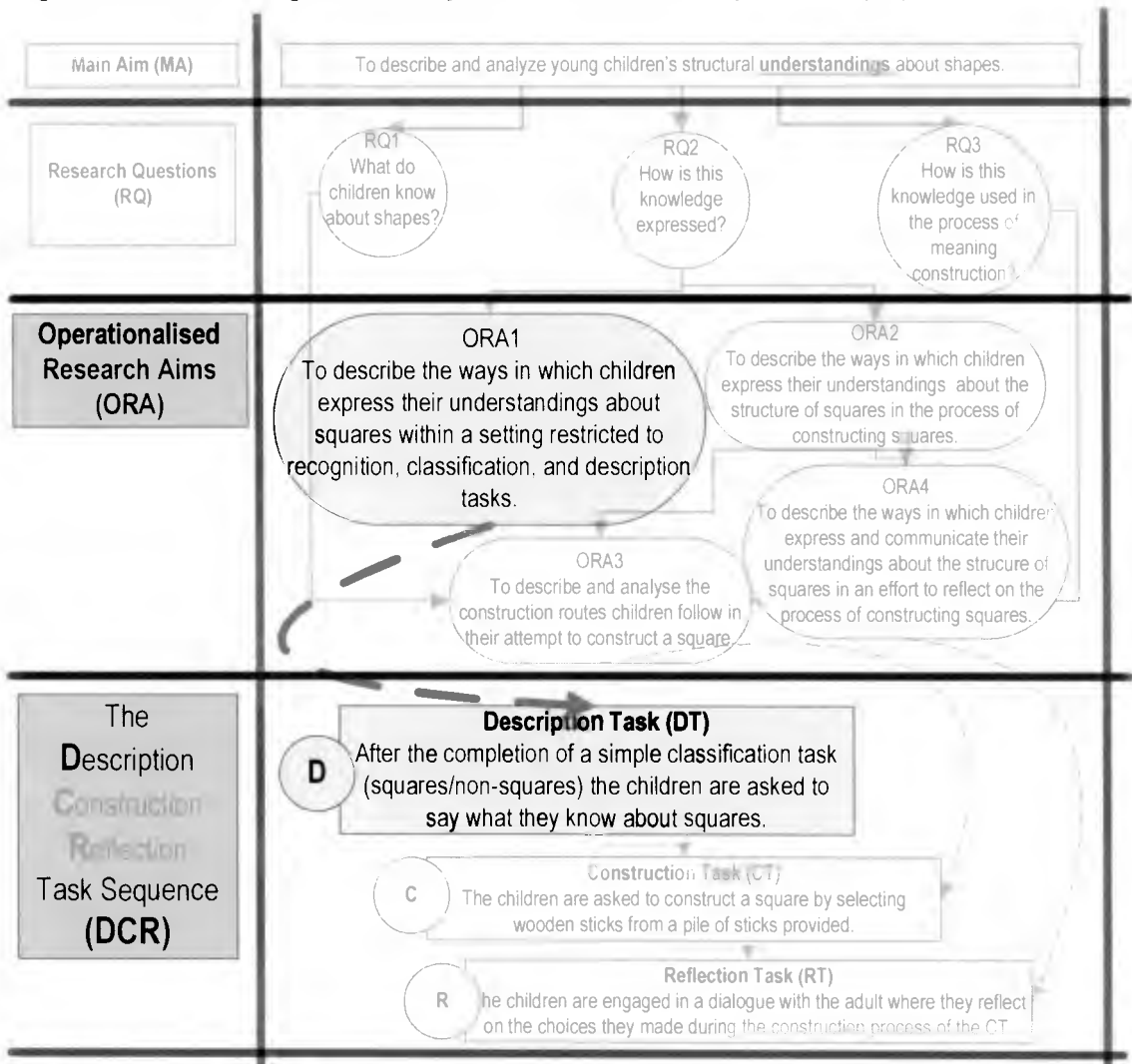
# CHAPTER 6

## FINDINGS: THE DESCRIPTION TASK (DT)

### 6.1. Introduction

Figure 8 illustrates the part of the aim and design of the study as described in Chapter 3 (Figure 2, p.39) which is the subject of this chapter. This chapter therefore portrays the findings in relation to the first task of the task sequence (Figure 6, DT) in an effort ‘to describe the ways in which children express their understandings (about squares) within a setting restricted to classification, recognition and description tasks’ (Figure 6, ORA1).

Figure 8: Aims and design of the study in relation to the Description Task (DT)



After the completion of a simple classification task, each child was asked to justify putting all squares in the same group and say whether s/he knew anything about squares. Each response to an interviewer's question was considered a unit of analysis and was assigned a matching code (Table 7, p.83). One hundred and thirty-eight responses were collected from the fifty-two children that took part in the study<sup>13</sup>. The interview script (Appendix C) included the following three questions which the interviewers were instructed to use to interview the children:

1. *'How did you know all these were squares?'*
2. *'Why do you say all these are squares?'*
3. *'Is there anything else you know about squares?'*

The reason for getting 138 responses from 52 children lies in the fact that some interviewers would ask the children all three questions whereas others would only ask the children questions 1 and 3 or 2 and 3 (presumably because they considered questions 1 and 2 similar). Thus, in the following section we have a description of the findings in relation to the ways the children expressed their understandings about squares during their involvement in the DT.

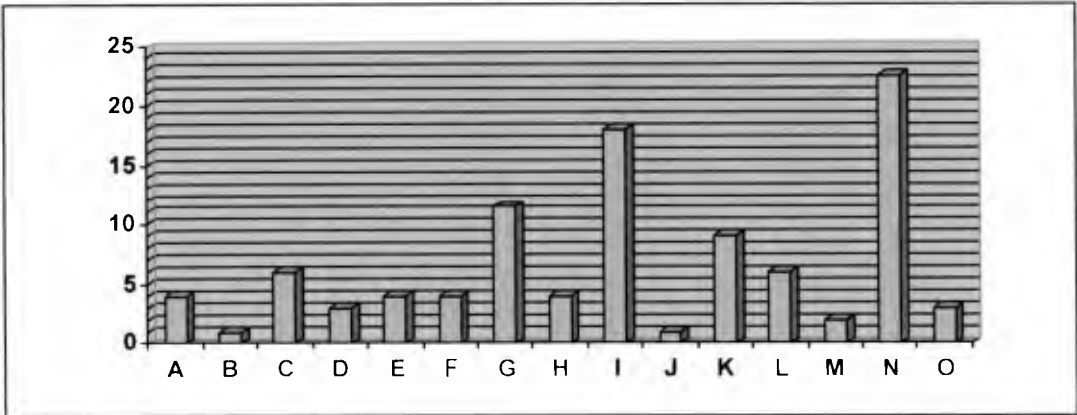
## **6.2. The ways in which the children expressed their understandings about squares during the Description Task (DT)**

Graph 1 provides the percentage of responses for each category identified in the data analysis process (Table 7, p.83). As one can see, the most frequently appearing type of response among the 52 subjects of the study was the 'NO' category. More than 20% of the responses were categorised as 'No' (Graph 1, N). The second most frequently appearing category of responses was the self-evident category (Graph 1, I). There were 25 such answers (18%). More than 10% of the children's responses were categorised as Simile Responses (Figure 7, G) and almost 10% were categorised as SELEV/CLASSINCL responses (Figure 7, K). All other categories of responses were much less frequent. The information provided by Graph 1 provides a first sense of the 'quality' of the children's understandings as expressed in a setting restricted to classification, recognition and description tasks.

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<sup>13</sup>A total of 137 responses were collected. There was one response which consisted of two parts. Each part could fit into a different data category and thus the response was counted twice. Reference to this response is made later on (p.109).

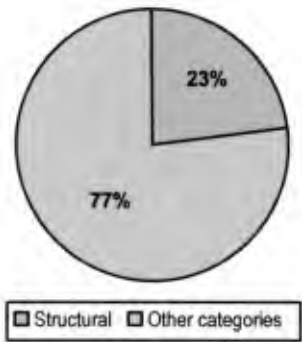
**Graph 1: Results in relation to the categories of responses given by the children during the DT**



All values are given in percentage rates.  
Key to response categories: A=(STR)CONVLANG, B=(STR)CONVLANG/MAN, C=(STR)UNCONVLANG, D=(STR)UNCONVLANG/MAN, E=(STR)MAN, F=(STR)CONSTR, G=SIM, H=SIM/CONSTR, I=SELFV, J=SELFV/MAN, K=SELFV/CLASSINCL, L=COL/SIZ, M=TEACH, N=NO, O=OTHER.

As explained in the previous chapter however, in the second stage of developing a coding scheme it was considered essential to define a system which would allow the distinction between expression means which implicitly included structural elements ('reference' to the shape's structure) and other means of expression. In Graph 2, one can see the results in relation to this distinction (structural - other categories). It is interesting to note that within the setting of the DT, 77% of the children's responses did not implicitly include structural elements.

**Graph 2: Percentage of 'Structural Answers'**

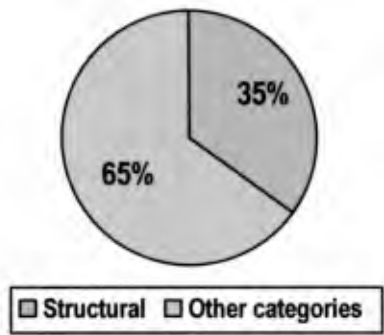


The picture is slightly different if we look at the results not in relation to the number of structural answers but in relation to the number of children that gave structural answers<sup>14</sup>. According to Graph 3, only 35% of the children involved in the study gave structural answers. This percentage

<sup>14</sup> It is interesting to pinpoint that all of the children that gave structural answers also gave answers which fall into other categories of responses. To be more precise, in 14 out of the 18 cases of children that gave a structural response during the DT, the structural response was first. On the contrary, we had the case of four children that initially gave a non structural answer and then proceeded with a structural answer.

represents 18 out of the 52 children that were involved in the study. Thus, what is important to highlight at this point is that 34 of the 52 subjects of the study (65%) expressed themselves in ways that did not exhibit any structural understanding of the shape's structure during the DT.

**Graph 3: Percentage of children that gave 'Structural Answers'**

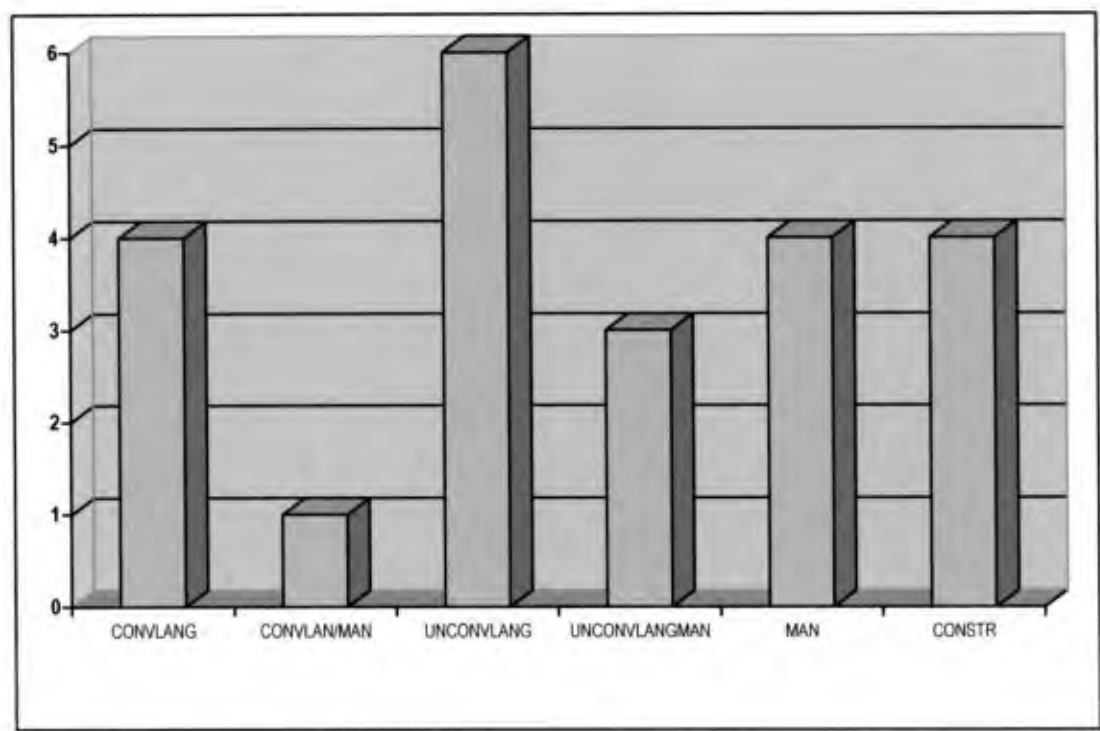


At this point, it is important to take a look at some examples of answers from the data which was categorised as structural. Based on the coding scheme that was described in Chapter 5, there were six different sub-categories of structural answers. Graph 4 provides the findings in relation to these sub-categories. The most frequently appearing structural responses were the type of responses that included the use of unconventional language (Graph 4, UNCONVLANG, UNCONVLANG/MAN). There were eight UNCONVLANG responses and five UNCONVLANG/MAN responses among the data. In these cases, the children would use their own words to describe the shape's structure. For example, they would use the word 'line' to refer to the sides and the word 'edge' to refer to the angles of the square.

Alexis, for example, (5,8)<sup>15</sup> stated that a square 'has four lines' whereas Marina (5,1) explained: 'They all have the same lines. They don't have two big and two small ones'. At the same time, she showed the parallel sides of a cutout square (first the one set of parallel sides and then the other). Furthermore, Melina (5,9) in an effort to describe the spatial arrangement of the sides of a square said: 'because it has one line up, one line down, one line right and one line left'. Similarly, Periklis (5,7) said: 'because we make a line, a line, a line' while moving his fingers along the sides of a cutout square. Periklis' and Marina's responses were categorised as (STR)UNCONVLANG/MAN. On the other hand, Costas (6,3) used the alternative word 'edge' to refer to the square's angles. Costas said 'it has four edges'.

<sup>15</sup> The parenthesis following a child's name indicates the subject's age (years, months)

Graph 4: Results in relation to the sub-categories of structural responses



Graph 4 shows the number of responses per category.

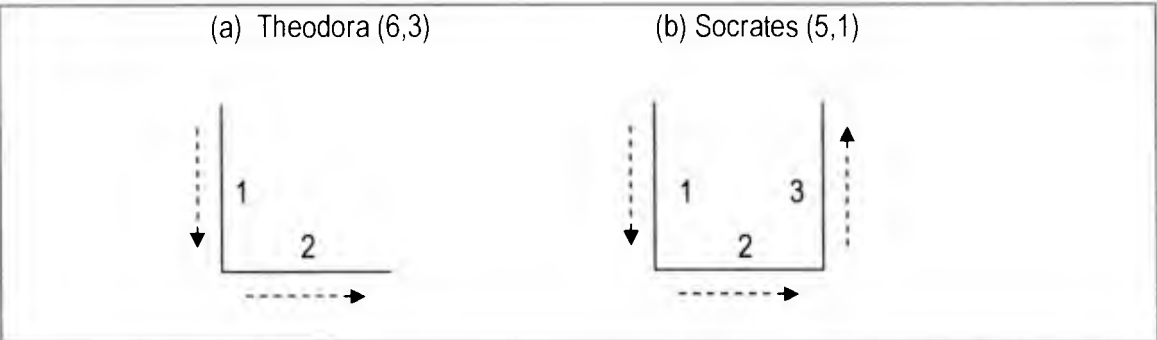
Furthermore, there was a limited number of responses which included the use of conventional language (Graph 4, CONVLANG, CONVLANG/MAN). Four responses were categorised as CONVLANG and only one response as CONVLANG/MAN. Demetra (5,7) said that squares 'have four sides that are the same' and Giota (5,3) said that 'they have four sides'. On the other hand, Christoforos (5,4) said that squares 'have four angles'. Giota (5,3) also gave a CONVLANG/MAN response during the DT. She said 'because it has one side, two, three, and four' and at the same time showed the sides of a cutout square.

It is highly interesting how the children used manipulatives and construction to respond to the interviewer's questions during the DT. As shown in Graph 4, besides the ways in which the children used manipulatives to support their verbal (conventional and unconventional) responses as we saw in some of the preceding examples (UNCONVLANG/MAN, CONVLANG/MAN), there were six responses which were categorised as MAN and five responses categorised as CONSTR. We had, for example, the case of Menelaos (6,0) who said 'it has like this' while sliding his hand around each of the sides of a cutout square.

The responses that were categorised as CONSTR are also interesting. Alexis (5,8) said: 'because it is like this' and drew a square with his finger on the table. Theodora (6,3) also said 'because it is like this'. But Thelma 'constructed' with her finger a right angle as shown in Figure

9a. Similarly. Socrates (5,1) said: 'because they are like these and like these and like these' while 'drawing' with his finger on the table the shape illustrated in Figure 9b.

**Figure 9: Examples of the ways in which the children used construction during the DT**



The dotted arrows show the direction of the children's moves whereas the numbers show the order in which they 'constructed' each line.

To conclude, in relation to the findings concerning the structural responses, there was limited use of conventional language by the children during the DT. Only 23% of the reported responses made some use of conventional language (Graph 5a). Most of the structural responses reported made use of alternative means of expression such as unconventional language, manipulatives, and construction. 55% of the structural responses included some kind of non-verbal justification (Graph 5b) whereas 35% of the structural responses were supported only by non-verbal means of expression (Graph 5c).

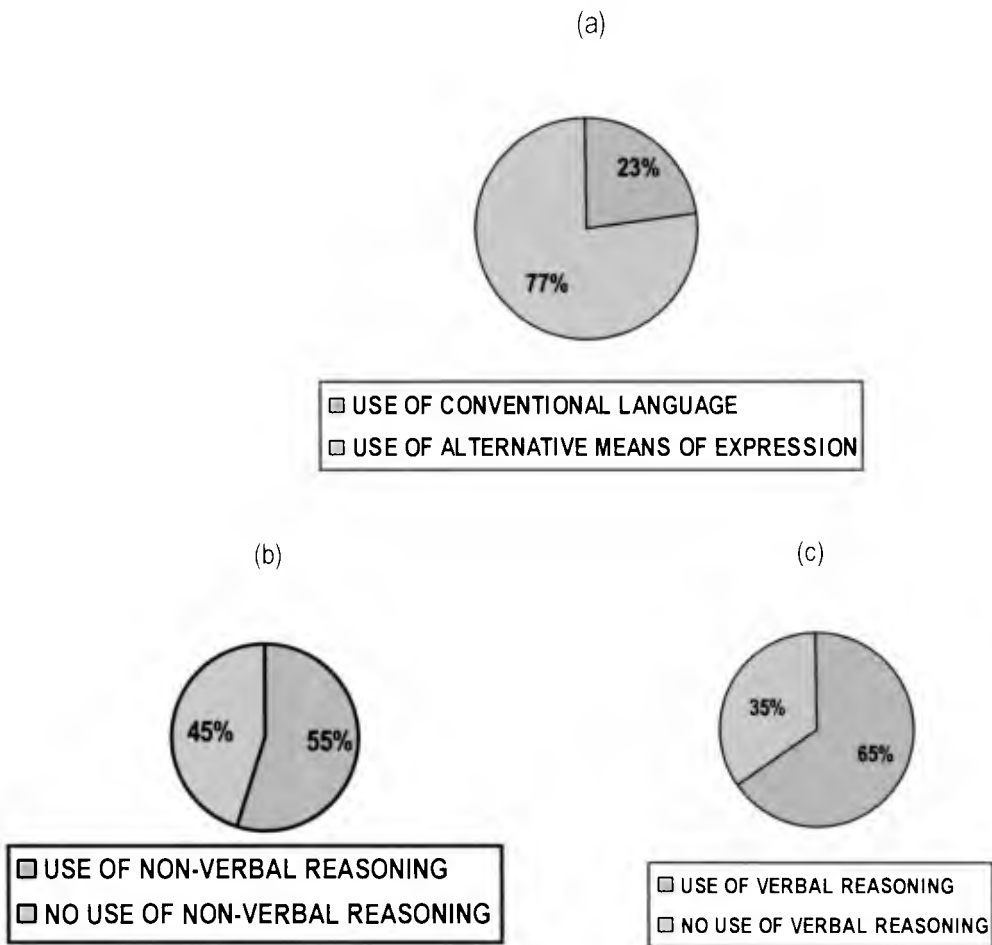
Now, let us proceed with the other categories of responses that did not include any implicit 'reference' to the shape's structure. As we saw in Graph 2, 77% of the children's responses during the DT made no implicit reference to the shape's structure. This percentage corresponds to 107 responses. Graph 6 presents the findings in relation to these categories. The results concerning the sub-categories of responses which included a simile (SIM, SIM/CONSTR) and those concerning the sub-categories of responses which included a self-evident justification (SELF-EV, SELF-EV/MAN, SELF-EV/CLASSINCL) are presented collectively in Graph 6 (SIMILE/SELF-EVIDENT).

As one can see from Graph 6, more than 25% of the non-structural responses included a self-evident justification. There were 39 such responses. Here are some examples of responses which were classified as SELF-EV:

- 'Because I know it!' (Michael, 6,0)
- 'Because I understand their shape!' (Savvas, 5,4)

- 'Because this is how a square is!' (Andys, 5,0)
- 'Because they have a square shape!' (Alexis, 5,8)

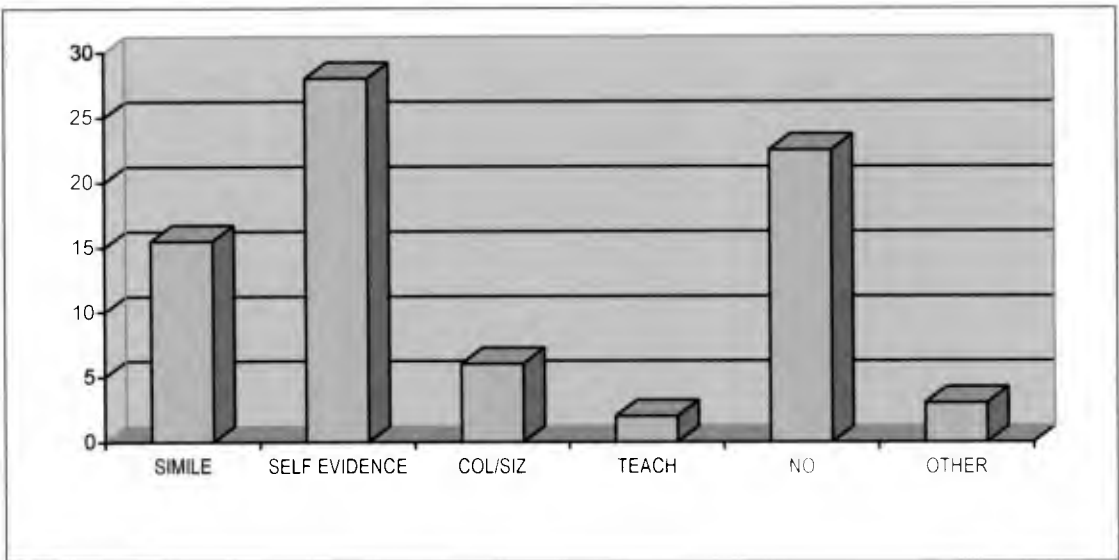
Graph 5: Categorisations of structural responses



Following are some examples of responses which were categorised as SELFV/CLASSINCL responses. Responses belonging to this category were highly connected to the objective of the preceding classification task.

- 'In order to make a group!' (Marianna, 6,2)
- 'I knew their name and you told me to put them in the same group!' (Rena, 6,3)
- 'Because you told me that the others were triangles so these had to be the squares!' (Riana, 5,1)

**Graph 6: Results in relation to the sub-categories of responses which were not classified as structural**



All values are given in percentage rates.

Finally, there was one response which was classified as SELFEV/MAN. This response was given by Michalis (6,0). Michalis pointed towards a cutout square while saying: 'because it is like this'.

Besides the responses which included a self-evident justification, there was a significant number of responses which were categorised as 'NO' responses. In these cases, children would state that they didn't know anything (else<sup>16</sup>) about squares or they would simply not reply to the interviewer's question. There were 31 such responses all together (Graph 6, NO). It is important to mention that there were four children among the subjects of the study that only gave answers classified as 'NO' answers.

Contrary to the emphasis given to the use of similes by existing research, only 16% of the children's responses in this study included a simile (Graph 6, SIM). Sixteen responses were categorised as SIM responses. The children would say that squares are like 'a house', 'a car', 'the yard', 'a kite', 'a chair', 'a carpet', 'windows', 'what we use to build', 'the underneath of a house'. Furthermore, six responses were categorised as Simile/Construction responses (SIM/CONSTR). In these cases, the children would use a simile to show that one can use a square to make/draw/construct something. For example, some of the subjects of the study stated that we use squares to make 'a house', 'a swing', 'an oven'.

<sup>16</sup> In 21 cases the 'No' response was the children's response to the question 'Is there anything else you know about squares?' which was either the second or the third question asked by the interviewer during the DT.



Moving on with the non-structural sub-categories of responses presented in Graph 6, there were eight COL/SIZ responses. One could argue that these responses refer to attributes of the shape (size and colour) which seem irrelevant to its structural appearance. In this category of responses we had answers like 'there are different colours, small and big', 'they are big and small', 'it has green ones and yellow ones'. But we also had the following response by Elisavet (5,2) which was very interesting: 'they have the same shape but not the same size'<sup>17</sup>.

To conclude with the non-structural responses and the results presented in Graph 6, there were also three responses (2%) which were categorised as TEACH (e.g. 'the teacher told me') and four answers classified as 'OTHER'. The 'OTHER' responses were the following:

- 'I know them from gym!' (Socrates, 5,1)
- 'When we draw a square it is still a square!' (Savvas, 5,4)
- 'I know names which start from the same letter!' (Michael, 6,0)
- 'Can I make the shape?' (Demosthenis, 6,4)

These four answers could not easily fit into one of the other categories for different reasons. The first two responses could not be easily classified because there is ambiguity in their meaning. The third response refers to something that is irrelevant to the shape (appearance and structure) and the last response is a special case since the child is responding with a question. Even though this last answer is expressing the child's need to use construction to express his understandings about the shape, it could not be classified as a (STR)CONSTR answer. Since the interviewers were specifically instructed to accept all of the children's responses with no further intervention, the interviewer in this case did not respond and thus the child did not proceed (presumably) with a construction process.

### **6.3 Reflections on the perspective of existing research**

How do the findings described in the previous section of the study match or contradict with findings from existing research? On the one hand, we have the findings and interpretations of the study conducted by Burger and his colleagues (Burger, 1985; Shaughnessy & Burger, 1985; Burger & Shaughnessy, 1986). As stated by Shaughnessy & Burger (1985) over seventy students in elementary and secondary school were involved in this project. The interview procedure used included an 'identifying and defining' task. The subjects were given a sheet of geometric shapes and were asked to identify all squares, all rectangles, all parallelograms and all rhombuses.

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<sup>17</sup> Elena's response was categorised both as a SELFV/CLASSINCL and a COL/SIZ response

Based on their data, Shaughnessy & Burger (1985) claim that younger students relied on 'predominantly visual sorts' and when asked what they would tell a friend so that she or he could pick up all squares from a sheet of paper they would answer 'I'd tell them to pick out all the squares' or 'look for the doors'. Based on these findings, Shaughnessy & Burger (1985) reconfirmed the existence of van Hiele level 0 as the level where descriptions are purely visual and no attention is given to shape properties.

If this study was to follow the same methodological framework as Shaughnessy & Burger (1985) it could reconfirm the existence of van Hiele level 0, since 34 out of the 52 subjects of the study used descriptions which did not include any structural understanding of the shape involved. The two examples of responses ('I'd tell them to pick out all the squares' and 'look for the doors') provided by Shaughnessy & Burger (1985) and which, as they state, were given by young children during the 'identifying and defining' task and are considered as characteristic responses of children belonging to van Hiele level 0, match with two of the main categories of responses that were identified in this study's data during the DT. The answer 'I'd tell them to pick out all the squares' can be categorised as a self-evident answer whereas the response 'look for the doors' is a simile. Going back to Graph 1, 28% of the children's responses in the DT included a self-evident justification whereas 16% of the responses included a simile.

The difference between this study's methodology and the methodology followed by Shaughnessy & Burger (1985) lies in the fact that here the aim is to describe the ways children express their understandings of shapes within and in correlation to a specific setting and not to evaluate children in order to place them in levels independently of the setting in which they express their understandings. That is why the existence of level 0 cannot be reconfirmed, at least not in the way it would be in most existing studies. But the study's findings can support the position that within a setting restricted to classification, recognition and description tasks, children often use 'appearance-based descriptions'.

There is also the study by Lehrer et al (1998). In this study 37 children were interviewed 18 times over a span of three years. In one of the tasks involved in the interviews the children were given nine triads of two-dimensional shapes and were asked to say which two were most alike and justify their choice. Lehrer et al (1998) identified three main categories of justifications: 'visual' (appearance-based justifications), 'property' (explicit reference to shape properties) and 'class' (justifications appealing to class membership). As already described briefly in Chapter 2, based on their data, Lehrer et al (1998) were led to some very interesting findings. Even though they

confirmed that 'children's reasoning most often involved the visual appearance of figures as suggested by van Hiele' this kind of justification included many sub-categories and types of reasoning. Some visual appearance-based responses (e.g. 'it's pointy') were attributed to some kind of property awareness (e.g. the properties of shapes). Based on these findings, visual reasoning does not exclude structural understanding.

It is also interesting to note that the three main categories of reasoning identified by Lehrer et al (1998) are given in the following order: 'visual', 'property' and 'class', and that 'class' justifications are considered a separate category to 'visual' justifications. Even though Lehrer et al (1998) give a different perspective to van Hiele's visual level, they do, to some extent, accept the existence of this level. In their discussion of their findings, they claim that 'the children's thinking of shapes can be characterised as appearance-based, as suggested by van Hiele (1986). However, children distinguish among many different features of form' (p.145). It is not clear whether Lehrer et al (1998) view the three types of reasoning (visual, property and class) as different levels of thinking where, for example, justifications including properties and class membership can be seen as more advanced than visual justifications. A response like 'these two are alike because they are both rectangles' was categorised as a 'class' justification.

It is interesting to think of how researchers such as Shaughnessy and Burger would categorise the response: 'these two are alike because they are both rectangles'. They would most probably categorise it as 'visual', as evidence that children see shapes as a whole and pay no attention to shape properties. This remark makes me reflect on this study's findings. Contrary to the analysis by Lehrer's et al (1998), in the data analysis system that was developed for this study there were two main categories of data: structural and non-structural. In Chapter 5, structural responses were defined as the responses which exhibited structural understanding (understanding of the shape's properties). We could therefore say that this category of data corresponds to Lehrer's et al (1998) second category of data ('property'). But whereas Lehrer et al (1998) have two additional categories of data ('visual' and 'class') there is only one ('non-structural') in the present study.

As explained previously (Graph 1, p.103), the majority of answers (28%) during the DT included self-evident justifications. This category of answers reminds us of Lehrer's et al (1998) class justification. During the DT, the children which gave self-evident answers would claim that squares are squares because they belong to the same class, they have a square shape, they have the same shape, etc. Self-evident answers were categorised as non-structural answers.

During the data analysis process (Chapter 5), I decided that I wanted a system that would allow the distinction between expression means which implicitly included structural elements and other means of expression. I was very careful in the way I defined structural and non-structural justification. I intentionally avoided defining non-structural justification as the kind of reasoning which excluded structural understanding. Thus, behind the analysis lies the acknowledgement that one cannot always tell, from a (single) verbal justification, what exactly a child knows about a shape's structure.

As already explained in Chapter 2, this has been confirmed by Potari et al (2003). According to Potari et al (2003), children often use similes to describe shapes but often similes 'are carriers of advanced thinking'. The data from the present study provided evidence that this is the case not only for similes but for other types of responses as well. The data analysis included data categories which combined different means of expression. In some cases, by combining their verbal utterances with gestures or mental constructions, the children would 'elevate' their answer from a simple self-evident answer to a structural response. There was, for example, the case of Pavlos' (5,1) response which was classified as SELF-EV: 'Because they are like this'. On the other hand, we had the following response by Aggeliki (5,11) which was classified as (STR)MAN: 'Because they are like this' while sliding her hands along the sides of a cutout square. The verbal response was the same by both of these children. But the way Aggeliki used the manipulative parallel to the verbal response exhibited awareness of the shape's structure.

Neither the studies by Burger and Shaughnessy (Burger, 1985; Burger & Shaughnessy, 1986; Shaughnessy & Burger, 1985) nor the study by Lehrer et al (1998) make any reference to the use by children of alternative means of expression like gestures, constructions and manipulatives. Only the children's utterances are reported. Even though in relation to the DT, the intention was to design a task that was restricted, as in the cases of aforementioned studies, the children used alternative means of expression like construction and manipulatives. One difference identified between the DT and the tasks used by other existing studies has to do with the material used. In the aforementioned studies, the shapes shown to the children were pictures drawn on pieces of papers, whereas in the DT the shapes were cut out and thus could more easily be 'manipulated' by the children.

As explained in Chapter 2, most existing research concerning children's understandings of shapes supports the point of view that usually young children use appearance-based justifications when trying to describe a shape. With few exceptions (e.g. Potari et al, 2003; Lehrer et al, 1998),

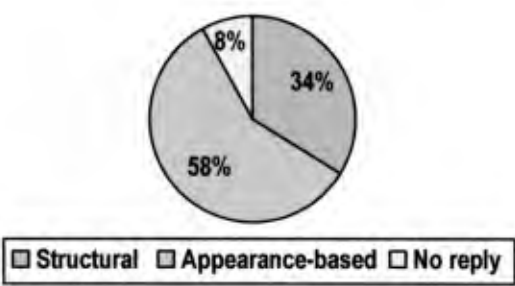
this appearance-based reasoning has been interpreted as the result of children seeing a shape as a whole and paying no attention to its components. Children using such appearance-based, or otherwise characterised as visual, reasoning were assigned to the first level of the van Hiele model of geometric thinking. In the categorising system used in this study, there was no explicit reference to appearance-based or visual responses. Alternatively, emphasis was given to the distinction between responses which included some reference to the shape's structure and those responses that did not.

Even though the DT was not designed to assess children's understandings, I will attempt to categorise the children based on their responses. This categorisation will allow, in a later stage, to compare the picture of the children's understandings as these were exposed during the DT with the children's understandings as these were exposed during the CT.

Since the children were asked to reply to more than one question during the DT, there is a need to find a way to categorise the children based on their answers on the whole. There were children that gave a structural response in one of the interviewer's questions but a non-structural response to one of the other questions. Similarly, there were children that responded in some way to one of the interviewer's question's but did not respond or gave a 'no' answer to one of the other questions. Thus, even though the results as these were presented in Graph 1 (p.103) provide a picture of how young children normally express themselves about squares within a setting restricted to traditional means of expression, it does not allow the classification of the children into categories.

A categorisation of the children based on their responses on the whole is attempted in Graph 7. In this Graph, the children are classified into three groups. First, we have the children that gave at least one structural response during the DT (Structural). Then, we have the children that responded with a self-evident answer or a simile to at least one of the interviewer's questions but who, at no point, gave a structural response (Table, Appearance-based). Finally, we have a group of children that did not reply to any of the interviewer's questions (Table, No Reply). Thus, as one can see in Graph 7, only 34% of the 52 subjects gave a structural answer during the DT. Based on this categorisation system, the majority of children are categorised as appearance-based. Finally, there were four children that did not reply to any of the interviewer's questions.

Graph 7: Percentage of children who gave at least one structural response, percentage of children who were restricted to non-structural responses and percentage of children who did not reply to any of the interviewer’s questions.



**6.4. Concluding remark**

In conclusion, one can argue that within a setting restricted to simple classification, recognition and description tasks, children exhibited poor structural understandings of shapes. Based on the analysis provided in this chapter in an attempt to compare this study's data with interpretations of similar data collected from other studies, I raised the question of whether it is safe to conclude that children who do not exhibit structural understandings within a setting restricted to traditional means of expression pay no attention to or know nothing of a shape's structure. In other words, can children's justifications within settings like the DT be taken as evident of what children know of shapes? This question brings us to the findings in relation to the children's understandings as these were expressed within a different setting; that of the CT. Thus, in the next chapter we will describe children's understandings of shapes as these were expressed through their attempt to construct a square.

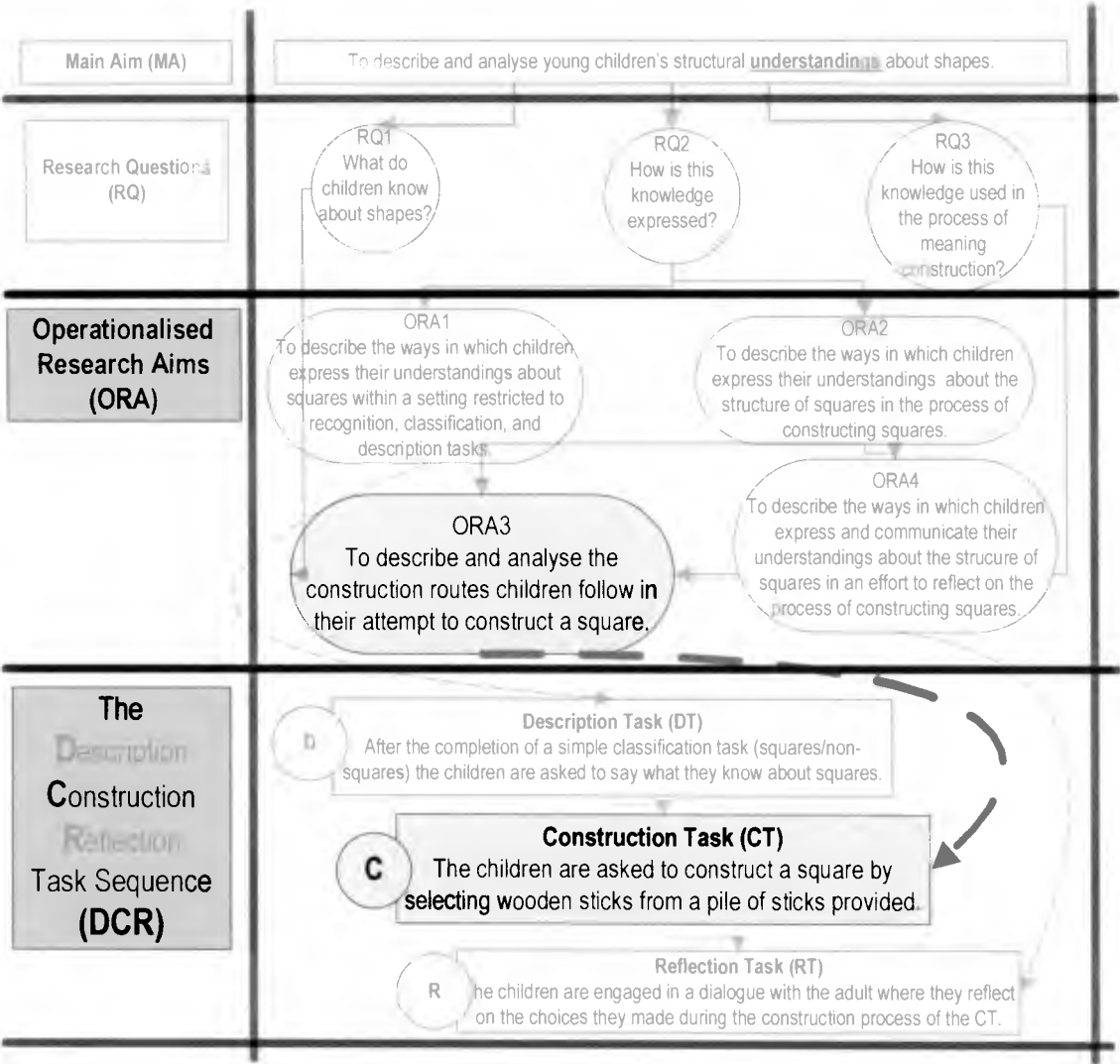
# CHAPTER 7

## FINDINGS: THE CONSTRUCTION TASK (CT)

### 7.1. Introduction

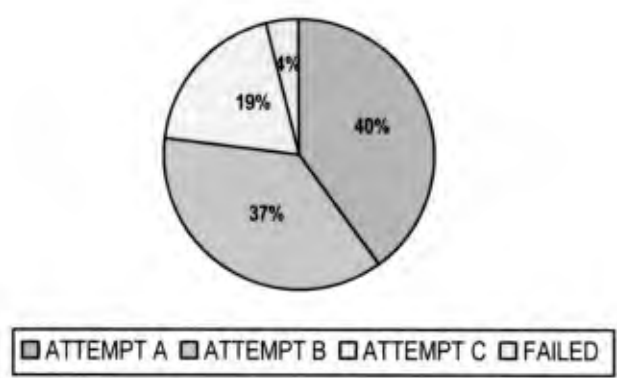
Figure 10 illustrates the part of the aim and design of the study as this was described in Chapter 3 (Figure 2, p.39), and which is the subject of this chapter. Hence, this chapter evolves around the findings in relation to the Construction Task (Figure 10, CT) so as to ‘describe and analyse the construction routes children follow in their attempt to construct a square’ (Figure 10, ORA3).

Figure 10: Aims and design of the study in relation to the Construction Task (CT)



As described in Chapter 5, in the CT the 52 children involved in the study were asked to construct a square with the use of the sticks provided. A number of children 'successfully' constructed a square with the use of four equal sticks from their first attempt. If a child 'failed' to construct a square in this first attempt the interviewer would follow the guidelines of the interview script (Appendix C) in asking the child to try again. In case the product of the second attempt was still not 'successful', the child was asked to try one last time after s/he had the opportunity to watch the interviewer carefully select four equal sticks and construct a square. Graph 8 shows the percentage of children that successfully constructed a square from their first attempt (ATTEMPT A), the percentage of children that had to try twice in order to be successful (ATTEMPT B), the percentage of children that had to try three times (ATTEMPT C) and the percentage of children that failed to construct a square after their third attempt (FAILED).

**Graph 8: Results in relation to the number of attempts the children needed in order to 'successfully' construct a square**



As shown in Graph 8, 40% of the subjects involved in the study (21/52) 'successfully' constructed a square from their first attempt (ATTEMPT A). 19 out of the remaining 31 children (37%) 'successfully' constructed a square after a second attempt (Graph 8, ATTEMPT B) and 10 children (19%) had to try once more in order to 'succeed' (Graph 8, ATTEMPT C). Two children (4%) failed to construct a square even after their third attempt (Graph 8, FAILED).

The following sections of this chapter provide an analysis of each of the children's attempts to construct a square.

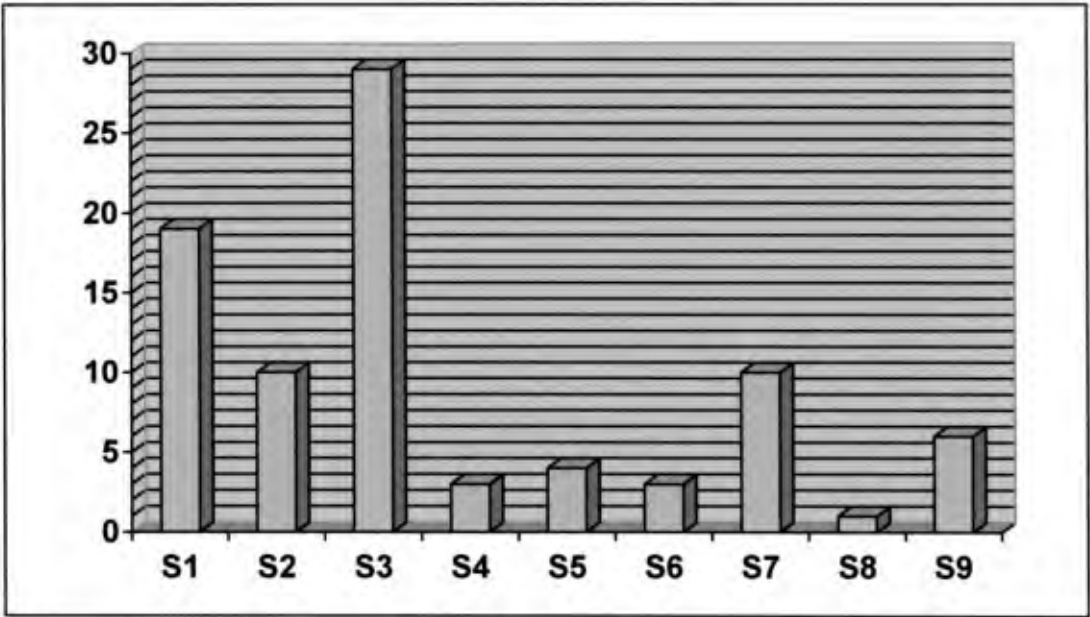


## 7.2. The children’s first attempt to construct a square

In describing the findings of the CT, emphasis is given to the children’s first attempt (Attempt A). For this attempt, results are provided in relation to the strategies used by the children, the product of their attempt as well as a detailed description of the routes the children followed.

First, let us take a look at which strategies the children followed and what products they ended up with in their first attempt to construct a square. Graph 9 shows the percentage of children that followed each of the Strategies (S) identified and described in Chapter 5 (Table 8, p.88). A first significant observation that can be made based on Graph 9 is that the percentage of children that followed S1, S2 and S3 is much higher than that of the children that followed other strategies. This observation is prominent for the discussion of Chapter 9.

**Graph 9: Results in relation to the strategies (S) followed by the children in their first attempt**

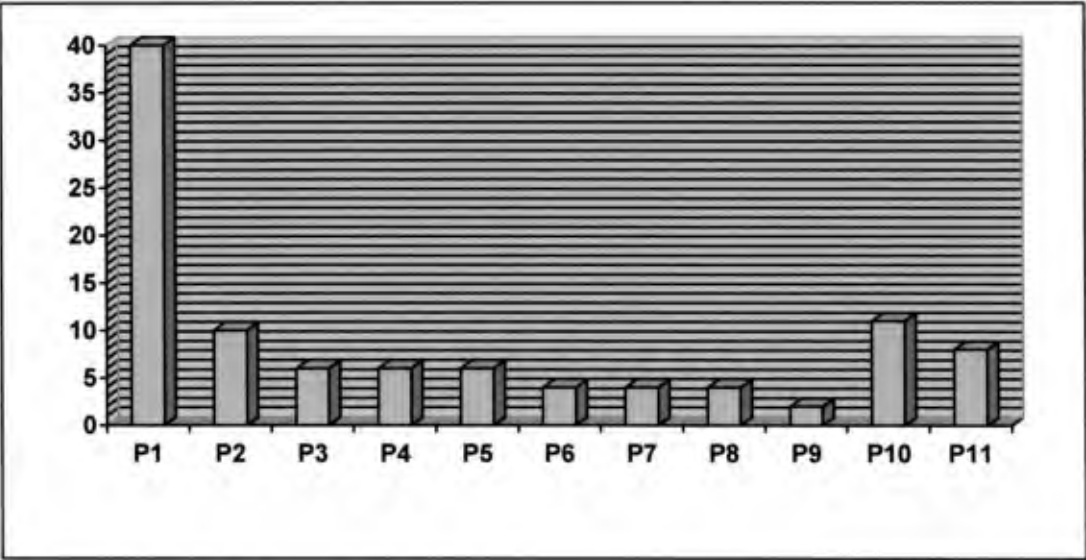


All values are given in percentage rates.  
Key to response categories: S1=selects four equal sticks and places them in such a way as to construct right angles, S2=selects three equal sticks and constructs an open shape with right angles, S3=selects two equal sticks and constructs a right angle, S4=selects two equal sticks and places them parallel and aligned, S5=Selects two unequal sticks and creates a right angle, S6=selects four unequal sticks and tries to construct a four-sided shape with right angles, S7=selects three equal sticks, S8=selects two equal sticks, S9=selects one stick at a time and tries to construct a four-sided shape which somehow resembles a square with no equal sides and no right angles.

As we can see in Graph 9, the most frequently used strategy among the 52 children was S3. 29% of the subjects used this strategy, in other words began their construction by selecting two equal sticks and constructing a right angle. A significant percentage of children (19%) followed S1 (selected four equal sticks and constructed a square with no experimentation required). The rest of the strategies were followed by smaller groups of children.

Graph 10 illustrates the findings in relation to the product of the children's first attempt to construct a square<sup>18</sup>. One first key observation is that there is a big difference between the findings in relation to P1 compared to other products. It is quite astonishing that 40% of the children involved in the study (21/52) successfully constructed a square by using four equal sticks from their first attempt (Figure 19, P1). The rest of the products were the result of the effort of much smaller groups of children.

**Graph 10: Results in relation to the products (P) of the children's first attempt**

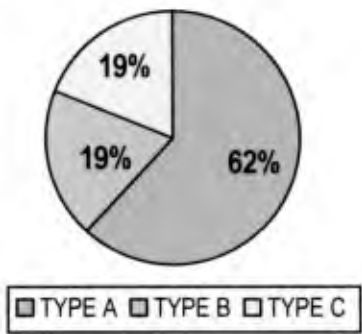


All values are given in percentage rates.  
 Key to response categories: P1=square with four equal sticks, P2=square with a gap, P3=square with extension, P4=square with more than four sticks, P5=rectangle with two sets of equal sticks, P6=rectangle with a gap, P7=rectangle with an extension, P8=rectangle with more than four sticks, P9=rectangle with gaps and extensions, P10=irregular quadrilateral that resembles a square but has no right angles, P11=irregular quadrilateral with some angles right and/or sides equal.

In Chapter 5, the eleven products identified among the data were categorised into three groups: Type A, Type B and Type C (Table 9, p.89). Graph 11 provides the results in relation to the three types of products (Type A, B and C). What is important to highlight in relation to Graph 11 is that the majority of the children involved in the study (62%) constructed a Type A shape (square with four equal sticks/gap/extension/more than four sticks). Thus, whereas in the DT only 35% (18/52) of the children gave structural responses (Graph 3, p.104), 62% of the children involved in the study (32/52) constructed a Type A product in their first attempt to construct a square. Half of the rest of the children (19%) constructed a Type B product and half (19%) constructed a Type C product.

<sup>18</sup> A description for each product is provided in Chapter 5 (Table 9, p.89).

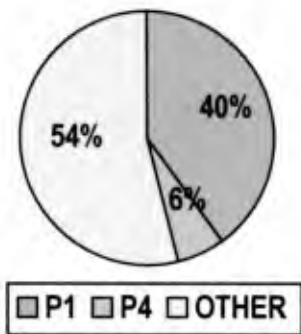
**Graph 11: Percentage of children that constructed a Type A, a Type B and a Type C product in their first attempt**



Key to response categories: Type A (P1/P2/P3/P4)=square with or without some 'flaw', Type B (P5/P6/P7/P8/P9)=rectangle with or without some 'flaw' and Type C (P10/P11)=irregular quadrilateral

Besides the number of children who successfully constructed a square with the use of four equal sticks (P1) there was a number of children that constructed a square (a quadrilateral with four equal sides and four right angles) with the use of more than four sticks (P4). So on the whole, 46% of the children involved in the study (24/52) constructed a square in their first attempt (Graph 12). This is a noteworthy percentage.

**Graph 12: Percentage of children that constructed a square with four equal sticks (P1) and a square with more than four sticks (P4) in their first attempt**



As was explained in Chapter 5, the identification of the strategies the children followed and the products of their attempt do not cover their whole construction route. During the process of data analysis (Chapter 5), two important observations were documented. First of all, that children following the same strategy did not necessarily end up with the same product and second, children that did follow the same strategy and end up with the same product did not necessarily follow the same route along the way. These observations reflect the variability that existed among

the data. Table 21 represents the way this variability was dealt with. It shows the relationship between the strategies followed by the children and the products of their first attempt to construct a square.

Table 21: The relationship between the strategies and the products of the children’s first attempt

CT(A)	Strategies									
Products	S1	S2	S3	S4	S5	S6	S7	S8	S9	Total
P1	10	1	6	-	-	-	4	-	-	21
P2	-	1	3	-	-	-	1	-	-	5
P3	-	1	2	-	-	-	-	-	-	3
P4	-	1	1	-	1	-	-	-	-	3
P5	-	-	-	2	1	-	-	-	-	3
P6	-	-	-	-	-	2	-	-	-	2
P7	-	1	-	1	-	-	-	-	-	2
P8	-	-	-	-	2	-	-	-	-	2
P9	-	-	-	-	-	1	-	-	-	1
P10	-	-	-	-	-	-	-	-	6	6
P11	-	-	3	-	-	-	-	1	-	4
Total	10	5	15	3	4	3	5	1	6	52

The analysis that follows describes each of the categories formed out of the process of corresponding the children's strategy with the product of their first attempt to construct a square, as presented in Table 21.

***The construction routes of children who began their attempt to construct a square by following S1 (selected four equal sticks and placed them in such a way as to construct right angles)***

Table 21 shows that, as was expected, the 10 children that followed S1 ended up with a square with four equal sticks (P1). Nevertheless, there was a certain variation among the children following this strategy, which was determined by two factors, described in Table 22.

As described in Table 22, Factor A concerns the sequence between the selection of the sticks and their spatial arrangement. Each construction attempt was the combining effort of selecting sticks and arranging them in space. In the case of S1, some children would first select the four sticks and then begin the spatial arrangement process (Table 22, A1), while others would select one stick at a time parallel to the process of arranging them in space (Table 22, A2). Factor B (Table 22) had to do with the process of selecting four equal sticks. Some children would compare sticks in order to find a set of four equal sticks (Table 22, B1), whereas others would select four equal sticks without the need for physical comparison (Table 22, B2).

Table 22: Variation factors in relation to the construction route of children who followed S1

	Factor A	Factor B
	Sequence between the selection of sticks and their spatial arrangement	Selection of four equal sticks
1	Children would first select four equal sticks and then proceed with the construction process.	Children would select four equal sticks through a process of comparing sticks (by putting them next to each other)
2	Children would select sticks parallel to constructing the square.	Children would select four equal sticks without the need for physical comparison.

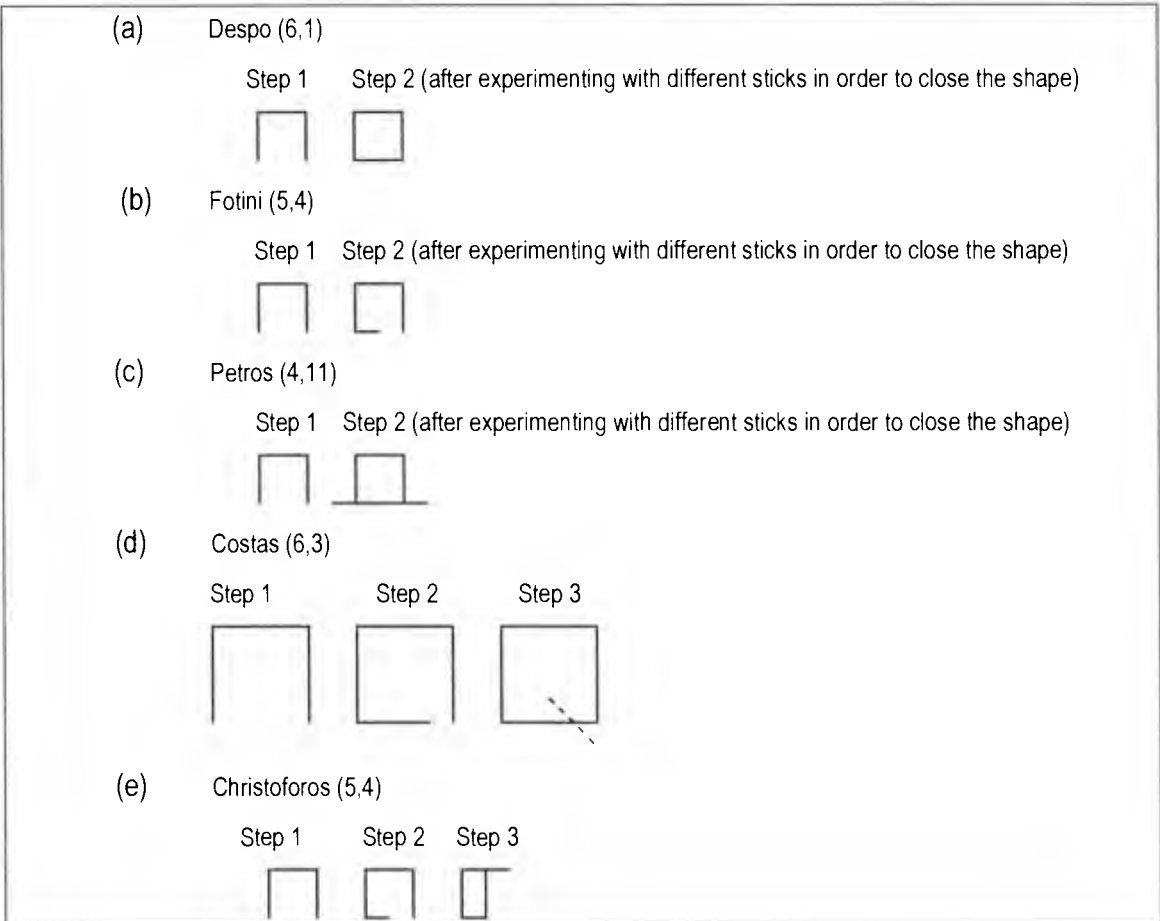
For example, in her attempt to construct a square, Rebecca (5,11) selected four equal sticks through a process of comparing them by putting them next to each other first and then proceeded with the construction process (Table 22, A1-B1). Charalambos (6,8) also selected the four equal sticks before constructing the shape but did not use physical comparison to choose the sticks as in the case of Rebecca (Table 22, A1-B2). Seraphima (6,8) selected the sticks through a process of physical comparison parallel to constructing the shape (Table 22, A2-B1). Finally, Periklis (5,7) selected four equal sticks parallel to constructing the shape without the need for physical comparison. (Table 22, A2-B2).

***The construction routes of children who began their attempt to construct a square by following S2 (selected three equal sticks and constructed an open shape with right angles)***

If we now go back to Table 21 (p.120) we can see that the five children that followed S2 all ended up with a different product. These children's routes are illustrated in Figure 11.

Despo (6,1), Fotini (5,4) and Petros (4,11) were all children who began their attempt to construct a square by selecting three equal sticks and creating an open shape with right angles. They then experimented with different sticks in order to close the shape (Figure 11a-c). As illustrated in Figure 11a, Despo was only satisfied when she found a stick that did not create gaps or extensions. Fotini (Figure 11b), on the other hand, completed her construction with a stick that was smaller than the other three, whereas Petros (Figure 11c) closed the shape with a larger stick, therefore constructing a square with extensions (P3).

Figure 11: The construction routes of children who followed S2



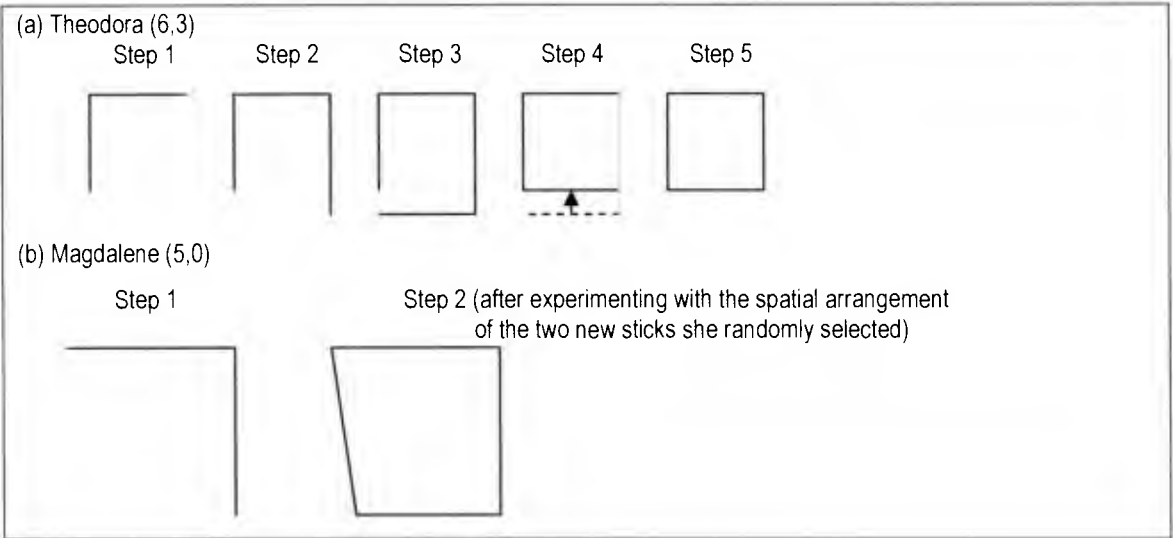
The slightly oblique dotted line crossing through the side of a construction indicates that that specific side is constructed with the use of two sticks.

Costas (6,3) on the other hand, as one can see in Figure 11d, after constructing the open shape with the three equal sticks, used more than one stick to close the shape (P4). The last child that used S2 was Christoforos (5,4). Christoforos was an interesting case. After he selected three equal sticks and constructed the open shape like the other four children, he picked up a smaller stick to close the shape and thus constructed a square with a gap. In his effort to close the gap he ended up with the shape illustrated in Figure 11e, Step 3.

***The construction routes of children who began their attempt to construct a square by following S3 (constructed a right angle with two equal sticks)***

Moving on to the findings in relation to S3, there was also a certain variation among the children. Five different products were identified among the 15 children that followed this strategy, (Table 21, p.120).

**Figure 12: The construction routes of children who followed S3**

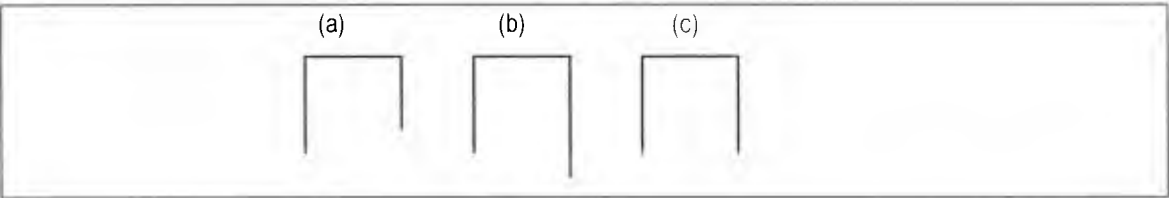


The arrow shows how the child moved one stick from one position to another.  
The light grey line shows that the child has removed a specific stick from the construction.

Figure 12a illustrates the construction route of Theodora (6,3) who followed S3 and ended up with a P1 at the end of her attempt. After constructing a right angle with two equal sticks, Theodora added to her construction a third stick, which was bigger than the first two (Step 2). She then took a fourth stick that was equal to the first two and constructed the shape shown in Figure 12a, Step 3. Her next move was to remove the one stick that was bigger than the other three and move the fourth stick as shown in Figure 12a, Step 4. Finally, she carefully looked at the pile of sticks provided, picked one that was equal to the three sticks she had already used and closed the shape (Step 5).

The remaining five children that followed S3 and ended up with P1 (square with four equal sticks), the three children that followed S3 and ended up with P2 (square with a gap) and the two children that followed S3 and ended up with P3 (square with an extension) (Table 21, p.120) all followed a similar route. After constructing a right angle with two equal sticks, they would try out different sticks for the third side of the shape. What was interesting about these children's attempt is that in experimenting with the third side of their construction, they would try out a stick and check the end of that stick in relation to the opposite one, thus mentally outlining the fourth side of the construction. If, at this point, the construction looked like Figure 13a or 13b, they would reject the third stick and try another one until the construction would look like Figure 13c. In this case, they would proceed with trying to find a stick to complete their construction.

Figure 13: Examples of the way in which most children experimented with sticks after following S3



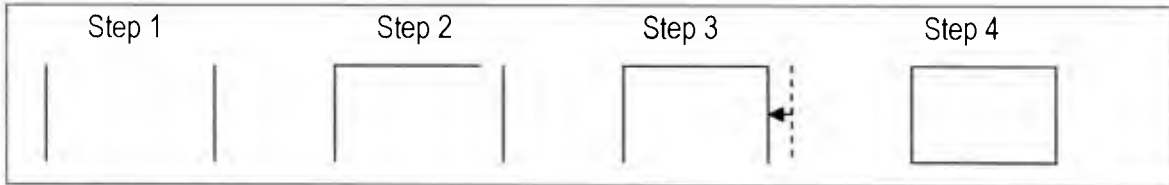
The remaining three children that followed S3 ended up with P11 (Table 21, p.120) after following similar routes. As illustrated in Figure 12b, after constructing a right angle with two equal sticks (Step 1), Magdalene (5,0) randomly selected two other sticks. Magdalene then experimented with the spatial arrangement of the two sticks she had randomly selected until she managed to close her construction (Step 2).

***The construction routes of children who began their attempt to construct a square by following S4 (selected two equal sticks and placed them parallel and aligned)***

Moving on to S4, as one can see in Table 21 (p.120), there were two different cases in relation to the outcome of the children's attempt to construct a square. Two children constructed a rectangle with two sets of equal sticks (P5) and one child constructed a rectangle with an extension (P7).

The two children that constructed a rectangle with two sets of equal sticks followed exactly the same route. First, they selected two equal sticks, placed them down parallel to each other (Figure 14, Step 1) and then added a third stick to their construction which was not equal to the other three (Step 2). They then adjusted the distance between the first two sticks in relation to the third stick (Step 3). Finally, they found a stick equal to the third stick and used it to close the shape thus constructing a rectangle with two sets of equal sticks (P5).

Figure 14: The construction route of children who followed S4



The arrow shows how the child moved one stick from one position to another.

The child that followed S4 and constructed a rectangle with an extension (P7) was Spiros (5,10). Spiros followed Steps 1-3 as shown in Figure 14. But to complete his construction he used a big stick and thus constructed a rectangle with an extension (P7).



***The construction routes of children who began their attempt to construct a square by following S5 (constructed a right angle with two unequal sticks)***

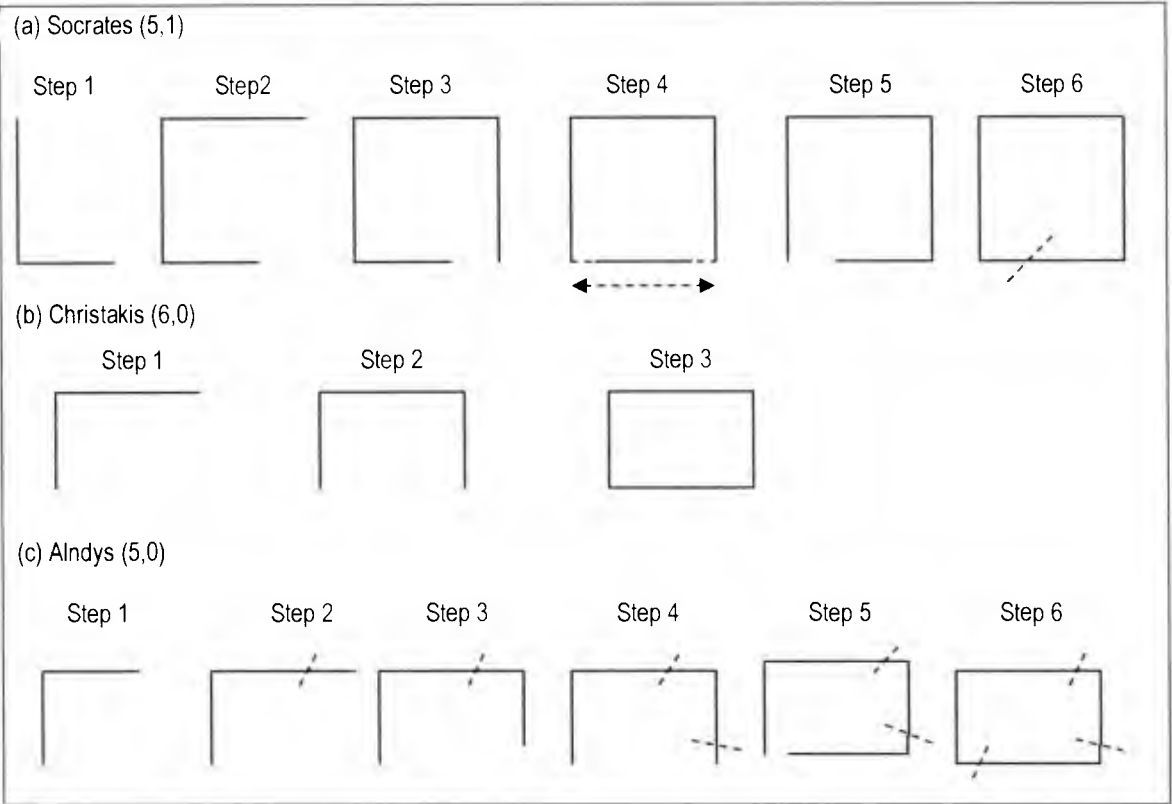
The four children that followed S5 constructed three different shapes (Table 21, p.120). One child constructed a square with more than four sticks (P4), one child constructed a rectangle with two sets of equal sticks (P5) and two children constructed a rectangle with more than four sticks (P8).

Socrates (5,1) was the child who constructed a square with more than four sticks (P4). First, he constructed a right angle with two unequal sticks (Figure 15a, Step 1). After experimenting with different sticks, Socrates added to his construction a third stick which was the same as one of the other two sticks (Figure 15a, Step 2). His third step was to add a fourth stick (equal to the third) to his construction. After this move, he ended up with a shape with right angles constructed with three equal sticks and a smaller one. He then moved the smaller stick forwards and backwards (Figure 15a, Step 4). Finally, he placed the smaller stick as shown in Figure 15a, Step 5, found a much smaller stick, closed the gap (Figure 15a, Step 6) and constructed a square with more than four sticks (P4).

Christakis (6,0) also followed S5 but constructed a rectangle with two sets of equal sticks (P5). Christakis followed the route illustrated in Figure 15b. After he constructed a right angle with two unequal sticks (Figure 15b, Step 1), he experimented with different sticks until he ended up with the shape shown in Figure 15b, Step 2. Finally, after trying out different sticks he closed the shape (Figure 15b, Step 3).

The remaining two children that followed S5 constructed a rectangle with more than four sticks (P8). Both of these children followed similar routes. Andys (5,0), for example, began his construction by constructing a right angle with two unequal sticks (Figure 15c, Step 1). It is interesting to note that the two sticks he used were only slightly different in length. He then added a third stick to make the smaller of the first two sticks bigger (Figure 15c, Step 2). Next, he added a fourth stick to his construction as illustrated in Figure 15c, Step 3. He looked at his construction and then experimented with adding different sticks to the last stick he had added to his construction until he constructed the shape shown in Figure 15c, Step 4. In the same way, he used two sticks to form the last side of the shape so that it would close (Figure 15c, Steps 5-6). He therefore ended up with a rectangle constructed with the use of seven sticks.

Figure 15: The construction routes of children who followed S5

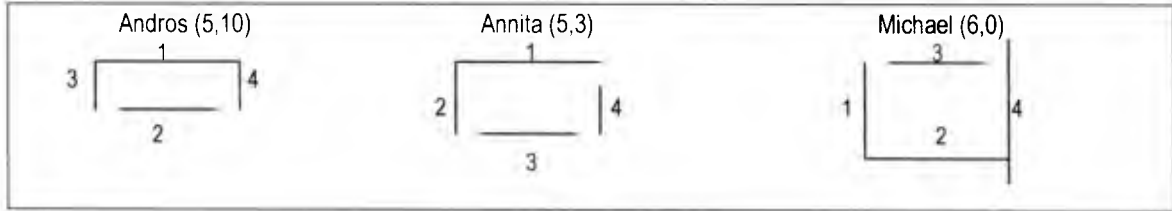


The slightly oblique dotted line crossing through the side of a construction indicates that that specific side is constructed with the use of two sticks.  
The arrow shows how the child moved one stick from one position to another.

**The construction routes of children who began their attempt to construct a square by following S6 (randomly selected sticks and tried to construct a four-sided shape with right angles)**

Going back to Table 21 (p.120), two of the children that followed S6 constructed in the end a rectangle with a gap (P6) whereas the other child that followed S6 constructed a rectangle with gaps and extensions (P9). These three children followed similar routes. They would pick up one stick at a time and add it to their construction whilst keeping the angles of their construction right. Figure 16 illustrates the routes followed by these three children. The numbers in each case indicate the order in which the children placed each stick.

Figure 16: The construction routes of children who followed S6



***The construction routes of children who began their attempt to construct a square by following S7 (selected three equal sticks)***

According to Table 21 (p.120), five children followed S7. Four of these children ended up with a square with four equal sticks (P1) while the other one constructed a square with a gap (P2). Figure 17 illustrates the routes followed by four of these children.

Vaggelis (5,9) used the three equal sticks he selected at the beginning of his attempt to construct the shape as illustrated in Figure 17a, Step 1. He then selected from the pile of sticks provided a fourth stick equal to the other three and added that to his construction (Figure 17a, Step 2). He then replaced the last stick he had added to his construction with another one that was smaller (Figure 17a, Step 3) and moved the stick that was not forming a right angle to close the shape (Figure 17a, Step 4). Next, he moved the same stick again as illustrated in Figure 17a, Step 5. After that, he removed from his construction the stick that was smaller than the other three and moved the stick he had previously tried to different positions as shown in Figure 17a, Step 6. Finally, he chose a stick that was equal to the other three from the pile of sticks provided and constructed a square with four equal sticks (Figure 17a, Step 7).

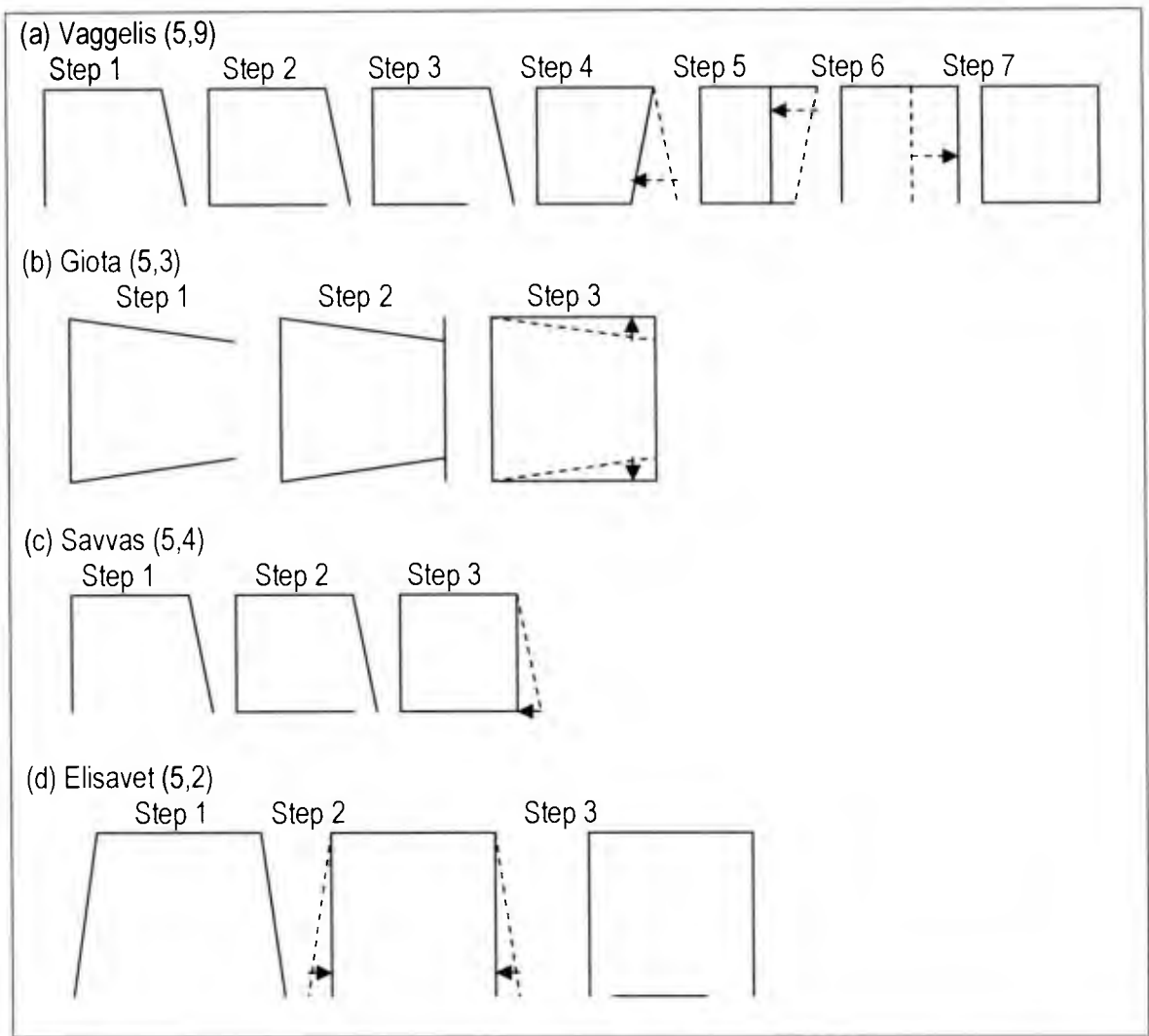
Two of the other children that followed S7 and constructed a square with four equal sticks (P1) followed similar routes. One of these children was Giota (5,3). Giota began her attempt to construct a square by selecting three equal sticks and arranging them as illustrated in Figure 17b, Step 1. She then added a fourth stick to her construction as shown in Figure 17b, Step 2 and finally moved the two sticks that were not forming right angles to end up with a square with four equal sticks (Figure 17b, Step 3).

Savvas (5,4) also followed S7 and constructed a square with four equal sticks (P1). In Figure 17(c), one can follow Savvas' steps in constructing a square. Savvas began his construction (Figure 17c, Steps 1-2) in exactly the same way as Vaggelis (Figure 17a, Steps 1-2). But Savvas did not go through all the different phases that Vaggelis went through to complete the construction. After Savvas had added to his construction the fourth stick that was equal to the other three (Figure 17a, Steps 2), he moved the stick that was not forming a right angle as illustrated in Figure 17a, Steps 3) and thus successfully completed his construction.

Finally, Elisavet (5,2) was also a child that followed S7. But contrary to the other four children, Elena ended up with a square with a gap (P2). She began her construction by arranging the three equal sticks as shown in Figure 17d, Step 1. She then moved the two sticks that were not parallel

so that they would form right angles with the other stick (Figure 17d, Step 2) and completed her construction by adding a fourth stick that was smaller than the other three, as illustrated in Figure 17d, Step 3.

**Figure 17: The construction routes of children who followed S7**



The slightly oblique dotted line crossing through the side of a construction indicates that that specific side is constructed with the use of two sticks.  
The arrow shows how the child moved one stick from one position to another.

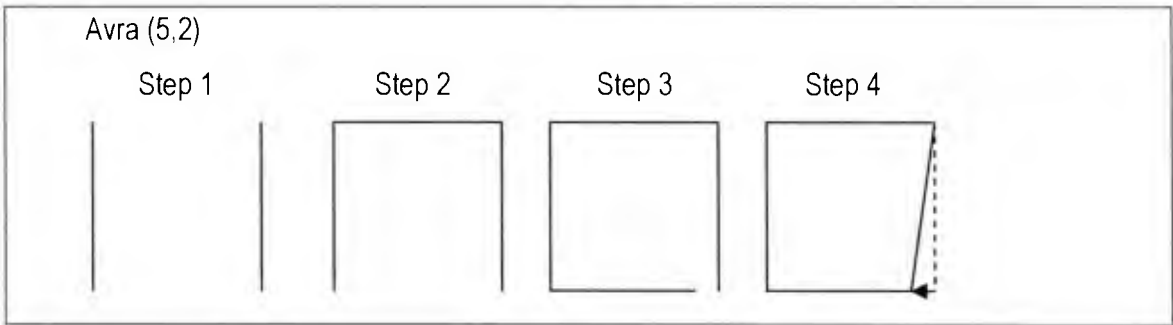
***The construction route of the child who began her attempt to construct a square by following S8 (selected two equal sticks)***

As illustrated in Table 21 (p.120) there was the case of one child that followed S8 and ended up with P11 (irregular quadrilateral with some angles right and some sides equal). This child (Avra) was indeed a special case. Avra (5,2) began her construction by selecting two equal sticks and placing them parallel to each other. The outcome of her attempt was a quadrilateral with some angles right and some sides equal. Consequently, Avra originally appeared in Table 21 (p.120) as a child that followed S4 and constructed a P11. While going through this earlier version of Table

21 (p.120) it seemed strange that there was the case of a child that had followed S4 and had constructed a P11. This was paradoxical because of the way S4 and P11 were defined. In the definition of S4, the foundational action was the selection of two equal sticks and their parallel arrangement. Products categorised as P11 were quadrilaterals constructed with the use of only four sticks with some angles right and some sides equal. This led me back to the raw data (videotape and transcript of the interview).

What I discovered was that even though Avra began her attempt by selecting two equal sticks and placing them parallel to each other, this was not her foundational action. Avra originally selected two equal sticks and placed them parallel to each other, but in the process of constructing the shape she did not maintain the two sticks in parallel positions. The case of Avra led to the need for the creation of S8, which was characterised by the foundational action of selecting two equal sticks. Thus, Avra was re-coded as a child that had followed S8 and constructed a P11.

Figure 18: The construction route of the child who followed S8



The arrow shows how the child moved one stick from one position to another.

After selecting two equal sticks and placing them parallel to each other (Figure 18, Step 1), Avra chose another stick that was equal to the other two and added that to her construction (Figure 18, Step 2). She then used a stick that was smaller than the other three to close the shape (Figure 18, Step 3). Finally, she closed the shape as shown in Figure 18, Step 4.

***The construction routes of children who began their attempt to construct a square by following S9 (selected one stick at a time and constructed an irregular quadrilateral that resembled a square)***

To conclude with the children's first attempt to construct a square, there were six children that used S9 (Table 21, p.120). All of these children constructed a quadrilateral that resembled a square but had neither right angles nor equal sides (P10). These six children followed a strategy

of randomly selecting one stick at a time and adding it to their construction. Figure 19 illustrates some examples of products of children that followed S9.

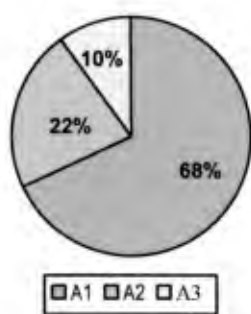
**Figure 19: Examples of products of children that followed S9**



**7.3. The children’s<sup>19</sup> second attempt to construct a square**

As we saw in the previous section, only 21 children ‘successfully’ constructed a square from their first attempt. The remaining 31 children were asked to try again. Here follow the findings from analysing the data collected from these children’s second attempt to construct a square. As one can see in Graph 13, the majority of children tried to fix the construction of their first attempt. To be more precise, 68% of the children that had to try a second time (21/31) made an effort to correct the product of their first attempt (A1), 29% (7/31) undid the construction of their first attempt and started all over again (A2) and 13% (3/31) originally tried to fix the product of their first attempt but then decided to undo it and start all over again (A3).

**Graph 13: Results in relation to the approaches used by the children in their second attempt**



Key to response categories: A1=the child tries to fix the construction that was the product of his/her first attempt, A2=the child undoes the construction of his/her first attempt and tries all over again, A3=the child tries to fix the construction that was the product of his/her first attempt but then decides to undo it and try all over again.

It is noteworthy that the percentage of children that considered that their original construction could, with some adjustments or corrections, be turned into a square is very high. A total of 78%<sup>20</sup>

<sup>19</sup> This section of the thesis relates only to the group of 31 children that failed to construct a square with the use of four equal sticks in their first attempt and thus had to try again.  
<sup>20</sup> This percentage is the sum of adding the percentage of children that followed A1 and A3 (Graph 13).

of the children (28/31) belong to this category. This might mean that these children had some understanding of the fact that their original construction had some attributes that were correct and some that were wrong and needed to be fixed.

A more analytical picture of the way the children tried to construct a square in their second attempt is illustrated in Table 23. As shown in Table 23, eight of the children that followed A1 fixed their original construction, in other words they fixed the mistake in the product of their first attempt in order to turn it into a square with four equal sticks (Table 23, Fix). Nine children made a number of adjustments (change of sticks and their spatial arrangement) to their original construction (Table 23, Adjust) and four children made certain improvements (changed the spatial arrangement of the already used sticks) to their original construction and transformed it into a different shape (Table 23, Improve). The children that followed A2 (undid their original construction and tried again from the beginning) followed different strategies. The strategies these children followed are presented in Table 23.

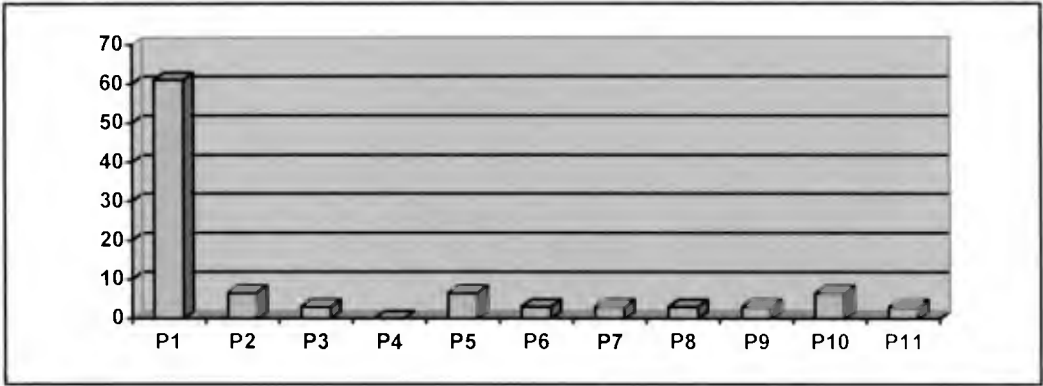
**Table 23: Results in relation to the sub-categories of the approaches used by the children in their second attempt**

CT(B)			
	Sub-categories	No	%
A1	Fix	8	26
	Adjust	9	29
	Improve	4	13
A2	S1	3	10
	S2	1	3
	S3	1	3
	S4	1	3
	S9	1	3
A3	Adjust/S1	1	3
	Improve/S3	2	7
Total		31	100

Finally, out of the three children that followed A3 (first tried to transform their original construction and then decided to undo it and try all over again), one tried to adjust the original construction (replaced sticks and their spatial arrangement) and then started all over again by following S1 (selected four equal sticks and constructed a square) and two tried improving their original construction (changed the spatial arrangement of existing sticks) and then followed S3 (selected two equal sticks and constructed a right angle) in an effort to try again (Table 23).

Graph 14 presents the results in relation to the product of the children's second attempt to construct a square. 61% of the children that had to try and construct a square for a second time (19/31) successfully constructed a square with the use of four equal sticks (Graph 14, P1).

**Graph 14: Results in relation to the product (P) of the children's second attempt**



All values are given in percentage rates.  
 Key to response categories: P1=square with four equal sticks, P2=square with a gap, P3=square with extension, P4=square with more than four sticks, P5=rectangle with two sets of equal sticks, P6=rectangle with a gap, P7=rectangle with an extension, P9=rectangle with more than four sticks, P9=rectangle with gaps and extensions, P10=quadrilateral that resembles a square but has no right angles, P11=quadrilateral with some angles right and/ore sides equal.

Table 24 illustrates the relationship between the product of the children's first attempt and the product of their second attempt. It is interesting to note that, as illustrated in Table 24, the eleven children that constructed a Type A product that was not a square with four equal sticks in their first attempt (CT(A)) successfully constructed a square (P1) in their second attempt (CT(B)). Table 24 provides an illustration of the routes the children followed in their second attempt to construct a square, which is the subject of the remaining of this section of the thesis.

**Table 24: The relationship between the product (P) of the children's first and second attempt**

CT(A) \ CT(B)		Type A			Type B					Type C		Total
		P1	P2	P3	P5	P6	P7	P8	P9	P10	P11	
Type A	P2	5	-	-	-	-	-	-	-	-	-	5
	P3	3	-	-	-	-	-	-	-	-	-	3
	P4	3	-	-	-	-	-	-	-	-	-	3
Type B	P5	1	-	-	1	-	-	-	1	-	-	3
	P6	1	-	-	-	-	1	-	-	-	-	2
	P7	1	-	-	-	-	-	1	-	-	-	2
	P8	1	1	-	-	-	-	-	-	-	-	2
	P9	-	-	-	-	-	-	-	-	-	1	1
Type C	P10	3	-	-	-	1	-	-	-	2	-	6
	P11	1	1	1	1	-	-	-	-	-	-	4
Total		19	2	1	2	1	1	1	1	2	1	31



***The construction routes followed by children that constructed a Type A product in their first attempt, but 'successfully' constructed a square in their second attempt***

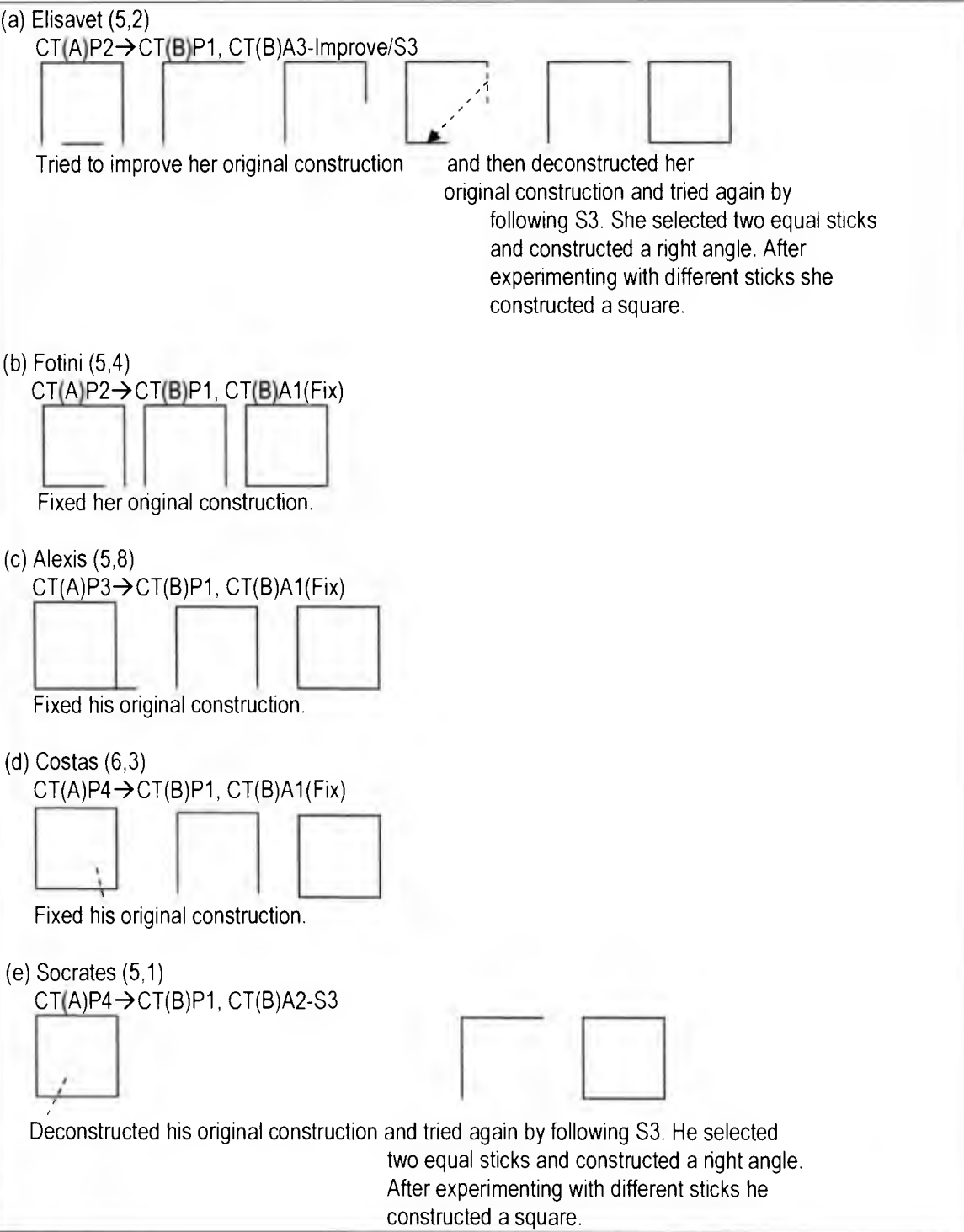
According to Table 24 (p.132), we had eleven children that constructed a square with a 'flaw' in their first attempt, but successfully constructed a square with four equal sticks in their second attempt. Figure 20 illustrates some examples of the routes followed by these children.

It is worth taking a closer look at the cases of children who followed A1 and A3 in their second attempt to construct a square (CT(A)A1, CT(A)A3). As was suggested earlier, by following these approaches children exhibited the ability to distinguish in their original constructions some correct and some wrong attributes. Thus, in their second attempt they tried to maintain the correct attributes intact and change the wrong attributes. In Figure 20, one can identify such construction routes.

First of all, there was the case of Elisavet (5,2). Elena (Figure 20a) had constructed a square with a gap in her first attempt (CT(A)P2). In her second attempt, she initially tried to improve her original construction by following the route described in Figure 20a. Throughout Elisavet's attempt, we see her persistence in maintaining the two equal sticks which create a right angle intact. Even when she decided to undo her original construction and try all over again, she began her new attempt by selecting two equal sticks and constructing a right angle. After experimenting with different sticks, she successfully constructed a square.

Fotini (5,4), Alexis (5,8) and Costas (6,3) were all children that followed A1 in their second attempt to construct a square. These children identified the slight 'mistake' they had done in their first attempt, fixed it and ended up with a square constructed with four equal sticks. Fotini (Figure 20b) simply replaced the stick that was shorter than the other three with one that was equal, Alexis (Figure 20c) replaced the longer stick of his original construction with one that was equal to the other three and Costas (Figure 20d) transformed his original construction which was a square with five sticks (CT(A)P4) into a square with four equal sticks (CT(B)P1) by simply replacing the two sticks that he used for one of the sides, with one that was equal to the other three sticks.

**Figure 20: The construction routes followed by children that constructed a Type A product in their first attempt but 'successfully' constructed a square in their second attempt**



The slightly oblique dotted line crossing through the side of a construction indicates that that specific side is constructed with the use of two sticks.  
The arrow shows how the child moved one stick from one position to another.

Finally, Socrates (5,1) as in the case of Costas, had constructed a square with five sticks (CT(A)P4) in his original construction and ended up with a square with four equal sticks (CT(B)P1) in his second attempt (Figure 20e). But unlike Costas, Socrates undid his original construction and started all over again by following S3. He selected two equal sticks and

constructed a right angle. He then experimented with different sticks until he successfully completed his construction. If we go back to the description of the children's first attempt to construct a square, we can see the construction route Socrates followed in his first attempt (Figure 15a, p.126). So whereas Socrates chose two unequal sticks to construct a right angle at the beginning of his first attempt, he chose two equal sticks and constructed a right angle at the beginning of his second attempt. The consistency which characterised children's actions from one attempt to the other is highly interesting. Socrates' choice of two unequal sticks in his first attempt was what led him to construct a square with five sticks. Thus, in his second attempt, he began by selecting two equal sticks. This choice led him to a successful construction of a square (with the use of four equal sticks) in his second attempt.

***The construction routes followed by children that constructed a Type B product in their first attempt and a Type A product in their second attempt***

Going back to Table 24 (p.132), there were five children that constructed a Type B product in their first attempt (CT(A)) and ended up with a Type A product in their second attempt (CT(B)). Four of these children successfully constructed a square with four equal sticks in this second attempt whereas one child constructed a square with a gap (P2). Figure 21 illustrates some examples of children that fall in this category.

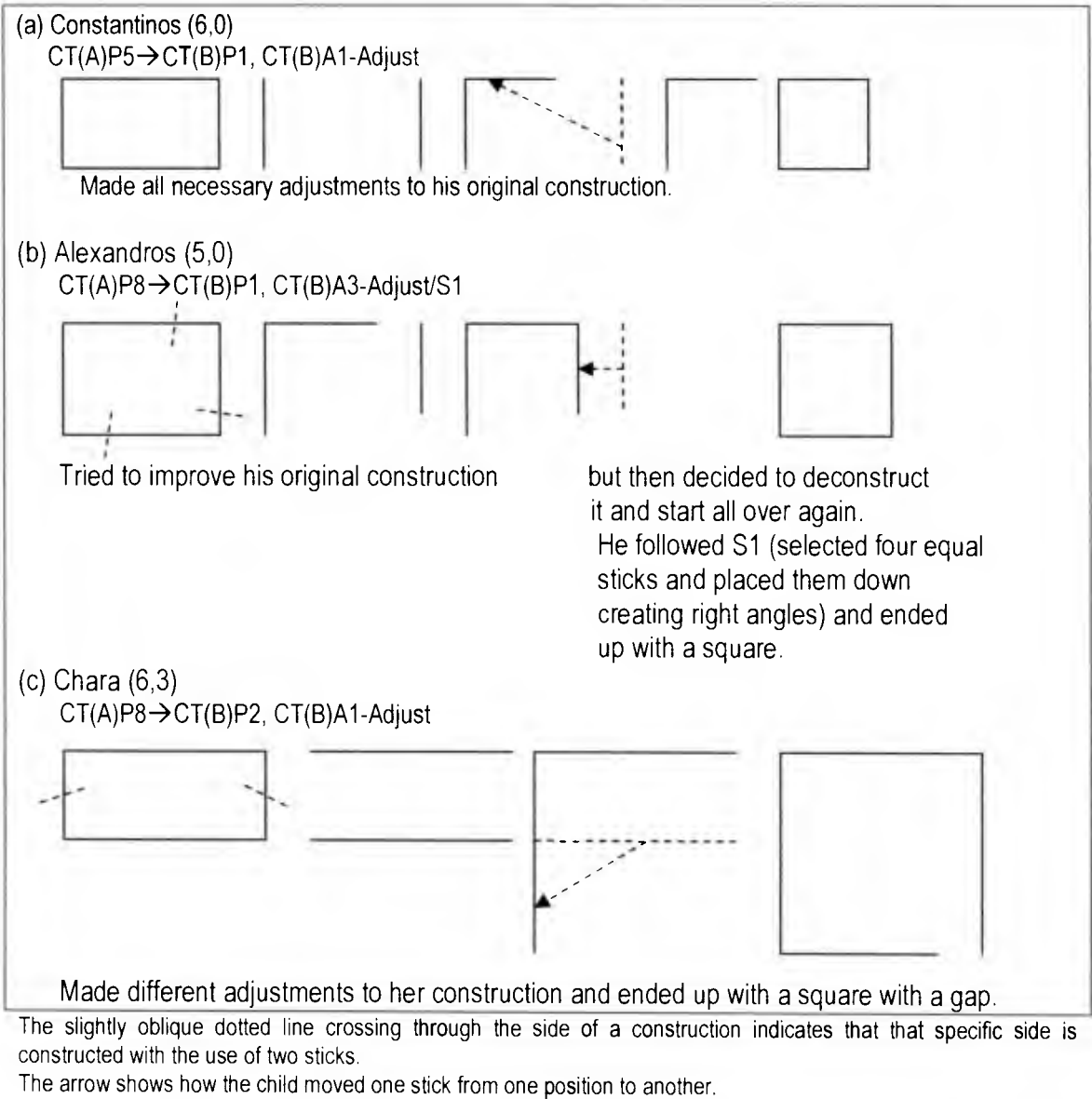
Christakis' (6,0) route in his second attempt to construct a square is highly interesting (Figure 21a). Christakis had constructed a rectangle with the use of two sets of equal sticks in his first attempt (P5). In his second attempt, he made all the necessary adjustments and transformed it into a square (P1). He removed the two horizontal sticks and used one of the vertical sticks as shown in Figure 21a to form a right angle. He then completed his construction with a new set of equal sticks. Therefore, for Christakis a square was no longer a shape with its opposite sides equal but a shape with all sides equal. Thus, any two adjacent sides of the shape had to be equal.

A similar approach was followed by Chara (6,3). Chara used the two vertical sticks of her original construction to make a right angle (Figure 21c). She then found a third stick that was equal to the other two and created an open shape with right angles. She completed her attempt with a stick that was smaller than the other three and thus ended up with a square with a gap (P2).

In his first attempt, Andys (5,0) had constructed a rectangle with seven sticks (P8). At first, he tried to improve his original construction in order to transform it into a square following the route

illustrated in Figure 21b. Even though Andys seemed to be aware of what was wrong with his original construction, at some point he decided to deconstruct it and start all over again. In this new attempt he followed S1. He selected four equal sticks, placed them in such a way as to create right angles and thus ended up with a square with four equal sticks (P1). In his first attempt to construct a square (Figure 15c, p.126), Andys had used two unequal sticks, which were only slightly different, to begin his construction. In his second attempt, he originally tried to construct a square by leaving these two sticks intact. So it might be that he thought the two sticks were actually equal but was not satisfied with the visual result. The ease with which he finally constructed the square by following S1 might support this hypothesis.

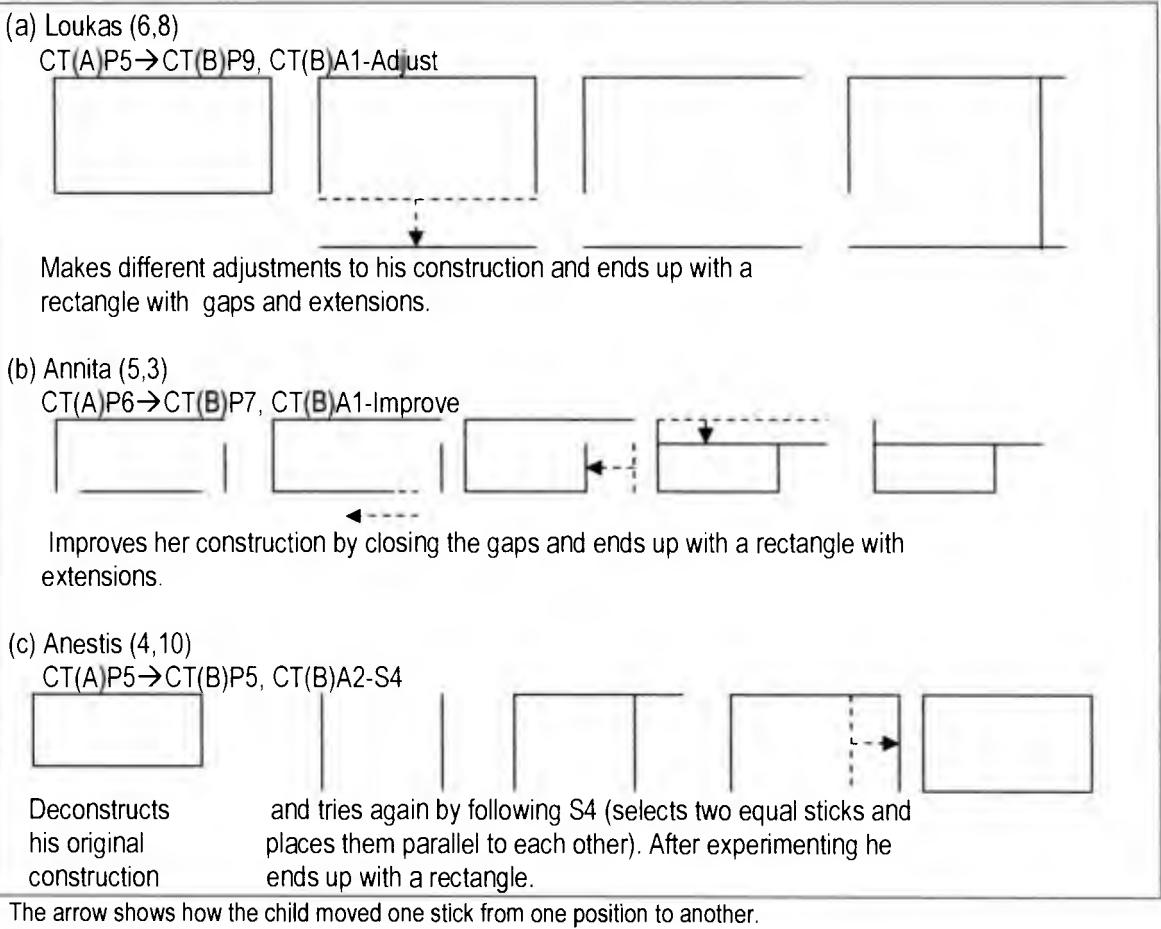
**Figure 21: The construction routes of children that constructed a Type B product in their first attempt and a Type A product in their second attempt**



**The construction routes followed by children that constructed a Type B product in their first and second attempt**

Four children constructed a Type B product in both the first and the second attempt (Table 24, p.132). Figure 22 provides some examples of children that fall in this category. Loukas (6,8) made different adjustments to his original construction, which was a rectangle with four sticks (P5), and transformed it into a rectangle with more than four sticks (P9). As illustrated in Figure 22a, he added to the vertical sides of his original construction sticks that were shorter than the horizontal sticks in order to make them longer. Through his actions, Loukas tried to minimise the difference in the length of the vertical and horizontal sides of his construction and thus ended up with a quadrilateral which somehow resembled a square.

**Figure 22: The construction routes of children that constructed a Type B product in their first and second attempt**



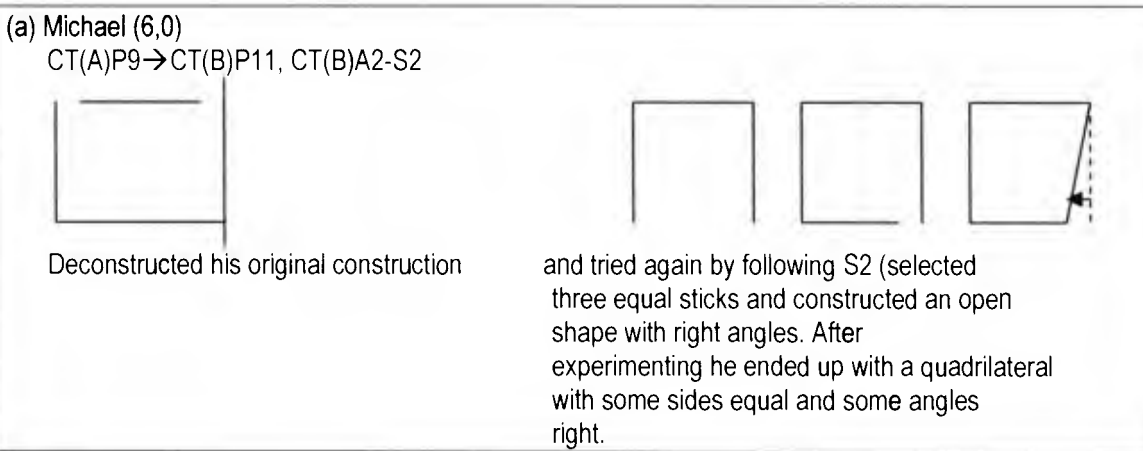
Annita (5,3), who had constructed a rectangle with gaps (P6) in her first attempt, 'improved' it by changing the position of some of the sticks in order to close the shape (Figure 22b). As a result, she ended up with a rectangle with extensions (P7). Finally, Anestis (4,10) deconstructed the product of his original construction and tried again by following S4 (Figure 22c). He selected two

equal sticks and placed them parallel to each other. Even though the distance between the two parallel sides was a sign that Anestis was ‘doing well’, after experimenting with different sticks, he picked a third stick which was longer than the other two and placed it as shown in Figure 22c. Instead of searching for a stick that would fit, Anestis adjusted the distance between the two sticks he had used at the beginning so that the new stick would fit. Then he found a fourth stick that was equal to the third stick he had used and closed the shape.

***The construction route followed by the child that constructed a Type B product in his first attempt and a Type C product in his second attempt***

Only one child constructed a Type B product in the first attempt and a Type C product in the second attempt (Table 24, p.132). This child was Michael (6,0). As illustrated in Figure 23, Michael had constructed a square with gaps and extensions (P9) in his first attempt (CT(A)). In his second attempt he deconstructed his original construction and tried again by following S2. He selected three equal sticks and placed them in such a way as to create an open shape with right angles. Then he used a shorter stick to complete the construction. He moved one of the first sticks he had used in order to close the shape and thus ended up with a quadrilateral with some angles right and some sides equal (P11).

**Figure 23: The construction route of the child that constructed a Type B product in his first attempt and a Type C product in his second attempt**



The arrow shows how the child moved one stick from one position to another.

***The construction routes followed by children that constructed a Type C product in their first attempt and a Type A product in their second attempt***

Figure 24 illustrates the routes followed by children that constructed a Type C product in their first attempt and a Type A product in their second attempt. Five children belonged to this category (Table 24, p.132). Aggeliki (5,11) had constructed a quadrilateral with no right angles in her first

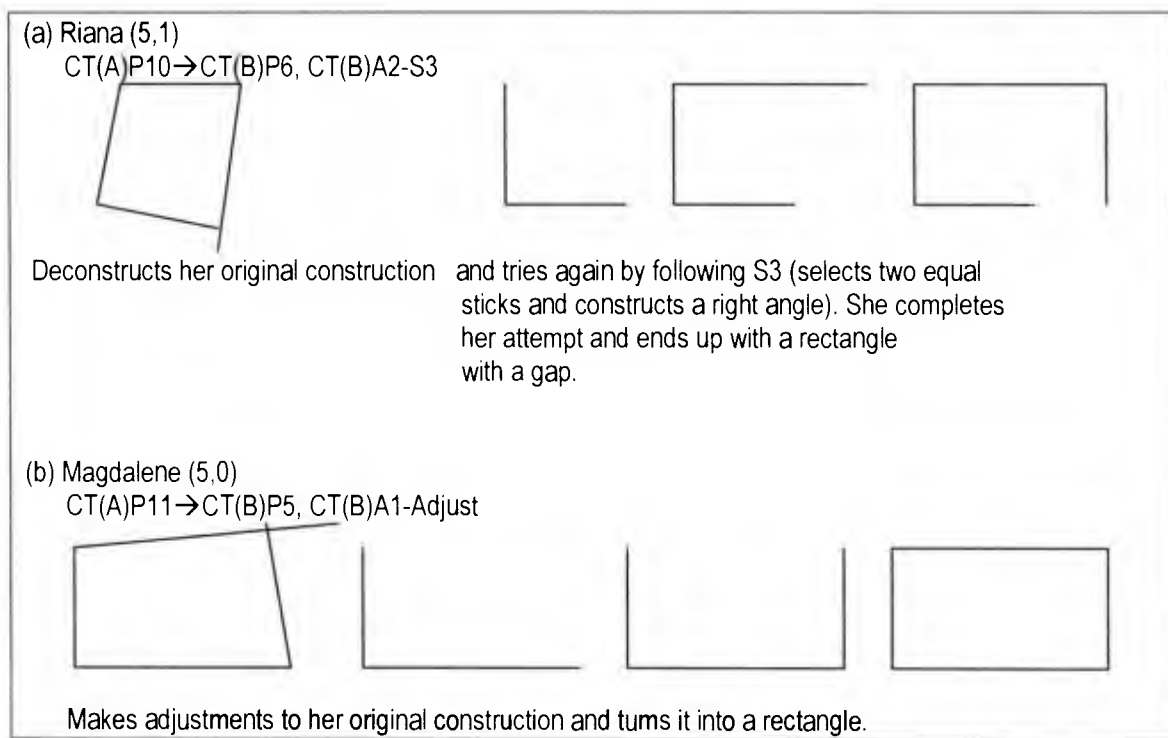


In the case of Aggeliki and Marilena, it is astonishing how from a vague 'square' construction they constructed a perfect square with relative ease. It is amazing how they made a shift from a free-style expression of the concept 'square' to a more strict mathematical representation. Avra improved her original construction in a similar way. She changed the position of one of the sticks so that all of the angles would be right. As a result, the product of her second attempt was a square with a gap (P2).

*The construction routes followed by children that constructed a Type C product in their first attempt and a Type B product in their second attempt*

Going back to Table 24 (p.132), there were two children that had constructed a Type C product in their first attempt and a Type B product in their second attempt. Figure 25 illustrates the routes followed by these children in their second attempt.

Figure 25: The construction routes of children that constructed a Type C product in their first attempt and a Type B product in their second attempt

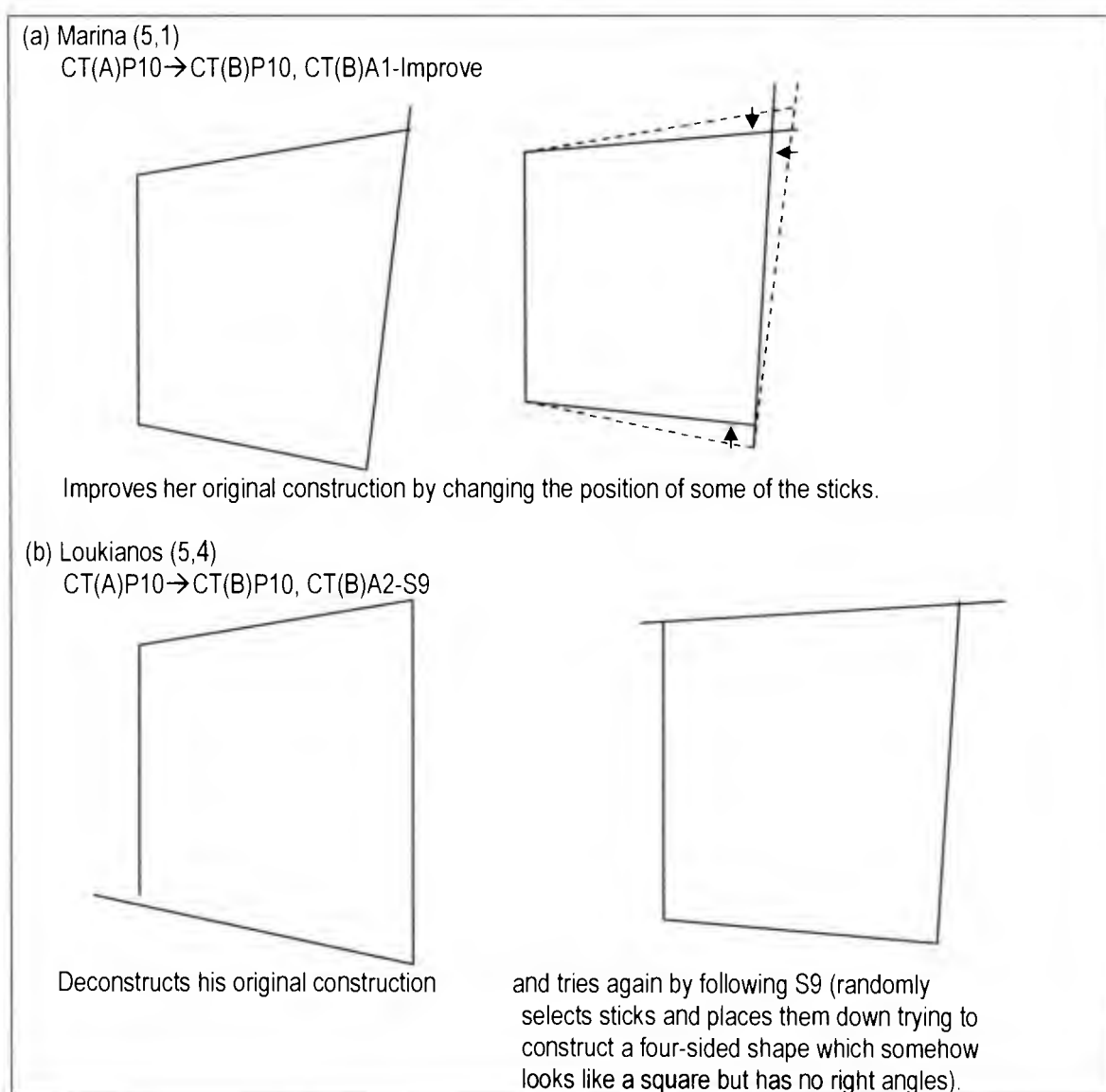


Riana (5,1) deconstructed her original construction, a quadrilateral with no right angles, and tried all over again by following S3 (Figure 25a). She selected two equal sticks and constructed a right angle. She then added a third stick to her construction which was longer than the other two as illustrated in the figure. She completed her attempt with a stick that was equal to the first two she had used. Thus, she ended up with a rectangle with a gap (P6). Magdalene (5,0), on the other hand, made several adjustments to her original construction which was a quadrilateral with some



angles right (P11) and transformed it into a rectangle (P5). To be more precise, she removed two sticks while keeping the two sticks that were creating a right angle and added a third stick that was equal to one of the existing sticks. Then, she found a fourth stick that closed her shape (Figure 25b). So these two children moved from a shape with no right angles to a shape with right angles.

To conclude with the children's second attempt, there were two children that had constructed a Type C product in both the first and the second attempt (Table 24, p.132). The routes followed by these children in their second attempt are illustrated in Figure 26.



First we had the case of Marina (5,1). In her first attempt, Marina had constructed a quadrilateral with no right angles (P11). In her second attempt, she slightly moved three of the sticks of her original construction closing the angles of the shape (Figure 26a). The angles were still not right angles and thus she ended up with a P11 once more. Finally Loukianos (5,4), as Illustrated in Figure 26b, who had also constructed a quadrilateral with no right angles (P11) in his first attempt, deconstructed his original construction in his second attempt and tried all over again. In this attempt, he followed S9 (randomly selected sticks and placed them down trying to construct a four-sided shape which somehow looked like a square but had no right angles). Thus he ended up with a P10 once again.

### 7.4. The children’s<sup>21</sup> third attempt to construct a square

This section focuses on the twelve children that failed to construct a square with four equal sticks in their second attempt. In these cases, the interviewers, according to the interview script (Appendix C), were instructed to construct a square in front of the child and ask the child to try again. Table 25 shows the approaches these children followed in this third attempt and the product of their final effort to construct a square.

**Table 25: Results in relation to the process and products of the children’s third attempt**

From CT(B) to CT(C)	
Process → Product	Total
A3-Adjust/S1→P1	1
A3-Improve/S1→P1	1
A1-Fix→ P1	1
A2-S1→P1	6
A2-S2→P1	1
A2-S5→P7	1
A2-New Strategy→P11	1
Total	12

As illustrated in Table 32, two children followed A3 (tried to transform the construction of their second attempt but then decided to deconstruct it and start all over again), one child followed A1 (attempted to turn the construction of the second attempt into a square) and nine children followed A2 (undid the construction of their second attempt and tried all over again).

<sup>21</sup> This section of the thesis relates only to the group of twelve children that failed to construct a square with the use of four equal sticks in their second attempt and thus had to try again.

The information provided by Table 25 also shows that half of the children followed S1 and their attempt was successful (P1). In other words, after these children observed the interviewer constructing a square, they selected four equal sticks and placed them in such a way as to create right angles. In addition, there was one child that tried to make adjustments (Adjust) to the construction of the second attempt and a child that tried to improve (Improve) the construction of the second attempt. In the end, both of these children decided to deconstruct the product of their second attempt and tried again by following S1. Therefore, a total of eight out of the twelve children followed S1 and successfully constructed a square (P1). There was also the case of one child that fixed (Fix) the product of the second attempt and transformed it into a square with four equal sticks (P1). Furthermore, there was one child that followed S2 (selected three equal sticks and constructed an open shape with right angles) and successfully constructed a square (P1).

Thus, ten out of the twelve (84%) children that failed to construct a square in their second attempt successfully constructed a square with four equal sticks in their third attempt (Table 26). Finally, there were two children that failed to construct a square in this third attempt (Table 26). One child followed S5 (used two unequal sticks and created a right angle) and constructed a rectangle with extensions (P7) and one child followed a new strategy (selected four equal sticks but placed them in a way that they were not creating right angles) and ended up with a quadrilateral with some angles right and some sides equal (P11).

**Table 26: Results in relation to the products of the children's third attempt**

CT(C)		Total	
		No	%
A	P1	10	84
B	P7	1	8
C	P11	1	8
Total		12	100

Table 27 shows the relationship between the product of the children's second attempt and the product of their third attempt. Two of the three children that constructed a Type A product (square with some 'flaw') in their second attempt (CT(B)) successfully constructed a square with four equal sticks (P1) in their third attempt (CT(C)). The third child constructed a quadrilateral with some sides equal and some angles right (P11). Out of the six children that constructed a Type B product (rectangle with or without some 'flaw') in their second attempt, five successfully constructed a square with four equal sticks in their third attempt (P1) and one constructed a Type B product (P7). Finally, the three children that constructed a Type C product (quadrilateral with

some or no right angles) in their second attempt (CT(B)) successfully constructed a square with four equal sticks (P1) in their third attempt (CT(C)).













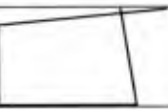



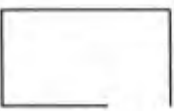
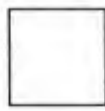

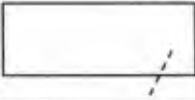


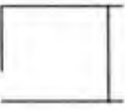




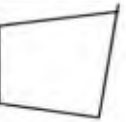
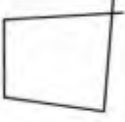

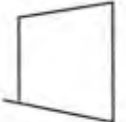
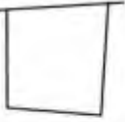
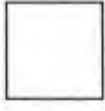



**Table 27: The relationship between the product (P) of the children's second (CT(B)) and third attempt (CT(C))**

CT(B) \ CT(C)		Type A P1	Type B P7	Type C P11	Total
Type A	P2	1	-	1	2
	P3	1	-	-	1
Type B	P5	2	-	-	2
	P6	1	-	-	1
	P7	-	1	-	1
	P8	1	-	-	1
	P9	1	-	-	1
Type C	P10	2	-	-	2
	P11	1	-	-	1
Total		10	1	1	12

It is interesting to follow these twelve children through their three attempts to construct a square. Table 35 provides all the necessary information. It shows, for each child, the route followed (strategy and/or approach) and the product of their first (CT(A)), second (CT(B)) and third attempt (CT(C)). Observing how each child progressed through the three attempts is highly interesting. None of these children had constructed a Type A product (square with gaps/extensions/more than four sticks) in their first attempt (CT(A)). Half of these children had constructed a Type B product (rectangle with or without gaps/extensions/more than four sticks) in their first attempt (CT(A)) and the other half a Type C product (quadrilateral with no right angles/some right angles and some equal sides) in their first attempt.

As shown in Table 28, Chara (6,3) had constructed a rectangle with the use of six sticks (P8) in her first attempt (CT(A)). In her second attempt (CT(B)), her product developed into a square with a gap (P2) and finally in her third attempt (CT(C)) she constructed a square with the use of four equal sticks (P1). On the other hand, Anestis (4,10) had constructed a rectangle with the use of two sets of equal sticks (P5) in his first two attempts (CT(A-B)) and finally constructed a square with four equal sticks (P1) in his third attempt (CT(C)). Loukas (6,8) constructed a rectangle with the use of two sets of equal sticks (P5) in his first attempt (CT(A)) and in his second attempt (CT(B)), he constructed a rectangle with gaps and extensions (P9) by following the route described in Figure 15a (p.126) to which we referred in the previous section of this chapter. Finally, in his third attempt (CT(C)) he constructed a square with four equal sticks (P1).

**Table 28: A synoptic table of the products and the strategies/routes of the children's three attempts (CT(A), CT(B), CT(C))**

Children	CT(A)		CT(B)		CT(C)	
	Code	Visual Description	Code	Visual Description	Code	Visual Description
Chara (6,3)	S5 P8		A1- Adjust P2		S1 P1	
Pavlos (5,1)	S3 P11		A1- Improve P3		A1-Fix P1	
Avra (5,2)	S4 P11		A1- Improve P2		A2-New Strategy P11	
Anestis (4,10)	S4 P5		A2-S4 P5		A2-S1 P1	
Magdalene (5,0)	S3 P11		A1- Adjust P5		S1 P1	
Riana (5,1)	S9 P10		A2-S3 P6		A2-S2 P1	
Stelios (5,10)	S4 P7		A1- Improve P8		A3- Improve/ S1 P1	
Loukas (6,8)	S4 P5		A1- Adjust P9		A2-S1 P1	
Annita (5,3)	S6 P6		A1- Improve P7		A3-S5 P7	
Marina (5,1)	S9 P10		A1- Improve P10		S1 P1	
Loukianos (5,4)	S9 P10		A2-S9 P10		S1 P1	
Michael (6,0)	S6 P9		A2-S2 P11		A3- Adjust/S1 P1	

The slightly oblique dotted line crossing through the side of a construction indicates that that specific side is constructed with the use of two sticks.

Pavlos (5,1), as shown in Table 28, had constructed a quadrilateral with some right angles and some equal sides (P11) in his first attempt (CT(A)). This developed into a square with an extension (P3) in the second attempt (CT(B)) and finally into a square with four equal sticks (P1) in the third attempt (CT(C)). Maria (5,0), like Pavlos, started with a quadrilateral with some right angles and some sides equal (P11) in her first attempt (CT(A)). This turned into a rectangle with two sets of equal sticks (P5) in the second attempt (CT(B)). Finally, in her third attempt (CT(C)) Magdalene (5,0) constructed a square with four equal sticks (P1). Marina (5,1) and Loukianos (5,4) followed similar routes. They constructed quadrilaterals with no right angles (P10) in their first two attempts (CT(A-B)) but successfully constructed a square with four equal sticks (P1) in their third attempt (CT(C)).

To conclude with the descriptions of Table 28, we have the cases of Avra (5,2) and Annita (5,3). Avra and Annita were the two children that failed to construct a square with four equal sticks in their third attempt (CT(C)). Avra progressed from a square with some right angles and some equal sides (P11) in her first attempt (CT(A)) to a square with a gap (P2) in her second attempt (CT(B)) and again to a quadrilateral with some angles right and some sides equal (P11) in her third attempt (CT(C)). Annita, on the other hand, progressed from a rectangle with gaps (P6) in her first attempt (CT(A)) to a rectangle with extensions (P7) in her last two attempts (CT(B-C)).

## 7.5. Concluding remark

This Chapter provided a detailed description of the children's actions during the CT. This exhaustive description was considered necessary since, as was explained in Chapter 3, the CT is the main source of data that will allow us to address this study's research questions. In Chapter 9, this thesis' discussion will be built mainly in connection and continuous reference to the descriptions provided in this chapter.

At first sight one can argue that the children based their attempt to construct a square on specific structural understandings. These structural understandings unfolded through the strategies the children followed and the products of their attempts. Through the course of the children's attempts one can sketch the ways in which the children's understandings evolved and changed. The picture sketched in relation to the children's understandings in this chapter is rather different to the equivalent picture sketched in Chapter 6. Thus, at this point, we can support the point of view that even though within a setting restricted to simple classification and recognition tasks the

children exhibited poor structural understandings about squares, through their involvement in the CT they exhibited rich structural understandings. This will be discussed thoroughly in Chapter 9. But first let us see how the children reflected on the process of constructing a square which was the objective of the Reflection Task (RT).

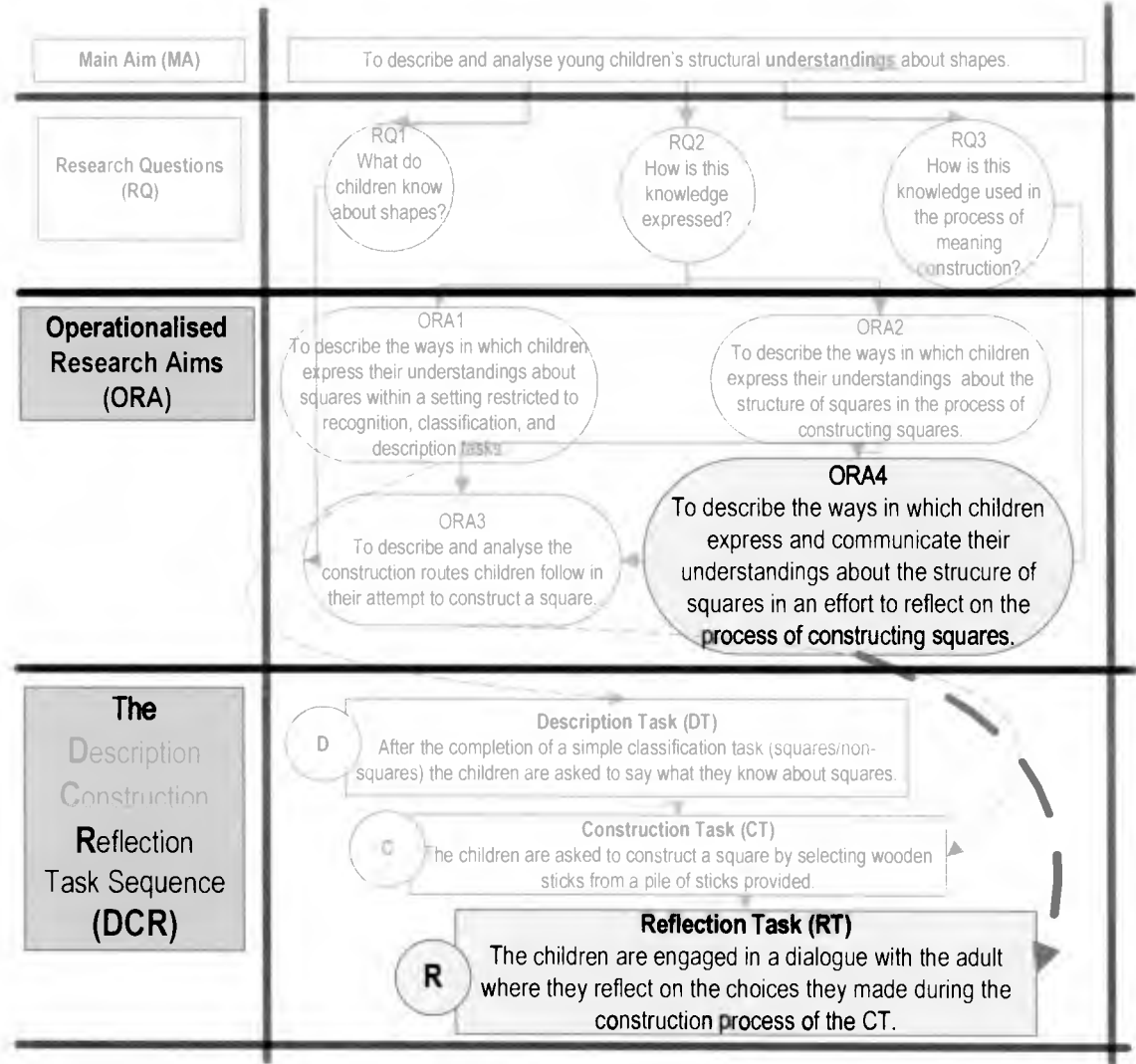
# CHAPTER 8

## FINDINGS: THE REFLECTION TASK (RT)

### 8.1. Introduction

Chapter 8 is the final chapter which deals with the findings of the study. Figure 43 highlights the part of the study's design which is the subject of this chapter. The chapter therefore portrays the findings of the Reflection Task (Figure 27, RT), so as to 'to describe the ways in which children express and communicate their understandings about the structure of squares in an effort to reflect on the process of constructing squares' (Figure 27, ORA4).

Figure 27: Aims and design of the study in relation to the Reflection Task (RT)



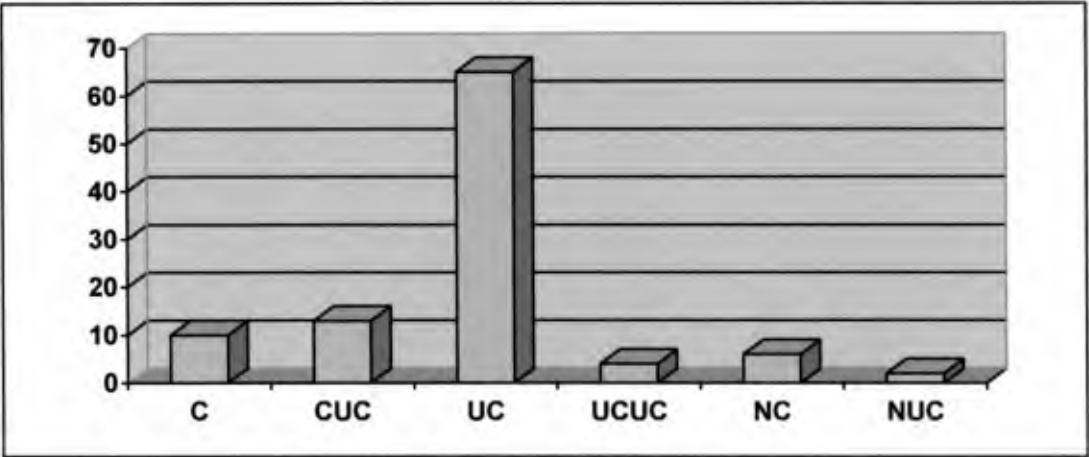


As was explained in Chapter 5, the RT was completed in two parts. Part A focused on the children's reflections in relation to the choice of sticks while Part B dealt with the children's reflections in relation to the spatial arrangement of sticks in view of the construction of a square. The findings for each part of the RT are provided in separate sub-sections.

**8.2. Children's reflections on the process of constructing squares in relation to the choice of sticks**

Graph 15 illustrates the findings in relation to the communication routes<sup>22</sup> the children followed in RT(A).

**Graph 15: Results in relation to the communication routes of the children's attempt to reflect on the process of constructing squares concerning the choice of sticks.**



All values are given in percentage rates.  
Key to response categories: C=Conventional, U=Unconventional, N=No Reply.

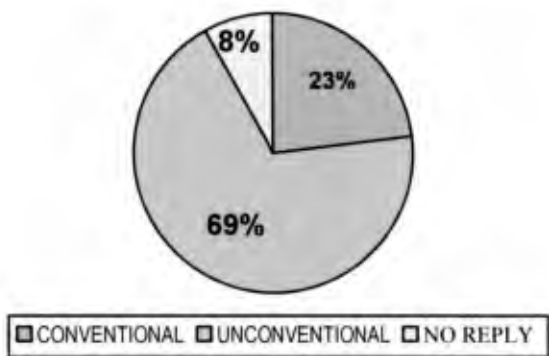
As one can see, the majority of the discussions during the RT(A) followed a route where the children would begin by giving unconventional answers and end up with providing a conventional answer after interacting with the adult (Graph 15, UC). 65% of the discussions (34/52) followed this communication route. The rest of the communication routes were much less frequent among the dialogues reported during the data collection. The children gave a conventional answer right from the beginning in only 10% of the discussions and, in those cases, the interviewer proceeded with the second part of the RT (Graph 15,C). But there were also discussions which were categorised as CUC. Based on the information of Graph 15, 13% (7/52) of the discussions fall in this category. In these cases, even though the children had replied in a conventional manner at the interviewer's opening question, the interviewer's persistence in repeating the question or rephrasing the question 'forced' the children to switch to unconventional responses before giving

<sup>22</sup> A description for each communication route is provided in Chapter 5 (Table 15, p.95).

a conventional answer again. Similarly, 4% of the dialogues (2/52) were categorised as UCUC (Graph 15). So in a total of nine cases, the adult conducting the interviews would keep interviewing the child even after s/he had given a conventional answer. This 'forced' the children to switch to the use of unconventional expressions. Furthermore, 6% of the dialogues were categorised as NC and 2% as NUC (Graph 15). In these cases, the children failed to reply in any way to the interviewer's opening question, but were able to finally give a conventional answer after interacting with the adult.

The effort to categorise the children based on their answer to the interviewer's opening question ('Why did you take these sticks in order to make your square?')<sup>23</sup> led to the results illustrated in Graph 16.

**Graph 16: Results in relation to the children's answer to the question: 'Why did you take these sticks in order to make a square?'**



It is interesting to note that even after a construction activity where the majority of children had successfully constructed a square with the use of four equal sticks, only 23% of the children's answers were conventional answers (Graph 16, CONVENTIONAL). Only 12 out of the 52 subjects of the study answered in a conventional manner to the interviewer's question regarding the choice of sticks. If we go back to the results of the CT, we will see that 40% of the children involved in the study successfully constructed a square with the use of four equal sticks in their first attempt (Graph 8, p.116). So even though these children's understandings allowed them to make all the right choices and successfully construct the shape, most of them replied in an unconventional manner to the interviewer's opening question.

<sup>23</sup> In the two cases where the children failed to construct a square after their third attempt, and thus the interviewer had to construct a square in order to proceed with the RT, the question was rephrased as follows: 'Why did we use these sticks in order to make a square?'

Following there is a more detailed analysis of some Dialogues<sup>24</sup> between the interviewer (Int) and the children (Ch) during the RT(A).

***Examples of dialogues where children gave a conventional answer to the interviewer's opening question and the interviewer proceeded with the second part of the RT.***

Following there are two Dialogues that belong to the CONV category. In these two cases, the children gave conventional answers and the interviewer proceeded with the second part of the discussion (RT(B)). Despo (Dialogue 1), who had successfully constructed a square in her first attempt, said that she used the specific set of sticks to make her square 'because they were equal' whereas Marilena (Dialogue 2), who had to try twice in order to construct a square, said that she used the specific set of sticks 'because they were as big as each other'.

Dialogue 1

---

Despo (6,1), CONV, followed S2 and successfully constructed a square in her first attempt

---

- |   |  |
|---|--|
| 1 | Int: <i>Why did you take these sticks to make your square?</i> |
|   | Ch: <b><i>Because they were equal.</i></b>                     |
- 

Dialogue 2

---

Marilena (5,3), CONV, successfully constructed a square in her second attempt

---

- |   |  |
|---|--|
| 1 | Int: <i>Why did you take these sticks to make your square?</i> |
|   | Ch: <b><i>Because they were as big as each other.</i></b>      |
- 

***Examples of dialogues where the children gave a conventional answer to the interviewer's opening question but the interviewer would insist on interviewing the children in relation to the choice of sticks instead of moving on to the second part of the RT***

Even though Periklis also gave a conventional answer to the interviewer's opening question (Dialogue 3, 1), the interviewer continued interviewing him in relation to the choice of sticks. Thus as we can see in Dialogue 3 (2-10), Periklis started giving unconventional answers before repeating a conventional answer again (Dialogue 3, 11). At the beginning of the dialogue; Periklis explained that he used the specific set of sticks to construct a square '*because they are the same*' (Dialogue 3, 1). In the discussion that followed; Periklis used the simile '*because they look like matches*' (Dialogue 3, 3) to justify why a set of sticks is suitable for the construction of a square. Furthermore; to the interviewer's question 'What do the sticks have to be?' Periklis twice

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<sup>24</sup> The children's and the interviewer's verbal behavior during the discussion is presented in the cited dialogues with the use of *italics*. Everything in insertions represents the children's and the interviewer's non-verbal behavior.

replied 'big' (Dialogue 3, 5/8). Finally; at the end of the dialogue, Periklis gave once again a conventional answer ('*they are equal*').

Dialogue 3

---

Periklis (5,7) CONV→UNCONV→CONV, followed S1 and successfully constructed a square in his first attempt

---

- 1    Int: *Why did you take these sticks to make your square?*  
      Ch: ***Because they are the same.***
- 2    Int: (The interviewer holds one set of equal sticks in one hand and a set of unequal sticks in the other hand). *With which sticks can I make a square?*  
      Ch: (The child shows the set of four equal sticks).
- 3    Int: *Why?*  
      Ch: ***Because they look like matches.***
- 4    Int: *Can I make a square with these sticks* (shows the child the four equal sticks again)?  
      Ch: *Yes.*
- 5    Int: *What are these sticks?*  
      Ch: *Big.*
- 6    Int: *If I take these sticks can I make a square* (shows him a set of relatively big unequal sticks)?  
      Ch: *We can join them together* (The child takes the sticks and tries to construct a shape with them. He uses the biggest stick and a smaller one to construct a right angle. He then takes the third one and adds it to his construction as an extension to the second stick).
- 7    Int: *With which sticks will our square be nice?*  
      Ch: *These ones* (the child selects four equal sticks and constructs a square).
- 8    Int: *So what do our sticks have to be?*  
      Ch: ***Big.***
- 9    Int: *So if I take these sticks can I make a square* (shows him a set of four relatively big unequal sticks).  
      Ch: *This is short* (points to one of the sticks the interviewer is holding which is smaller than the other three).
- 10   Int: *With which sticks will our square be better? These sticks* (shows him a set of four equal sticks) *or these sticks* (shows him the set of unequal sticks)?  
      Ch: (Shows her the set of equal sticks.)
- 11   Int: *Why?*  
      Ch: ***They are equal.***
- 

The unconventional answers<sup>25</sup> given by Periklis are very interesting. The use of the simile 'because they look like matches' is attention grabbing. Why did the child use this simile? Back in Chapter 2, we supported the point of view that by using a simile one attributes a characteristic of one thing to another. One of the properties of matches is that they all have the same length so it is possible that with the use of the simile 'like matches' Periklis was pointing towards the fact that in order to make a square one needs sticks of the same length.

The emphasis given by Periklis to the fact that the sticks have to be 'big' is also very interesting since it was a pattern that kept reappearing in most of the discussions collected during the RT.

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<sup>25</sup> A definition of 'unconventional answers' is given in Chapter 5 (Table 14, p.95).

This was a way that many children used to justify why they used the sticks they used in order to construct a square. In almost all the dialogues we cite in this section of the study, at some point the children support the view that a set of sticks is suitable for the construction of a square 'because they are big'. This was anticipated based on the data collected from the phases of the piloting procedure. This is why in designing the interview script (Appendix C) special attention was given to guiding the interviewers on how to react to this kind of answer.

From the beginning of this thesis (Chapter 1), I supported Margaret Donaldson's idea that, as far as young children are concerned, adults should be careful with '*what is said and what is meant*'. So what did the children mean when they said that the sticks were big? One possible interpretation is that what they actually meant was that the sticks are as big as each other. In the example presented in Dialogue 2 above, we saw Maria giving this exact justification. In the following pages, we will have the opportunity to see how, in a number of cases, the children themselves tried to explain what they meant when they said 'because they are big'.

***Examples of dialogues where the children gave an unconventional answer to the interviewer's opening question and the interviewer continued interviewing the children in relation to the choice of sticks until the children replied in a conventional manner.***

Moving on to Dialogue 4, we have an example of a dialogue that was categorised as UNCONV→CONV. To the interviewer's opening question, Aggeliki replied: 'because *they were big*' (Dialogue 4, 1). Thus, the interviewer started interacting with her until she finally gave a conventional answer; '*the same*' (Dialogue 4, 12). During her interaction with the interviewer, Aggeliki kept replying in unconventional ways in order to explain why a set of sticks was suitable or not for the construction of a square. In the discussion that follows we can see how Aggeliki tries to communicate with the interviewer and explain what she meant when she said that the sticks have to be big. When the interviewer holds a set of relatively big sticks one of which is bigger than the other three, Aggeliki points towards the bigger stick and explains that 'this stick has to be smaller' (Dialogue 4, 4). After this, the interviewer repeated the opening question ('So what do the sticks have to be?') and Aggeliki changed her reasoning from 'the sticks have to be big' to 'the sticks have to be small' (Dialogue 4, 5). Here we see Aggeliki using the words small and big to explain the same thing; that the sticks have to be the same (as small as each other or as big as each other). Aggeliki's struggle to communicate with the adult becomes even more interesting along the way.

Aggeliki (5,11), UNCONV→CONV, successfully constructed a square in her second attempt

- 1 Int: *Why did you take these sticks to make your square* (the square is made of relatively big sticks)?  
Ch: **Because they were big.**
- 2 Int: *If I take big sticks will I be able to make a square?*  
Ch: Yes.
- 3 Int: *Ok.. I will try* (the interviewer selects four relatively big unequal sticks and constructs a shape that is not a square). *Is this a square?*  
Ch: No.
- 4 Int: *Why?*  
Ch: **This stick has to be smaller** (the child shows a stick from the construction which is bigger than the other three).
- 5 Int: *So what sticks do I need in order to make a square?*  
Ch: **Small sticks.**
- 6 Int: *Ok.. I will try* (the interviewer selects four relatively small unequal sticks and constructs a shape that is not a square). *Is this a square?*  
Ch: No.
- 7 Int: *Why. What can we do to turn it into a square?*  
Ch: (The child replaces one stick and turns the shape into a non-square rectangle.)
- 8 Int: *Is this a square?*  
Ch: No.
- 9 Int: *We need to make a square.*  
Ch: (The child selects two of the parallel sticks of the non-square rectangle and uses them to make a right angle. She then finds two more sticks that are equal to the other two and completes her construction which is now a relatively small square).
- 10 Int: *Well done. Now it is a square* (referring to the smaller square Aggeliki had just constructed). *Why did you use these sticks to make the square?*  
Ch: **Because they were the same as these** (points towards the sticks of the relatively big square she had constructed earlier). **This is a small square** (shows the small square she had constructed) **and this is a big square** (shows the big square she had constructed earlier).
- 11 Int: *So what sticks do we need in order to make a square* (points towards the relatively big square)?  
Ch: **Big ones.**
- 12 Int: *What sticks do we need?* (the interviewer deconstructs the relatively big square and holds the four sticks in her hand showing them to the child).  
Ch: **The same.**
- 13 Int: *Well done. The same.*

After Aggeliki's claim that the sticks have to be small, the interviewer constructed a shape with four relatively small but unequal sticks so Aggeliki 'decided' to take over and transformed the shape constructed by the adult into a square (Dialogue 4, 7). After this, Aggeliki had in front of her two squares (a big and a small one) and the interviewer repeated the question: 'Why did you use these sticks to make the square?' (Dialogue 4, 10). Aggeliki's reply was astonishing. Whereas the interviewer's question was referring to the sticks of the smaller square, Aggeliki pointed towards the big square and explained 'because they were the same as these'. And continued: 'This is a small square (points towards the small square she had just constructed) 'and this is a big square' (points towards the big square she had constructed earlier). So what did Aggeliki mean when she said 'because they were the same as these'? The common characteristic between the sticks used

for the big square and the sticks used for the small square is that both sets consisted of sticks of the same length.

After this, once again Aggeliki replies 'big ones' when asked by the interviewer 'so what sticks do we need' (Dialogue 4, 11) and only when the interviewer deconstructs a square construction and holds the equal sticks in her hand does Aggeliki finally give a conventional answer ('The same'). The way Aggeliki finally used the conventional answer at the end of the discussion shows that this is what Aggeliki was saying from the very beginning.

Melina's dialogue (Dialogue 5) with the interviewer was also categorised as UNCONV→CONV. Melina gave exactly the same answer to the interviewer's opening question as Aggeliki (Dialogue 5, 1). But, as opposed to Aggeliki, Melina found the magic word ('equal') that would make the interviewer 'happy' much more easily than Aggeliki.

Dialogue 5

---

Melina (5,9), UNCONV→CONV, followed S1 and successfully constructed a square in her first attempt

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- |   |      |  |
|---|------|--|
| 1 | Int: | <i>Why did you take these sticks to make your square (the square is made of relatively big sticks)?</i>  |
|   | Ch:  | <b><i>Because they were big.</i></b>   |
| 2 | Int: | <i>So I need four big sticks to make a square?</i>   |
|   | Ch:  | <i>Yes.</i>  |
| 3 | Int: | <i>I will make a square. I will take four big sticks to make my square (the interviewer selects four relatively big unequal sticks and starts constructing a shape).</i> |
|   | Ch:  | <i>One of them is big.</i>   |
| 4 | Int: | <i>One of them is big? So what do the sticks have to be?</i>   |
|   | Ch:  | <b><i>Equal.</i></b>   |
| 5 | Int: | <b><i>Equal! Well done Melina.</i></b>   |
- 

A similar route to the one we saw in the case of Aggeliki (Dialogue 4) was followed by Fotini (Dialogue 6). Even though Fotini gave a different answer to the interviewer's opening question (Dialogue 6, 1), at some point she also said that the sticks she used to construct a square were 'all big' (Dialogue 6, 3) and continued her reasoning in a similar way to Aggeliki's. In her effort to communicate with the adult, she used phrases like 'they have to be bigger', 'they have to be smaller', 'all big' and 'all small' to explain what she meant. At some point, when she said that the sticks she used to make a square were 'all small' (Dialogue 6, 9) the interviewer said 'But didn't you tell me previously that I need big sticks to make a square?' (Dialogue 6, 10). This question made Fotini give a conventional answer. To be more precise, she explained that: 'You can do it with really small ones as well. This is big, this is small, this is bigger and this is smaller (the child

points respectively to the set of unequal sticks the interviewer is holding in one hand). These are all the same though (points to the set of equal sticks the interviewer is holding)'.

#### Dialogue 6

---

Fotini (5,4), UNCONV → CONV, successfully constructed a square in her second attempt

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- 1 Int: *Why did you take these sticks to make your square?*  
Ch: **There were four.**
- 2 Int: *So if I take four sticks can I make a square (the interviewer selects four unequal sticks from the pile of sticks and shows them to the child)?*  
Ch: **No, because there are small and big ones.**
- 3 Int: *How are the ones you used?*  
Ch: **They are all big.**
- 4 Int: *So if I select four big sticks can I make a square (the interviewer selects four relatively big and unequal sticks from the pile of sticks provided and shows them to the child)?*  
Ch: **No, because they are all a little smaller, all of them have to be all big.**
- 5 Int: *But the ones I am holding are all big.*  
Ch: *But these are smaller (the child points to the two sticks the interviewer is holding and which are smaller than the other two).*
- 6 Int: *So what do the sticks have to be in order to be able to make a square?*  
Ch: **They have to be a bit bigger.**
- 7 Int: *So what do I have to do?*  
Ch: *You have to put this in order to make it bigger (the child picks up a really small stick from the pile of sticks provided and uses it to extend the smaller stick the interviewer is holding).*
- 8 Int: *Let me show you something else. (The interviewer selects four relatively small equal sticks and holds them in one hand while still holding the four unequal ones in the other hand.) With which sticks can I make a square? These sticks (shows the set of equal sticks) or these ones (shows the set of unequal sticks)?*  
Ch: *(The child points to the set of relatively small equal sticks.)*
- 9 Int: *Why?*  
Ch: **They are all small.**
- 10 Int: *But didn't you tell me previously that I need big sticks to make a square?*  
Ch: *You can do it with really small ones as well. This is big, this is small, this is bigger and this is smaller (the child points respectively to the set of unequal sticks the interviewer is holding in one hand). These are all the same though (points to the set of equal sticks the interviewer is holding).*
- 

It is also important to look at Fotini's opening claim. At the beginning of the interview, Fotini claimed that she used a specific set of sticks because 'there were four?' (Dialogue 6, 1). Even though at a first look this seems to be an 'out-of-place' answer, Fotini's ability to reason during the discussion 'forces' us to believe that this answer was not accidental or wrong. The question is what did Fotini mean? The way the interviewer reacted (Dialogue 6, 2) shows that the interviewer thought the child was saying that one needs four sticks to make a square ('So if I take four sticks can I make a square?'). But could it be that Fotini meant that she chose those specific sticks because she was able to find four of that specific length?



It is interesting to go back and see the construction route that Fotini followed in order to construct a square in the CT (Chapter 7). In her first attempt to construct a square, Fotini had constructed a square with a gap using three equal sticks and one that was smaller (Figure 11, p.122). She initially used the three equal sticks to construct an open shape with right angles and then experimented with different sticks to close the shape. A few trials later, she was finally satisfied with one stick that was smaller than the other three and ended up with a square with a gap. After the interviewer had interfered she finally found a stick that was equal to the other three and corrected her construction by replacing the smaller stick (Figure 20, p.134). Immediately after she had finally found a fourth stick that was equal to the other three and fixed her original construction the interviewer asked: 'Why did you take these sticks to make your square?' and Fotini replied: 'There were four'. So because of the construction route Fotini followed during the CT, it is possible that when she said 'there were four' what she meant was that there were four of the same length. Like Fotini, Stylianos (Dialogue 7) followed an UNCONV→CONV route in the RT(A).

#### Dialogue 7

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**Stylianos (5,9), UNCONV→CONV, successfully constructed a square in his second attempt**

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- 1    Int: *Why did you take these sticks to make your square?*  
       Ch: **Because they were lines.**
  - 2    Int: *Can I take any sticks and make a square?*  
       Ch: (Nods yes.)
  - 3    Int: *Shall I take any sticks and make a square?*  
       Ch: **No. Take four.**
  - 4    Int: *Four what? Tell me what sticks to take. Shall I take this one, this one, this one and this one*  
       (she picks up four unequal sticks)?
  - 5    Ch: (Nods no. He takes some of the smaller sticks the interviewer is holding and gives her  
       other sticks that are bigger but the sticks are still unequal.)
  - 6    Int: *Can I make a square with these sticks now?*  
       Ch: (Nods yes.)
  - 7    Int: *Try and make a square with these sticks.*  
       Ch: (The child constructs a non-square rectangle with gaps. He undoes the shape, **selects  
       four equal sticks and constructs a square**).
  - 8    Int: *Why did you take these sticks to make a square?*  
       Ch: .....
  - 9    Int: *With which sticks can you make a square? These* (she shows him four equal sticks) *or*  
       *these* (she shows him four unequal sticks)?  
       Ch: (Shows the equal sticks.)
  - 10   Int: *So how must the sticks be in order to make a square?*  
       Ch: **Good!**
  - 11   Int: *How are these sticks you showed me?*  
       Ch: **Equal.**
- 

Even though during the procedure Stylianos showed some evidence that he knew that four equal sticks are needed for the construction of a square (e.g. selects four equal sticks and constructs a

square, chooses the correct set of sticks when shown a set of equal sticks and a set of unequal sticks) he showed very poor ability (and/or intention) to reason. Thus, there is not much evidence of what he meant when he said 'because they were lines' at the beginning of the interview (Dialogue 6, 7) or exactly what he meant when he said 'take four' at some point (Dialogue 6, 3). What is interesting though is his penultimate answer. After he chose a set of equal sticks from a set of unequal sticks the interviewer asked him 'So how must the sticks be in order to make a square? Stylianos replied: 'Good' (Dialogue 6, 10). Immediately after that, when the interviewer asked him 'How are these sticks you showed me?' he said 'Equal' (Dialogue 6, 11). So when he said 'Good' he actually meant 'Equal'. After all, the 'good' (right) sticks to make a square are the ones that are equal.

Dialogue 8 was also categorised as UNCONV→CONV and concerns Anastasia. As opposed to Fotini (Dialogue 6) and Stylianos (Dialogue 7), who had to try twice in order to successfully construct a square, Anastasia successfully constructed a square in her first attempt after following S1. Nevertheless, as one can see from the length of the dialogue, it took Anastasia much longer to get to a conventional answer. To the interviewer's opening question Anastasia replied with a somehow self-evident answer; 'so that I could make its shape' (Dialogue 8, 1). In the discussion that followed, Anastasia, like many other children we saw, used expressions like 'you have to have big ones' (Dialogue 8, 5) and 'they are all big' (Dialogue 8, 16).

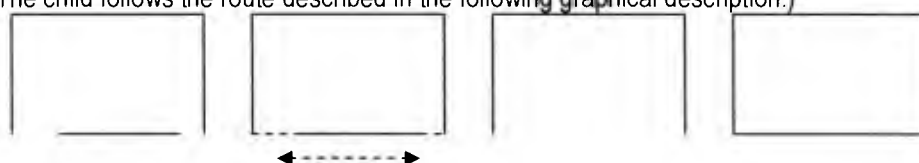
Anastasia used construction and pointed towards sticks to explain what she meant (as other children did) but what is interesting is her persistence throughout the dialogue to take a specific stick and say 'like this one' (Dialogue 8, 5/14). So she keeps following this inventive strategy where she is holding one stick in her hand and tries to explain that all the sticks have to be like that one in order to be able to make a square. And again in line 19 (Dialogue 8) Anastasia says: 'It's not the same shape. It's not the same as this one'. Twice in the dialogue Anastasia is asked to comment on a shape made of two sets of equal sticks. In the first case she has a rectangle in front of her (Dialogue 8, 8) and, when asked if she likes it, she says 'Bigger' (Dialogue 8, 9). Upon observing her following actions we can support the point of view that the word 'bigger' was not referring to the shape as a whole but to the set of vertical sticks which were bigger than the horizontal sticks. So she removed the vertical sticks from the construction (Dialogue 8, 10). And again in line 21 when the interviewer asked Anastasia if she can make a square with a set of big (equal) sticks and a set of small (equal) sticks, Anastasia replied 'It will be small'. And even though it seems as if she is referring to the shape as a whole, she is probably referring to the fact

that because one of the two sets is smaller than the other, we will end up with something that, as she explains in line 22, will look 'like a road'.

#### Dialogue 8

Anastasia (6,3), UNCONV→CONV, followed S1 and successfully constructed a square in her first attempt

- 1 Int: *Why did you take these sticks to make your square?*  
Ch: **So that I could make its shape.**
- 2 Int: *What shape?*  
Ch: *The square.*
- 3 Int: *So I will take some sticks and you will tell me if I can make a square. 1,2,3, (picks up unequal sticks while counting)...+*  
Ch: *One more and that's enough.*
- 4 Int: *Yes one more and that's enough (picks up one more stick). Can I make a square with these sticks?*  
Ch: *No.*
- 5 Int: *Why?*  
Ch: **You have to have big ones. Like this one** (she takes one of the sticks she used earlier to construct her square and shows it to the interviewer).
- 6 Int: *So what sticks do I need to take (still holding the set of four unequal sticks)?*  
Ch: **Hold this one** (Takes all sticks the interviewer is holding but one. She leaves the bigger one. She finds a set of three equal, relatively big sticks that are not equal to the one the interviewer is holding and gives them to the interviewer).
- 7 Int: *Can I make a square now?*  
Ch: *Yes.*
- 8 Int: *Why don't you try then?*  
Ch: (The child follows the route described in the following graphical description.)



- 9 Int: *Do you like your square?*  
Ch: *Bigger!*
- 10 Int: *There is something wrong with your square?*  
Ch: **They are bigger** (removes the two vertical sticks from her construction).
- 11 Int: *They are bigger? Let me find four sticks (picks up two sets of equal sticks that are all relatively big). Four big sticks.*  
Ch: **But these are small.**
- 12 Int: *Which are small?*  
Ch: (She takes the set of small sticks from the interviewer's hand.)
- 13 Int: *The interviewer takes them back.. Can I make a square with these sticks?*  
Ch: *No.*
- 14 Int: *Why?*  
Ch: **You need like these ones** (shows the set of big sticks the interviewer is holding).
- 15 Int: *The big ones? (she selects two more sticks that are equal to the ones Anastasia showed her).*  
Ch: *her).*
- 16 Int: *Now it's ok..*  
Ch: *So what are these sticks?*
- 17 Int: **They are all big.**  
*Let me take small sticks (she selects four equal relatively small sticks). Can I make a square with these sticks?*  
Ch: *with these sticks?*
- 18 Int: *Yes.*  
Ch: *So can I make a square with small sticks and with big sticks?*

- 19 Int: *Big square and small square* (shows cutout squares respectively).  
 Ch: *Let me take small sticks again* (she picks up one stick and then another stick that is unequal to the first one).  
 Int: ***It's not the same shape. It's not the same as this one*** (she starts selecting sticks that are equal to the one the interviewer had selected first and gives them to her).
- 20 Int: *Why are you giving me these sticks?*  
 Ch: *So that you can make a square.*
- 21 Int: *What if I take two small sticks and two big ones* (picks up two sets of equal sticks respectively)? *Can I make a square?*  
 Ch: ***It will be small.***
- 22 Int: *It will be small?*  
 Ch: ***Like a road.***
- 23 Int: *So what sticks must I take to make a square?*  
 Ch: ***(The child selects four equal sticks and makes a square.)***
- 24 Int: *If I replace this stick with this one* (she replaces one of the sticks of the child's construction with a bigger one thus transforming it into a square with extensions)?  
 Ch: *No.*
- 25 Int: *Why?*  
 Ch: ***It's big. Take this one*** (gives her the one she used earlier to construct her square).
- 26 Int: *Why did you use this stick?*  
 Ch: ***Because it's the same.***
- 27 Int: *So you are saying that it has to be the same? So you selected four sticks that were the ....*  
 Ch: ***.... same*** (she says this loudly).
- 

In this lengthy dialogue between Anastasia and the interviewer we see the child's effort to explain in different, multiple and inventive ways what she finally said in a conventional way at the end of the discussion (Dialogue 8, 27).

***Examples of dialogues where the children gave an unconventional answer to the interviewer's opening question and the interviewer continued interviewing the children in relation to the choice of sticks even after the children gave a conventional answer***

Following we have a dialogue involving Riana (Dialogue 9). Riana was one of the children that had to try three times in order to construct a square. Her discussion with the interviewer in relation to the choice of sticks was categorised as UNCONV→CONV→UNCONV→CONV. Riana started by replying to the interviewer's questions in unconventional ways. After interacting with the interviewer she finally gave a conventional answer, but the adult kept asking Riana in relation to the choice of sticks. As a result, Riana started giving unconventional answers before repeating the conventional answer.

Riana's answer to the interviewer's opening question was 'because they were like this' while pointing towards the square she had previously constructed (Dialogue 9, 1). In her interaction with the interviewer she used different interesting ways to explain what sticks are needed for the construction of a square. When the interviewer showed her a set of four unequal sticks and asked if she (the interviewer) could make a square with them Riana replied 'No, because you took big,

small and they are not convenient' (Dialogue 9, 6). And when the interviewer insisted ('so what sticks should I take?') Riana said: 'Big-Big'. Presumably, this was another way to say that the sticks had to be as big as each other. Riana also used at some point the same reasoning as Anastasia (Dialogue 8). She picked up one stick and said that all sticks had to be 'only like these sticks' (Dialogue 9, 18).

#### Dialogue 9

---

Riana (5,1), UNCONV→CONV→UNCONV → CONV, had to try three times to successfully construct a square

---

- 1 Int: *Why did you take these sticks to make your square?*  
Ch: **Because they were like this** (she points to the square she had previously constructed).
  - 2 Int: *So what sticks do I have to take in order to make a square?*  
Ch: **Four**
  - 3 Int: *Any four sticks?*  
Ch: **Four in order to make this shape** (she points to a cutout square).
  - 4 Int: *So if I take four sticks I can make a square?*  
Ch: *Yes, if you know it.*
  - 5 Int: *OK.. I will take four sticks. Can you count with me* (picks up four unequal sticks)  
Ch: *1,2,3,4.*
  - 6 Int: *Can I make a square with these sticks?*  
Ch: **No, because you took big, small and they are not convenient.**
  - 7 Int: *So what sticks should I take?*  
Ch: **Big-Big!**
  - 8 Int: (The interviewer picks up four relatively big sticks of which three are equal and one is smaller). *Can I make a square with these sticks?*  
Ch: *No, you made a mistake and you took a small one.*
  - 9 Int: (The interviewer replaces the smaller one with one that is bigger than but not as big as the other three). *Now can I make a square?*  
Ch: **Now they are the same.**
  - 10 Int: *So do you think you can make a square?*  
Ch: *Yes.*
  - 11 Int: *Ok then make a square with these sticks.*  
Ch: (The child constructs an open shape with the three equal sticks and changes the smaller one with one that is equal to the other three). *I'm changing the small one.*
  - 12 Int: *What stick will you take?*  
Ch: **A big one.**
  - 13 Int: *So what sticks do I need in order to make a square? Any sticks?*  
Ch: (She points to the sticks she used just now to make a square.) **No, big-big, like this, all the same sticks so that you can make a square.**
  - 14 Int: *So if I take small sticks I cannot make a square?*  
Ch: *You will make a small square*
  - 15 Int: *So I have to take four sticks ...*  
Ch: **... that are big to make a big square ...**
  - 16 Int: *And four small sticks to make a small square?*  
Ch: *Yes.*
  - 17 Int: *But what do my sticks have to be? Can I take any small sticks and any big sticks?*  
Ch: **No only like these sticks** (she takes one specific stick and shows it to the interviewer)
  - 18 Int: *Can I make a square with these sticks* (holds a set of four unequal sticks)?  
Ch: **No, they have to be equal.**
-

**Examples of dialogues where the children did not reply to the interviewer's opening question and the interviewer continued interviewing the children until they gave a conventional answer**

In Dialogues 10 and 11 we have examples where the children initially did not reply to the interviewer's opening question. Demetra (Dialogue 10), for example, did not reply to the interviewer's opening question, but when the interviewer proceeded with the scaffolding intervention the child said that the sticks '*have to be equal*' in order to make a square.

Dialogue 10

Demetra (5,7), NO REPLY→CONV, followed S1 and successfully constructed a square in her first attempt

- |   |      |   |
|---|------|---|
| 1 | Int: | <i>Why did you take these sticks to make your square?</i>                               |
|   | Ch:  | .....   |
| 2 | Int: | <i>If I take these sticks can I make a square (holds a set of four unequal sticks)?</i> |
|   | Ch:  | <i>No.</i>  |
| 3 | Int: | <i>Why?</i>   |
|   | Ch:  | <b><i>Because they have to be equal.</i></b>  |

The last example of dialogue included in this section of the thesis involves Annita (Dialogue 11). Annita was one of the two children that failed to construct a square after the third attempt. Annita's dialogue with the interviewer in relation to the choice of sticks was the only one categorised as NO REPLY→UNCONV→CONV. At the beginning of the dialogue, Annita repeatedly did not reply to the interviewer's questions (Dialogue 11, 1-2). When the interviewer started using sticks and constructions to interact with her, she started taking part in the discussion (Dialogue 11, 3).

At some point, when the interviewer showed Annita a set of four equal sticks and a set of four unequal sticks and asked which set was more appropriate for the construction of a square, the child pointed towards the set of unequal sticks. This is not surprising if we look at Annita's actions during the CT (Figure 16, p.126). In her first attempt to construct a square (CT(A)) Annita followed S6. She randomly selected four unequal sticks and placed them down trying to construct a quadrilateral with four right angles. As a result, she ended up with a rectangle with gaps. In her second attempt (Figure 22b, p.137) she improved her original construction by closing the gaps showing an understanding that what was wrong with her original construction was simply the fact that it was open. In her third attempt (Table 28, p.145), she followed S5 (selected two unequal sticks and constructed a right angle) and once more ended up with a rectangle with extensions. It is therefore not surprising that she considered that with a set of any four unequal sticks one can make a square.

Annita (5,3), NO REPLY→UNCONV→CONV, one of the two children that failed to construct a square after the third attempt

The child has in front of her the product of her third attempt to construct a square, which was not a square, and a square which the interviewer had constructed to help her before her third attempt.

- 1 Int: *What sticks do we need to use in order to make a square?*  
Ch: (The child looks at her own construction, the interviewer's construction and at some cutout squares.)
- 2 Int: *What sticks do we need to use in order to make a square like this one (points towards the square she constructed earlier to help the child?)*  
Ch: .....
- 3 Int: (The interviewer selects four relatively small equal sticks and four unequal sticks and holds one set in each hand.) *With which sticks can we make a square? These sticks (shows the set of relatively small sticks) or these ones (shows the set of unequal sticks)?*  
Ch: (The child points towards the set of unequal sticks)
- 4 Int: *These sticks (shows the set of unequal sticks)?*  
Ch: (The child nods yes.)
- 5 Int: *Let's see (the interviewer constructs a square with the relatively small equal sticks and an open shape that is not a square with the set of unequal sticks). With which sticks did we make a square?*  
Ch: (The child shows the square the interviewer has just constructed with the use of the set of the four relatively small equal sticks).
- 6 Int: *So what sticks do we need to use?*  
Ch: **Small ones.**
- 7 Int: *So if I take four small sticks can I make a square?*  
Ch: (The child nods yes.)
- 8 Int: *Let's see (the interviewer selects four relatively small unequal sticks and constructs a shape that is not a square). Is this a square?*  
Ch: (The child nods no.)
- 9 Int: *Why did we manage to make a square with these sticks (the interviewer deconstructs the small square she had constructed earlier and places the sticks down in front of the child) but we didn't manage to do a square with these sticks (the interviewer deconstructs the construction she had made with the set of four relatively small sticks she had constructed earlier and places the sticks down in front of the child as well)?*  
Ch: **These are not the same** (the child shows the set of unequal sticks).
- 10 Int: *Well done. These are not the same. So why could we make a square with these sticks (the interviewer shows the set of equal sticks)? Because they were the ...*  
Ch: **...same.**
- 11 Int: *So we need four sticks that are the same to make a square.*

During the interaction, the interviewer constructed a square with a set of relatively small, equal sticks and another open shape with a set of relatively small unequal sticks and asked Annita with which sticks did she manage to make a square. (Dialogue 11, 5). Annita showed the square. Considering Annita's actions during the CT we cannot be sure of whether Annita had shown the square because it was a square or because it was closed, as opposed to the other shape.

After interacting with Annita for a while (Dialogue 11, 6-8), the interviewer deconstructed the small square she (the interviewer) had constructed earlier and the construction she had made with the

set of four relatively unequal small sticks, placed the sticks down in front of the child and asked: 'Why did we manage to make a square with these sticks but we didn't manage to make a square with these sticks (points towards the two groups of sticks respectively)? To this question, Annita replied with a conventional answer: 'these are not the same' (Dialogue 11, 9).

Annita is a special case since she failed to construct a square after her third attempt. Because of Annita's 'failure to construct a square, we cannot be sure whether when Annita said 'these are not the same' (Dialogue 11, 9) she was actually answering the interviewer's question ('Why did we manage to make a square with these sticks ....but we didn't manage to make a square with these sticks') or whether at that point she had just realised that the difference between the two sets of sticks the interviewer was showing her was that in one case the sticks were equal and in the other they were not. It might be that it was only towards the end of the discussion (Dialogue 11, 10), where Annita completes the interviewer's sentence ('Because they were the ....') with the word 'same', that Annita actually made a connection between the set of equal sticks and the successful construction of a square. Annita's incapability to choose the set of equal sticks from the set of unequal sticks in line 3 (Dialogue 11) reinforces the conviction that this might indeed be the case.

### **8.3. Children's reflections on the process of constructing squares in relation to the spatial arrangement of sticks**

This section of Chapter 8 describes the findings in relation to the last part of the interview. As was explained in Chapter 5, this part of the interview (RT(B)) concerned the spatial arrangement of sticks for the construction of a square and was completed in four steps. Here are the findings after analysing the data collected from each step.

#### ***The children's responses to the interviewer's question in relation to the spatial arrangement of sticks for the construction of a square***

RT(B-ST1) had to do with the children's reply to the interviewer's following question: '*How must we put these sticks [four equal sticks] in order to make a square?*'<sup>26</sup>. Figure 46 provides the relevant findings. As illustrated in the findings of Graph 17, 32% of the subjects of the study (17/52) constructed something with the use of sticks (CONSTR(Sticks)) to reply to the interviewer's question and 27% (14/52) constructed something with the use of their fingers (CONSTR(Fingers)). Additionally, 15% (8/52) of the children constructed something with the use

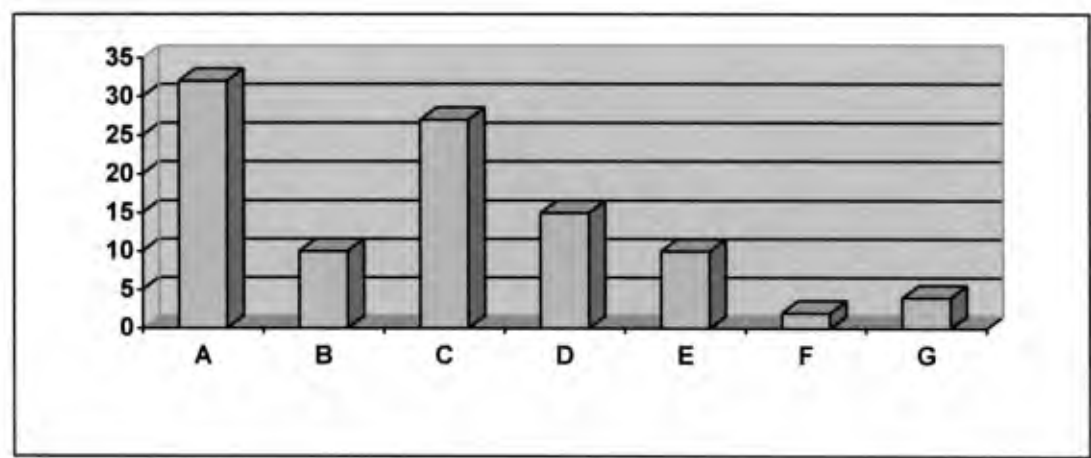
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<sup>26</sup> A description of the coding scheme that was used for the analysis of the RT(B-ST1) is provided in Chapter 5 (Table 17, p.99).



of their fingers and at the same time made some use of manipulatives (CONSTR(Fingers)/MAN). The rest of the categories were much less frequent among the subjects of the study.

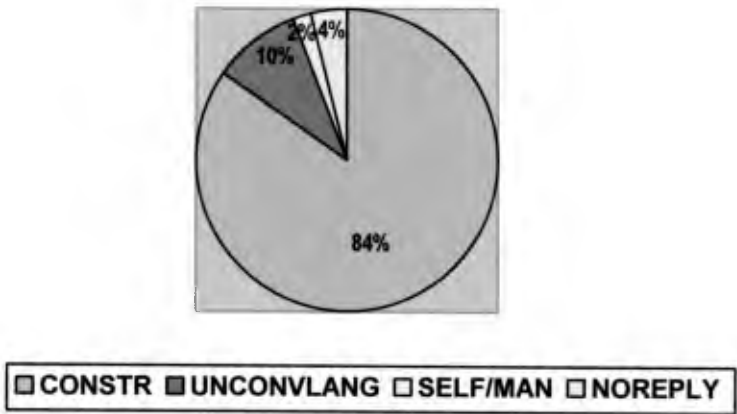
**Graph 17: Results in relation to the categories of the children’s responses to the question: ‘How must we place these sticks [four equal sticks] in order to make a square?’**



All values are given in percentage rates.  
Key to response categories: A=CONSTR(STICKS), B=CONSTR(STICKS)/MAN, C=CONSTR(Fingers), D=CONSTR(Fingers)/MAN, E=UNCONVLANG, F=SELFV/MAN, G=NO REPLY.

It is interesting to note the total percentage of children that used some kind of construction in order to explain how someone must put four equal sticks in order to construct a square. According to Graph 18, 84% of the children (44/52) used some kind of a construction to reply to the interviewer’s question. The high percentage of children that used a construction to reply to the interviewer’s question in this part of the interview was not surprising, since the question incorporated the element of construction.

**Graph 18: Percentage of children who used construction, unconventional language and self-evident responses in their attempt to answer the question: ‘How must we put these sticks [four equal sticks] in order to construct a square?’**



Here are some examples of children's responses which included some kind of a construction:

- Fotini (5,4) took the four equal sticks the interviewer was holding and constructed a square (CONSTR(Sticks)).
- Avra (5,2) 'constructed' a square on the floor with her finger and said '*like this*' (CONSTR(Fingers)).
- Pavlos (5,1) 'constructed' a square with his finger by sliding it around the perimeter of a cutout square (CONSTR(Fingers)/MAN).
- Chara (6,3) said '*like before*' and showed the square she constructed earlier (CONSTR(Sticks)/MAN).

The five responses which were categorised as UNCONVLANG were similar to each other. Demetra (5,7) for example said '*one up, one down, one right and one left*'. The response which was categorised as SELFEV/MAN was Despo's response. Despo (6,1) said '*so that it's like this*' and pointed towards a cutout square.

The examples cited above show that all answers which included some kind of a construction as well as all responses which made use of unconventional language were responses which explicitly referred to the shape's structure. We could therefore say that 94%<sup>27</sup> of the answers reported during the RT(B-ST) were structural responses. If we compare these results with the results of the DT (Graph 3, p.104) we can see that the percentage of structural answers is much higher in the case of RT(B-ST1). In the DT, only 35% of the children gave structural answers.

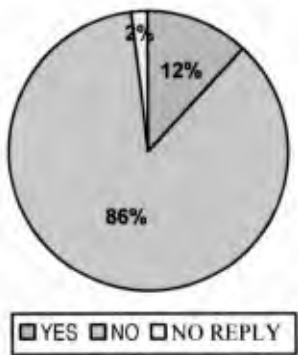
### ***The children's reaction to the construction of a rhombus by the interviewer***

Graph 19 illustrates the findings in relation to the second step of this part of the interview (RT(B-ST2))<sup>28</sup>. RT(B-ST2) had to do with the children's reaction to the rhombus the interviewer constructed with the use of four equal sticks after following the directions of the interview script (Appendix C). After the children responded to the interviewer's question on the spatial arrangement of four equal sticks (RT(B-ST1)), the interviewer would place the four equal sticks and construct a rhombus. Then the children were asked: '*Is this [the rhombus] a square?*' There were six children (12%) that said that the rhombus was a square (Graph 19, YES). On the other hand, 45 children (86%) said that the rhombus was not a square (Graph 19, NO). One child did not reply (Graph 19, NO REPLY).

<sup>27</sup> This percentage is the sum of adding the percentage of CONSTR and UNCONVLANG responses (Graph 18).

<sup>28</sup> A description of the coding scheme that was used for the analysis of the RT(B-ST2) is provided in Chapter 5 (Table 18, p.98).

Graph 19: Results in relation to the children’s response to the question: ‘Is this [a rhombus] a square?’



The six children that said that the rhombus was a square changed their answer immediately after the interviewers followed the interview script (Appendix C) and asked the children to think again. Similarly, the child that did not respond said that the rhombus was not a square after the interviewer proceeded with the scaffolding procedure. It is interesting that whereas none of the children had constructed a rhombus in the CT (a subject we will refer to extensively in the following chapter) there were six children who said that the rhombus was a square and one child who did not reply. One possible interpretation is that for the children there is often an issue in relation to the adult as the authority who is always right. This is more marked in student-teacher situations. It is therefore possible that the children did not want to tell the interviewer-teacher that their construction was wrong or that for a minute they could not accept that the adult’s construction was wrong. By following the interview script (Appendix, p), in these cases the interviewers seemed to manage to overcome this difficulty in the student-teacher relationship and help the children give a more ‘honest’ answer.

***The children’s explanations for why a rhombus is not a square***

Once the children had recognized that the rhombus was not a square, the interviewers proceeded with Step 3 (RT(B-ST3)) and asked them to justify their answer (‘Why [is this not a square]?’). This was the third step of RT(B). Graph 20 illustrates the findings in relation to the children’s responses as a result of their involvement in this third step of RT(B)<sup>29</sup>.

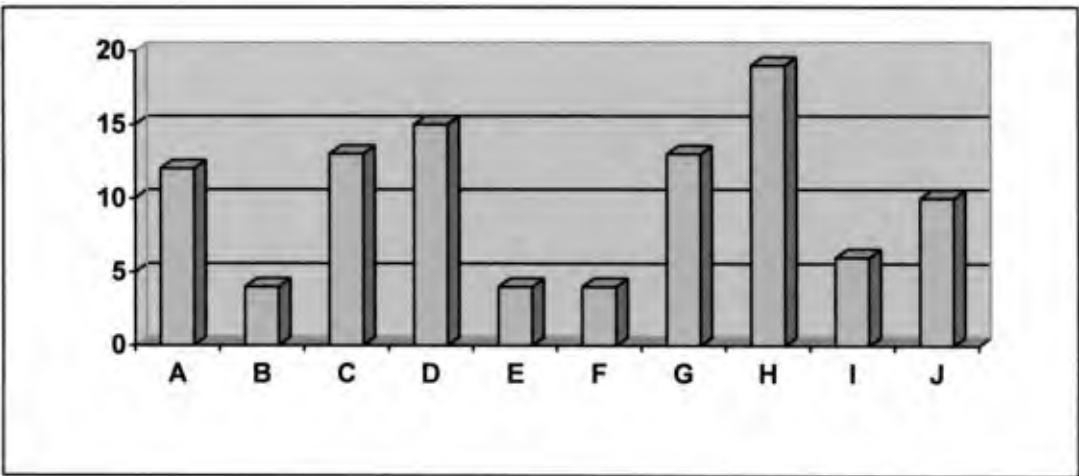
According to Graph 20, 12% of the children (6/52) fixed (FIX) the rhombus and turned it into a square in order to justify why it was not a square. This means that six children changed the

<sup>29</sup> A description of the coding scheme that was used for the analysis of the RT(B-ST3) is provided in Chapter 5 (Table 19, p.99)

position of all of the sticks of the rhombus in order to turn it into a square with a prototypical orientation. In addition, 4% of the subjects of the study (2/52) transformed the rhombus into a square while using unconventional language at the same time (UNCONVLANG/TRANSF). Periklis (5,7) for example said 'it is a bit crooked' and at the same time transformed the rhombus into a square (maintained one of the sticks stable and changed the position of the other three). Therefore, a total of eight children turned the rhombus into a square in order to explain to the interviewer why a rhombus is not a square.

Additionally, as shown in Graph 20, 4% of the children (2/52) constructed something with the use of sticks (CONSTR(Sticks)) and 13% (7/52) constructed something with their fingers (CONSTR(Fingers)) in order to explain why a rhombus is not a square. Marianna (6,2) for example picked up two sticks and constructed a right angle. At the same time, she said to the interviewer: *'Because it has to be like this'*. Marianna's response was categorised as CONSTR(Sticks). Alexis (5,8), on the other hand, said *'because a square is like this (constructs a square with his finger) whereas this (points towards the rhombus the interviewer constructed) is like this (constructs a rhombus with his finger)'*. Alexis' response was categorised as CONSTR(Fingers). Pavlos' (5,1) response was very interesting. He said *'because it is like this'* and 'drew' an acute angle with his finger. Again Pavlos answer was categorised as CONSTR(Fingers).

Graph 20: Results in relation to the categories of the children's responses to the question: 'Why is this [a rhombus] not a square?'



All values are given in percentage rates.  
Key to response categories: A = FIX, B = CONSTR(STICKS), C = CONSTR (Fingers), D = UNCONVLANG, E = UNCONVLANG/TRANSF, F = UNCONVLANG/MAN, G = SELF EV, H = SELF EV/MAN, I = SIM, J = NO REPLY.

In analysing the findings provided by Graph 20, we can identify three categories of answers to the question "Why [is this not a square]?", that involved the use of unconventional language

(UNCONVLANG). To be more precise, 15% of the children (8/52) used unconventional language (UNCONVLANG), 4% (2/52) used unconventional language and at the same time transformed the rhombus the interviewer had constructed into a square (UNCONVLANG/TRANSF) and 4% (2/52) used unconventional language and manipulatives at the same time (UNCONVLANG/MAN). Thus, twelve children used unconventional language in order to explain why a rhombus is not a square.

Fotini (5,4), for example, said *'because in order to make a square you have to place them differently'* (UNCONVLANG). Chrysanthos (5,11), on the other hand, said *'because 1,2,3,4 turns'* and showed the angles of the rhombus the interviewer had constructed respectively (UNCONVLANG/MAN). Theodora (6,3) said: *'this is a bit crooked'* (UNCONVLANG). Similarly, Savvas (5,4) said *'you made it a bit crooked. You have to put them straight'* (UNCONVLANG). Then again Giota (5,3) explained *'It's not a square. You have to put your sticks straight'* (UNCONVLANG). Magdalene (5,0), on the other hand, said *'It's upside down'* (UNCONVLANG). To conclude with the responses which included unconventional language, we had the case of Periklis (5,7) to which we referred earlier. He said *'it is a bit crooked'* and at the same time transformed the rhombus into a square (UNCONVLANG/TRANSF).

There were also two categories of answers to the question 'Why [is this not a square]?' that included self-evident answers (SELFEV). As shown in Graph 20, 13% of the children (7/52) used self-evident answers (SELFEV) whereas 19% (10/52) used self-evident answers and manipulatives at the same time (SELFEV/MAN). Thus, approximately one third of the children involved in the study (32%) used self-evident responses in this part of the interview.

Melina (5,9) for example said: *'Because this is not how a square is like'* whereas Loukas (6,8) said: *'It has a different shape'*. Melina's and Loukas' responses were classified as SELFEV. Aggelos (5,2) on the other hand said: *'Because this [the rhombus] is not a square. This [a square construction] is a square'*. Similarly, Andys (5,0) said: *'because this [the rhombus] is not like this [a square construction]'*. Despo (6,1) on the other hand used a cutout rhombus to support her self-evident answer. *'It is like this'* she said and showed the interviewer a cutout rhombus. Aggelos', Andys' and Despo's answers were classified as SELFEV/MAN.

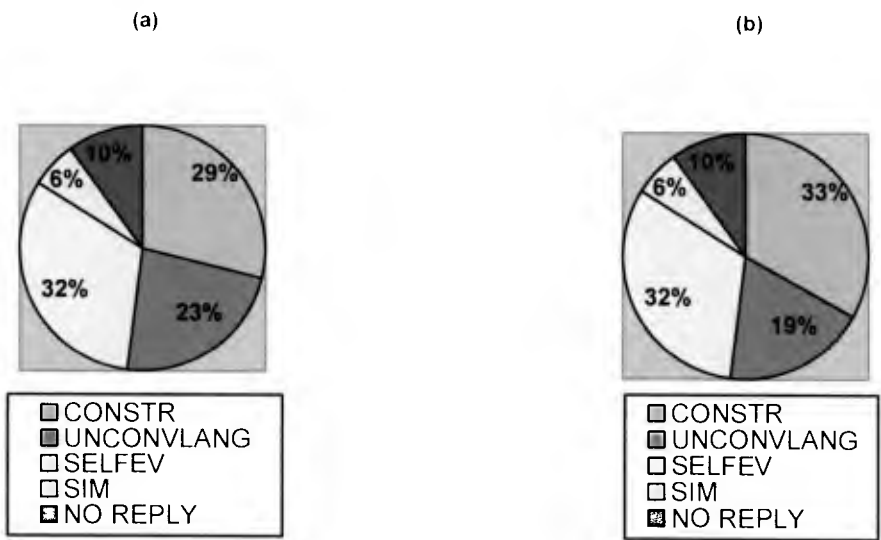
To conclude with the findings illustrated in Graph 20, 6% of the children (3/52) used a simile (SIM) to explain why a rhombus is not a square. These children said that a rhombus is not a

square *'because it's like a kite'*. Ten percent of the children (5/52) did not reply (Graph 20, NO REPLY) to the interviewer's question ('Why [is this not a square]?').

It is interesting to see the percentage of children that used some kind of construction to reply to the interviewer's question in this part of the study and the percentage of children that used unconventional language collectively. This information is provided by the pie charts of Graph 21. The reason for having two pie charts is that the UNCONVLANG/TRANSF responses can fit in both categories. In these cases, the children constructed something and used unconventional language at the same time. So Graph 21a shows the results if we include the UNCONVLANG/TRANSF responses in the UNCONVLANG category whereas Graph 21b shows the results if we include the UNCONVLANG/TRANSF responses in the CONSTR category.

A comparison between the results of RT(B-ST3), as these are presented in Graph 21, with the results of RT(B-ST1), as these were presented in Graph 17 (p.165), makes us realise that there are major differences. Whereas the percentage of answers which included a SELF EV justification in RT(B-ST1) was only 2% (Figure 47(a)) in RT(B-ST3), the corresponding rate is 32% (Graph 21). And whereas none of the children in RT(B-ST1) used a simile, in RT(B-ST3) 6% of the children used a simile (Graph 21).

**Graph 21: Percentage of children who used construction, unconventional language and self evident responses in their attempt to answer the question: 'why is this [a rhombus] not a square?'**



In relation to the findings of RT(B-ST1), we had commented that the high percentage of responses which made use of construction and the fact that even the unconventional responses made explicit reference to the shape's structure was closely related to the fact that the question the children had to respond to (*'How must we put these sticks [a set of four equal sticks] in order to make a square?'*) included the element of construction and thus urged the children towards these types of responses. We therefore see that when the question does not include the element of construction, as was the case of the question involved in RT(B-ST3), the quality of the children's responses was closer to the quality of the children's responses during the DT (Graph 1, p.103). When the children were asked why a rhombus is not a square they made more use of similes and self-evident responses than when they were asked how one must place a set of four equal sticks in order to construct a square.

Nevertheless, the element of construction and the reference to the shape's structure was still much more frequent in the RT(B-ST3) compared to the DT. Some of the cases which were categorised as SELF/MAN included the use of construction. As we have seen earlier, there were cases like Andreas and Alexandros (p.169) which were categorised as SELF<sub>EV</sub>/MAN and made use of already made square constructions to support their self-evident justification.

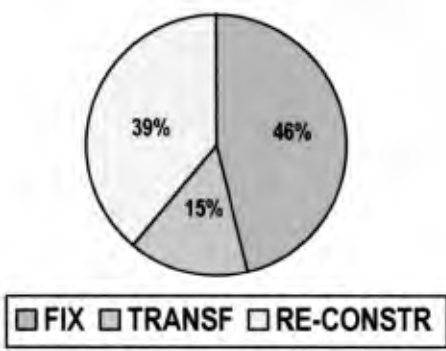
### ***The ways in which the children turned the rhombus into a square***

Moving on to RT(B-ST4), which was the last part of the interview, the results are shown in Graph 22<sup>30</sup>. This part of the interview had to do with the route the children followed in order to fix the rhombus the interviewer had constructed and turn it into a square. The majority of children changed the position of all of the sticks thus turning the rhombus into a square with a prototypical orientation (Graph 22, FIX). Twenty-four children (46%) followed this route. Eight children (15%) transformed the rhombus into a square by maintaining one of the sticks stable and changing the position of the other three sticks (Graph 22, TRANSF). Finally, twenty children deconstructed the rhombus and used the sticks to construct a square all over again (Graph 22, RE-CONTR).

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<sup>30</sup> A description of the coding scheme that was used for the analysis of the RT(B-ST4) is provided in Chapter 5 (Table 20, p.100).

**Graph 22: Results in relation to the ways in which the children transformed the rhombus into a square**



It is important to point out that all of the subjects of the study were able to construct a square at this point of the interview. It is also highly interesting to note that eight children transformed the rhombus into a square by maintaining one of the sticks stable and changing the direction of the other three.

**8.4. Concluding remark**

This chapter provided a description of the ways in which the children reflected on the process of constructing a square and thus a description of the ways the children expressed their understandings about squares after a construction experience. The general picture sketched by the findings was that, even after the CT, the children expressed themselves mainly in unconventional ways. Nevertheless, the CT enabled them to refer to the shape's structure more effectively than the DT where, as was presented in Chapter 6, the children made little reference to the structure of squares. These issues are dealt with in more detail in the next chapter.



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## CHAPTER 9

### DISCUSSION:

**What do children know about squares, how is this knowledge expressed and how is it used in the process of meaning construction**

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#### 9.1. Introduction

Based on the gaps identified in existing research concerning children and shapes (Chapter 2), I designed and described in the previous chapters a study which aimed to investigate and analyse children's structural understandings of shapes through an investigation of squares. After the presentation of the study's findings (Chapters 6-8), it is now time to address the study's research questions (RQ):

1. What do children know about shapes?
2. How is this knowledge expressed? and
3. How is this knowledge used in the process of constructing shapes?

Since the study's research questions were addressed through an investigation of squares, the discussion that follows refers specifically to squares and not to shapes in general. In Chapter 10, this discussion will be generalised from squares to shapes.

#### 9.2. Addressing Research Question 1 (RQ1): What do children know about squares?

When this research question was formulated, the intention was to identify children's existing structural knowledge of squares. As was explained in the review of the literature (Chapter 2), existing research supported widespread agreement on the existence of a visual level of thinking, where children see shapes as a whole and pay no attention to a shape's structure. This viewpoint was supported by findings from studies which examined children's understandings mainly within settings restricted to classification and shape recognition tasks. With the methodology design of this study, the intention was to investigate children's understandings about the structure of shapes through an investigation of the process children follow in order to construct squares.

Thus, the core part of the task sequence that was used for the interviews was the Construction Task (CT).

So the issue here is the following: What knowledge of shapes did the children exhibit through their involvement in the CT? Chapter 7 presented the results in relation to the strategies, products and routes the children followed throughout their attempt to construct a square. These findings were formulated based on the data analysis system that was developed and described in Chapter 5. As explained in Chapter 5, during the process of data analysis emphasis was given to the children's first complete (not necessarily 'successful') attempt to construct a square, where the children would expose and draw on their intuitive knowledge<sup>31</sup> in order to begin their construction. This is why the findings in relation to the children's first attempt to construct a square will allow us to answer the study's first research question: What do children know about squares?

In their first attempt to construct the shape the children followed different strategies that led them to a construction which was sometimes a square and sometimes not. The identification of nine strategies through the data analysis process (Table 8, p.88) was based on the children's foundational action. A foundational action was defined as a specific choice of sticks and/or their spatial arrangement, on which the children based their attempt to construct a square and which remained intact through to the end of their attempt. In Chapter 5, I claimed that a foundational action exhibited an understanding of specific properties of squares; a claim I did not analyse at the time.

Based on this process of data analysis, it is implied that through the strategies the children followed in their first attempt to construct a square, and consequently through the foundational action, one can detect children's existing knowledge of squares. But besides the strategies the children followed, the findings also concern the routes and the products of their attempts. This raises an important question: What should be used as an indicator of children's existing knowledge of squares: the strategy they followed in order to construct the shape, the whole route or the product of their attempt? This discussion will begin with the hypothesis that the strategies can indicate what children know about squares. As this discussion progresses I will try and investigate whether the routes and products of the children's first attempt to construct a square can also tell us something about their existing understandings.

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<sup>31</sup> 'Intuitive knowledge' was defined in Chapter 5 as the fragmented knowledge which children bring with them in a learning situation and are mostly able of expressing in ways which are contrary to what is formally acknowledged as correct.

9.2.1. What knowledge did the children exhibit about the structure of squares through the strategy they followed during their attempt to construct a square?

The simple concept map of 'square' illustrated in Figure 28 will facilitate the attempt of interpreting the strategies in terms of 'understandings'.

Figure 28: Concept map of 'square'.

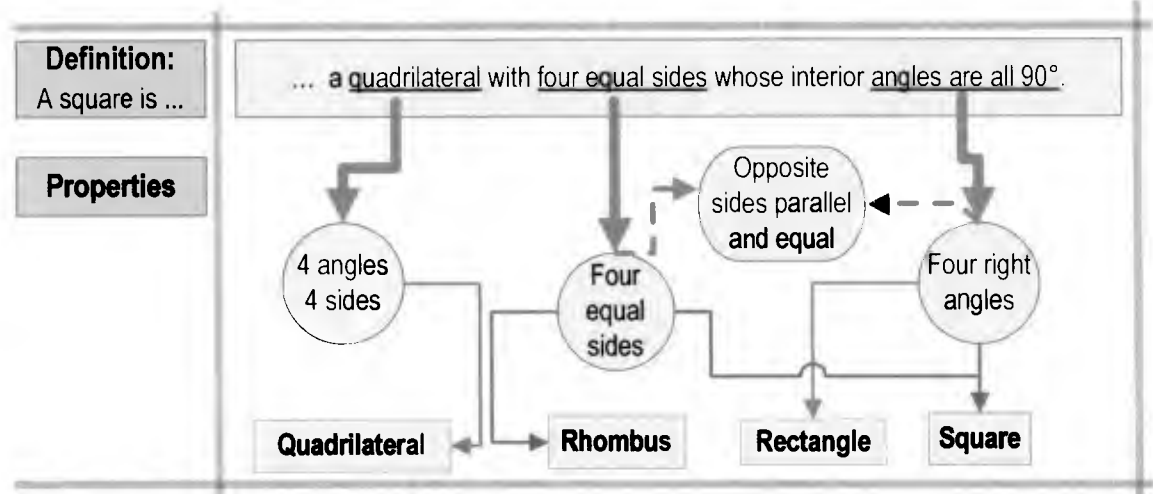


Figure 28 illustrates the properties of a square as these arise from the concept's definition and consequently how the concept square is a sub-category of broader shape categories (quadrilateral, rhombus and rectangle).

As illustrated in Figure 28 (Definition), 'a square is a quadrilateral with four equal sides whose interior angles are all 90°'. So, based on this definition, there are different things one can say about the shape's structure (Figure 28, Properties). First, as a 'quadrilateral', a square has 'four angles' and 'four sides'. Furthermore, a square has 'four equal sides'. This is why a square is also a 'rhombus'. Finally, a square has 'four right angles' and this is why a square is also a 'rectangle'. The combination of 'four equal sides' and 'four right angles' is what distinguishes a square from other quadrilaterals and makes it a special case of a rhombus and a special case of a rectangle. One last thing we can say about a square is that because of the equality of the angles the 'opposite sides are equal and parallel'. This attribute is also connected with the property of four equal sides. Now how can this analysis of the concept 'square' help us see what understandings are hidden behind each of the strategies the children followed during the CT?

An interpretation of each strategy in terms of understandings is provided in Figure 29, where the strategies the children followed during their first attempt to construct a square are connected with the properties of a square, as these were outlined in Figure 28. By following the arrows connecting the third section (Strategies) of Figure 29 with the second section (Properties) one can see how each strategy can be interpreted in terms of understandings. Based on this analysis,

the strategies that were identified during the data analysis process (Table 8, p.88) form four different categories. Each category of strategies is illustrated in Figure 29 with a different colour.

Figure 29: The interpretation of the strategies (S) in terms of ‘understandings’.

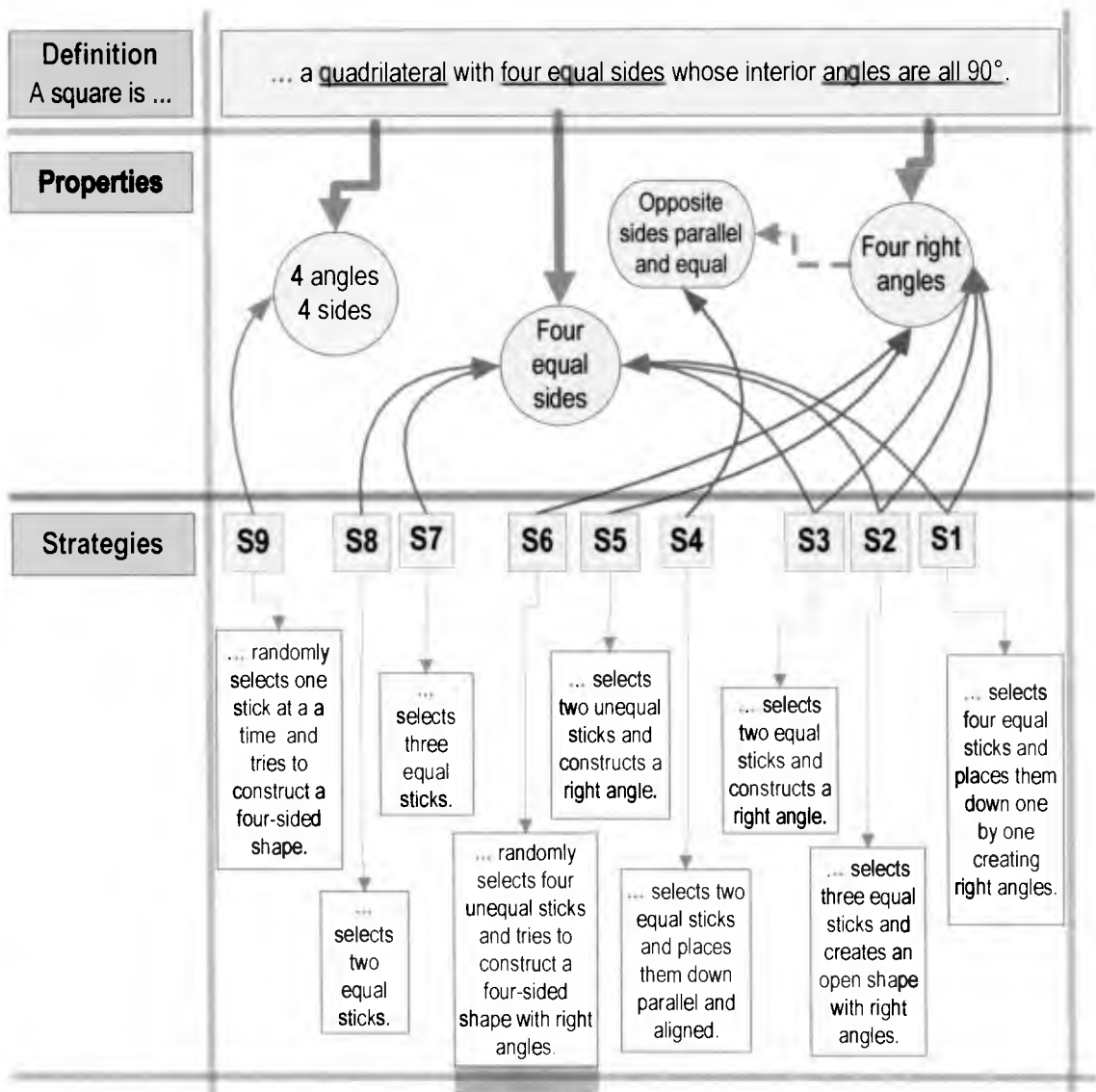


Figure 29 connects the properties of a square with the strategies the children followed in their first attempt to construct a square. The strategies connected to a specific property or a set of properties form a separate category. Each category of strategies is illustrated in the Figure with a different colour.

S1, S2 and S3 form the first category of strategies (Figure 29, red). Based on the interpretations of Figure 29, by following one of these three strategies the children exhibited a combined understanding of the two characteristics which distinguish the square from other quadrilaterals. In other words, by selecting equal sticks and constructing right angles, the children exhibited an understanding of the equality of the sides and the equality of the angles (equal to  $90^\circ$ ). We then have a second category of strategies: S4, S5 and S6 (Figure 29, blue). Children following these

three strategies gave emphasis to constructing a quadrilateral with four right angles, thus exhibiting an understanding of the fact that all the angles of a square are  $90^\circ$ .

Some clarification is required in relation to the interpretations of Figure 29 in terms of S4. Even though in Figure 28 (p.175) the fact that a square has its opposite sides parallel and equal was connected with two properties (four right angles, four equal sides), in Figure 29 this attribute is connected only with the property of four right angles. The children that followed S4 not only placed the two equal sticks parallel but also placed them so that they were aligned. Thus, the connections of Figure 29 imply that, by following this strategy, the children exhibited some understanding of the fact that a square has its opposite sides parallel in combination with the fact that a square has four right angles, independently of the fact that a square has four equal sides.

Moving on, there is S7 and S8 (Figure 29, green). Children following these two strategies began their attempt to construct a square by selecting equal sticks, therefore demonstrating some understanding of the equality of the sides. Finally, there is S9 (Figure 29, orange). Children following this strategy would randomly select sticks and try to construct a quadrilateral. These children exhibited an understanding of the fact that a square has four sides and four angles.

If we are to accept the interpretations of Figure 29 we can claim that, based on the strategy a child followed to construct a square, we can predict the final product of his/her attempt. Predictions based on this claim are illustrated in Figure 30. In this Figure the strategies are connected with some anticipated products based on the interpretations of Figure 29. Consequently, children following S1-3 would be expected to end up with a square at the end of their attempt, children following S4-6 would be expected to end up with a rectangle, children following S7-8 would be expected to end up with a rhombus<sup>32</sup> and finally children following S9 would be expected to end up with an irregular quadrilateral. Have these predictions been verified by the study's findings?

A quick and rough reflection on the findings may lead to three initial comments in relation to the predictions of Figure 30. First of all, none of the subjects of the study constructed a rhombus in their attempt to construct a square. Going back to the products of the children's attempt to construct a square, as these were identified and described in Chapter 5 (Table 9, p.89), one can

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<sup>32</sup> Any reference to a rectangle or a rhombus from this point forward concerns non-square rectangles and non-square rhombuses respectively.

see that the rhombus is not included. Thus, the prediction in relation to the children that followed S7-8 (Figure 30) cannot be confirmed by the findings.

**Figure 30: Prediction of the product of the children’s attempt to construct a square based on the Strategies (S) they followed.**

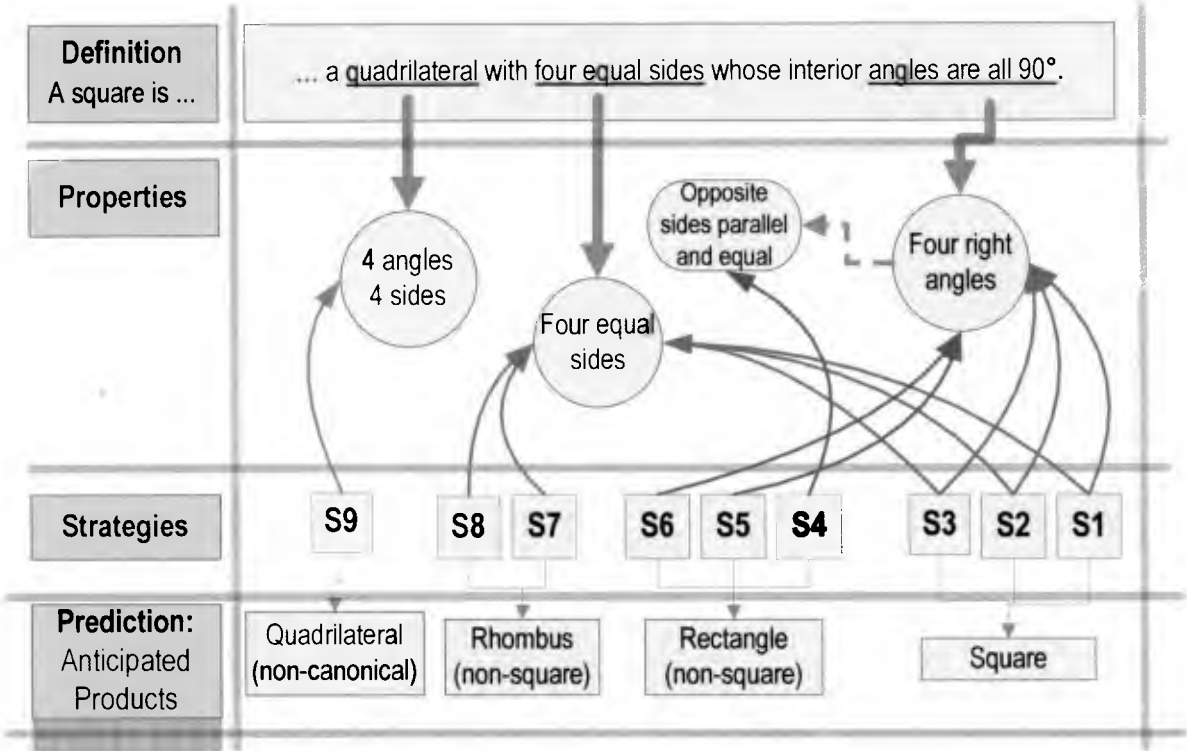


Figure 30 combines the analysis of the concept square (Figures 28) with the analysis of each strategy (Figure 29). Thus, by following the arrows from the strategies upwards one can see what property of the shape a child exhibited by following a specific strategy. Similarly, by following the arrows from each strategy downwards one can predict the product of a child’s attempt based on the strategy each child followed. This figure therefore suggests, for example, that children who followed S1 exhibited a combined understanding of the fact that a square has right angles and equal sides and thus it is expected that these children would have constructed a square at the end of their attempt.

Secondly, since S9 was defined as the strategy where children would select one stick at a time and try to construct a four-sided shape, which somehow resembles a square but has no equal sides and no right angles, one would expect that all of the children that followed this strategy would end up with an irregular quadrilateral. Thus, the prediction illustrated in Figure 30 in relation to S9 is expected to be verified 100%.

Thirdly, there was a certain variation in relation to the products that were identified among the data and which cannot, at first sight, match exactly the simplistic model of Figure 30. But the 11 products identified during the data analysis (Table 9, p.89) were categorised into three groups: Type A, Type B and Type C. These three types of products might allow, in some way, the verification of the predictions of Figure 30.

But let us see exactly to what extent the study's findings can verify or not the predictions and interpretations of Figure 30. What products did the children that followed each category of strategies (as these are described in Figure 29) end up with? In Chapter 7, I presented in detail the products constructed by the children that followed each strategy at the end of their first attempt to make a square (Table 21, p.120). These findings are reproduced in Table 29 in a more collective manner.

**Table 29: Results in relation to the connection between the type of strategy and the type of product of the children's first attempt to construct a square**

CT(A)		S1-3		S4-6		S7-8		S9		Total	
		No	%	No	%	No	%	No	%	No	%
Type A (P1-4)		26	86,7	1	10	5	83,3	-	-	32	61,6
Type B (P5-9)		1	3,3	9	90	-	-	-	-	10	19,2
Type C (P10-11)		3	10	-	-	1	16,7	6	100	10	19,2
<b>Total</b>		<b>30</b>	<b>100</b>	<b>10</b>	<b>100</b>	<b>6</b>	<b>100</b>	<b>6</b>	<b>100</b>	<b>52</b>	<b>100</b>

Table 29 shows, for each category of strategies, which type of products did the majority of the children end up with. Thus, 26 out of the 30 children (86,7%) that followed the first category of strategies (S1-3) ended up with a Type A product. Nine out of the ten children (90%) that followed S4-6 ended up with a Type B product. Five out of the six children that followed S7-8 (83,3%) ended up with a Type A product and, finally, all of the children that followed S9 (100%) ended up with a Type C product.

Figure 31 provides a synoptic picture of the discussion in relation to the interpretation of the strategies followed by the children in terms of their understandings (Figure 29, p.176), the prediction this interpretation led to in relation to the product of the children's attempt (Figure 30) and the verification based on the findings of the study (Table 29). Thus, even though there is no direct relationship between the strategy the children followed and the product of their first attempt to construct a square, in exactly the same simplistic way presented in Figure 30, there seems to be some kind of a consistency between the type of strategy the children followed and the type of product they ended up with. As illustrated in Figure 31 (Verification), a high percentage of children (>80%) that followed the same type of strategy ended up with the same type of product. Each type of product is characterised by a specific set of properties which match the way the strategies were interpreted in Figure 28.

For example, one can see from Figure 30 - by the way the 'Strategies' the children followed are connected with the 'Properties' of a square - that it is suggested that the children who followed

S1-3 exhibited a combined understanding of the fact that a square has four right angles and four equal sides. Based on the information of Figure 31 (Verification), 87% of the children that followed this type of strategies (S1-3) ended up with a Type A product (square with four equal sticks/more than four sticks/a gap/an extension).

Similarly, in the case of S4-6 it was suggested that the children exhibited some understanding of the fact that a square has four right angles (Figure 29). Based on this interpretation, one would expect these children to end up with a rectangle. As one can see from the information of Figure 31 (Verification), 90% of the children that followed S4-6 ended up with a Type B product (rectangle with two sets of equal sticks/more than four sticks/a gap/extensions). As was anticipated from the beginning, the prediction was confirmed in relation to the children that followed S9. All of the children that followed this strategy ended up with a Type C product (irregular quadrilateral). As illustrated in Figure 31, it was suggested that the children that followed this strategy exhibited an understanding of the fact that the square has four sides (and thus four angles).

In relation to the children that followed S7-8, the findings do not verify the prediction of Figure 30. As was explained earlier this was something to be expected since, based on the study's findings, none of the subjects of the study ended up with a rhombus in the CT. It is interesting to note that, as illustrated in Figure 31 (Verification), five out of the six children that followed S7-8 ended up with a Type A product (square with four equal sticks/more than four sticks/gap/extension). This does not contradict the suggestion that the children that followed S7-8 exhibited an understanding of the fact that a square has four equal sides. The type of product the majority of the children that followed this category of strategies ended up with was characterised by the property of the equality of sides. These children exhibited an additional understanding of the fact that a square has four right angles.

As one can see in Figure 31, the consistency between the type of strategy the children followed and the type of product they ended up with relates to 47 out of the 52 subjects of the study (90,4%)<sup>33</sup>. This consistency supports, to a great extent, the broader claim that children exhibited specific structural understandings of squares through the strategy they followed and the more specific interpretations of each strategy as these were illustrated in Figure 29 (p.176). But it also raises interesting issues.

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<sup>33</sup> Number 47 is the sum of the following addition: 7 (children that followed S9 and ended up with a Type C product) + 5 (children that followed S7-8 and ended up with a Type A product) + 9 (children that followed S4-6 and ended up with a Type B product) + 26 (children that followed S1-3 and ended up with a Type A product).



Figure 31: Verification of the predictions illustrated in Figure 30 (p. 179) based on the findings of the study.

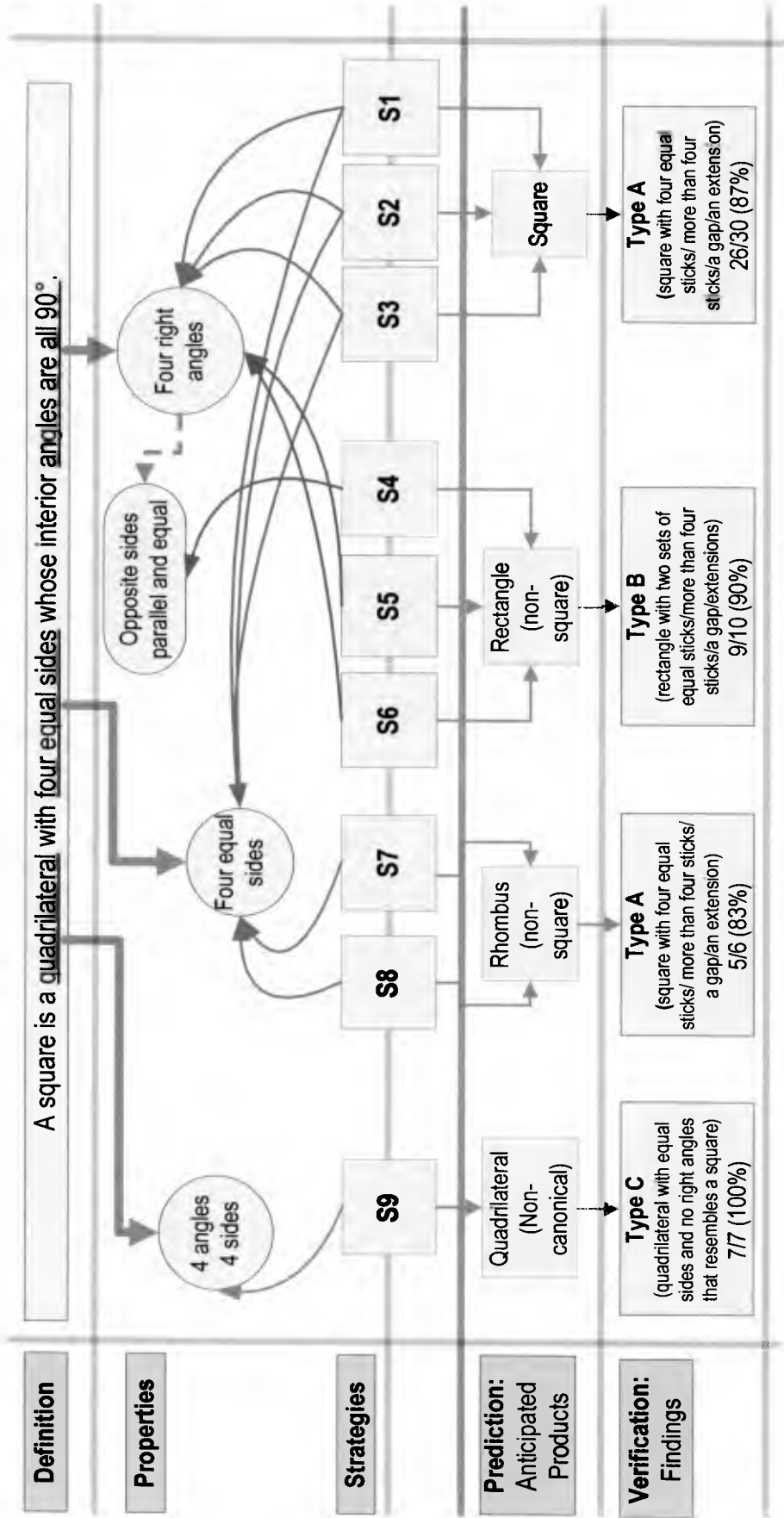


Figure 31 illustrates the findings of the study in relation to the products of the children's first attempt to construct a square in an effort to verify the predictions of Figure 30. Thus, for example, it shows that 26 out of the 30 children (87%) that followed S1, S2 and S3 and were expected to construct a square at the end of their attempt actually constructed a Type A product (square with four equal sticks/more than four sticks/a gap/an extension).

At the beginning of this discussion the following question was raised: What should be used as an indicator of what children know about squares: the strategy they followed, the whole route or the product of their attempt? In this section of the study, I tried to interpret the strategies the children followed in terms of understandings and supported these interpretations by the consistency that existed among the type of strategy the children followed and the type of product they ended up with. But despite the consistency identified between the type of strategy and the type of product, one cannot ignore the inconsistency that existed between strategies and products. Even though the majority of children that followed a specific type of strategy ended up with the same type of products, I cannot ignore the fact that there was no linear connection between a strategy and a product.

For example: the children that exhibited an understanding of the fact that a square has four equal sides and four right angles through the strategy they followed did not necessarily end up with a ('perfect') square at the end of their attempt. Should the children's 'failure' to construct a square at the end of their attempt erase the structural understandings they exhibited through the strategy they followed at the beginning of their attempt? If we go back to the findings, should we ignore for example the fact that children like Christoforos (Figure 11e, p.122) and Magdalene (Figure 12b, p.123) revealed structural understanding of the fact that a square has equal sides and right angles, at the beginning of their attempt, because of their failure to construct a square at the end of their attempt?

In addition, some of the children that exhibited an understanding of the fact that a square has four equal sides and four right angles at the beginning of their attempt and ended up with a square at the end of this attempt had to experiment. Again, does this experimentation imply that we should ignore the understandings these children exhibited at the beginning of their attempt? Even though, for example, Theodora (Figure 12a, p.123) displayed, through the strategy she followed, an understanding of the fact that a square has equal sides and right angles, and ended up with a square at the end of her attempt she had to experiment. Does this experimentation imply that we should ignore the understandings that Theodora demonstrated at the beginning of her attempt?

So here we are, as it shows, faced with a dilemma. Where should we focus on in order to determine children's existing knowledge: On their strategies, which included some 'correct' choices, on their need to experiment or on their products, which were sometimes 'faulty'? Is this really a dilemma that needs to be resolved or is it a finding in itself? This is the issue discussed in the next section of this chapter.

### 9.2.2. 'Correct' strategies or 'faulty' products: Is this a crucial dilemma or (another)<sup>34</sup> pseudo-dilemma?

In Figure 28 it was suggested that by following a specific strategy the children exhibited understandings in relation to specific aspects of the shape's structure. It was therefore suggested, for example, that children following S2 (selected three equal sticks and constructed an open shape with right angles) exhibited a combined understanding of the fact that the square has equal sides and right angles. If the children that followed this strategy, as well as the children that followed S3, 'knew' that a square has four equal sides and four right angles, as was suggested by the interpretations of Figure 28, why didn't they all end up with a square?

If we look carefully at the findings as these were presented in Chapter 7 (Table 21, p.120) we will see that only seven out of the twenty children that followed these strategies (S2-3) ended up with a P1 (square with the use of four equal sticks). Of course, there was also the case of two children that ended up with a P4 (square with the use of more than four sticks). But there were eleven children that constructed a variety of other products. Additionally, some of the seven children that did end up with a square at the end of their attempt had to experiment in different ways during the construction procedure. If they 'knew' that a square has equal sides and right angles, as was suggested by the interpretations of Figure 29, why did they have to experiment? Should we assume that it was wrong to interpret the strategies in terms of understandings the way we did?

On the one hand, it seems that it would be an exaggeration to claim that the children that followed S2<sup>35</sup> and 'failed' to construct a square, or had to experiment in order to construct a square, accidentally selected equal sticks and constructed right angles at the beginning of their attempt. On the other, one cannot ignore the fact that they did 'fail'<sup>36</sup> to construct a square at the end of their attempt or that they had to experiment during the construction procedure. So where should one focus: On the children's 'correct' choices or on their 'faulty' products?

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<sup>34</sup> We faced a similar issue in Chapter 2 (§ 2.6.1.) in relation to the hierarchical nature of the van Hiele model.

<sup>35</sup> Even though this discussion is referring to the children that followed S2-3, the same discussion is true for the children that followed other types of strategies as well. For example, based on the interpretations of Figure 52, it was suggested that children that followed S4-6 'knew' that a square has right angles' or (and thus) that the opposite sides are parallel and equal. So why didn't all of these children end up with a rectangle? And why did the children that did end up with a rectangle had to experiment?

<sup>36</sup> The word fail is used with caution. Seven out of these children constructed a square with a gap or an extension (P2-3) at the end of their attempt. Can these products be considered as a failure? We will come back to this question later on in §9.2.4.

In Chapter 2, after a critical review of existing literature, I concluded that substantial pieces of research concerning geometric thinking exhibit certain methodological weaknesses. One of the weaknesses identified had to do with the emphasis that was placed on children's 'wrong' responses and, consequently, on the elimination of children's important, rich, intuitive understandings. This weakness was identified as a result of the effort of existing studies to try and fit children into level systems. Thus, I consider that if I choose to focus on the children's 'faulty' products I face the risk of making the same 'mistakes' as the ones identified within existing research. On the other hand, I cannot ignore the 'difficulties' the children exhibited and focus solely on their correct choices. So it is not a matter of choosing between the two. If I do that, I feel that I will end up ignoring important aspects of children's understandings. So, we need to look towards another direction.

It is not a matter of choosing to focus on what children did correctly or incorrectly, but investigating what explanation might exist for this contradictory phenomenon which has been identified. At a first glance, it seems that during their attempt to construct the shape, the children were moving backwards and forwards between 'knowing' and 'not knowing'. Is this a rational explanation?

Based on the information of Figure 31 (p.181), I concluded that the children exhibited specific understandings of squares through the strategies they followed. This claim should, under different circumstances, lead to a more specific and explicit answer to the original questions of this discussion: 'What existing knowledge did the children exhibit through the strategies they followed?' But because of the contradiction identified between the strategies the children followed and the routes and products of their attempts, I feel that I cannot move towards this direction before I address another important issue: I cannot claim that children 'knew' specific things before I clarify what 'knew' means. In other words I cannot identify children's existing knowledge before I describe existing knowledge. Thus, even though this diversion from the study's data seems rather unconventional, I feel there is a need to talk in more detail about the nature of intuitive knowledge.

### **9.2.3. What can the findings tell us about the nature of intuitive knowledge?**

In the previous discussion, important issues were raised which pinpoint the contradiction that seems to exist among the knowledge the children exhibited through the strategy they followed and the routes and products of their attempts. This discussion keeps coming back to the variation that was

observed among the children in relation to their attempt to construct a square in the CT. This variation was obvious and emphasised in Chapter 7. We saw not only that the children that followed the same strategies ended up with different products, but also that children that followed the same strategies and ended up with exactly the same products followed different construction routes.

The 'contradiction' identified combined with the variation that characterised the children's attempt to construct a square leads us towards an important realisation. During the attempt to construct a square the children did not think in conventional ways. The word conventional is used here to refer to thinking that is broadly acceptable and recognisable.

For example to think that ...

*... a square has four equal sides and four right angles, thus I will select four equal sticks, place them in such a way as to construct right angles and thus construct a square*

in words, is a way of thinking in the conventional sense.

This is a widely acceptable and recognisable way of thinking. If there was evidence among this study's data that children did think in such ways no one could deny that these children were thinking and it would be easy to identify exactly what they knew. But it is clear from the data that, in most cases, the children did not think in such conventional ways.

Nevertheless, no one can deny that they made some right choices and were thinking in ways that allowed them to construct different shapes that had the same attributes as a square. In fact, 21 out of the 52 subjects of the study (40%) knew enough and developed their thinking in a way that allowed them to construct a 'perfect' square (with the use of four equal sticks) from their first attempt (Graph 8, p.116). Additionally, there were three children that also constructed a square but used, for one of the sides, more than one sticks (P4) (Graph 12, p.119). A total of 32 out of the 52 subjects (62%) knew enough and developed their thinking in a way that allowed them to construct a Type A product - a square with four equal sticks/more than four sticks/gap/extension (Graph 11, p.119). Furthermore, all of the remaining subjects of the study constructed shapes which shared some of the properties of a square.

One can therefore not deny that there were indications that the children 'knew' specific aspects of a square's structure which they used in an effort to construct the shape. But, at the same time, we cannot ignore the fact that this knowledge did not lead directly to the construction of the anticipated shapes. Likewise, we cannot ignore the variation that existed (a) among the children that followed the same strategy and (b) among the children that followed the same strategy and ended up with the same product. A quick look at the construction routes of the children's first attempt to construct a square, as these were illustrated in Chapter 7 (Figures 11-19, p.122-130), provides a representative picture of this variation.

The findings can support the point of view that through the strategies they followed - the choices of sticks and their spatial arrangement - the children exhibited a structural understanding of squares. This knowledge was, to a great extent, consistent with the product of their attempt on the one hand but, on the other, it did not always lead to, or directly to, the anticipated products. But the knowledge that the children exhibited through most<sup>37</sup> of the strategies they followed does not match what is traditionally recognised as knowledge. So what is the nature of this knowledge that the children exhibited through the strategies they followed and on which they developed their thinking?

The knowledge the subjects of this study exhibited about the structure of a square during the CT reminds us a lot of the way diSessa (1988, 2000) defines intuitive physics. This was briefly referred to in Chapter 2. Furthermore in Chapter 5, I borrowed some of the main characteristics which diSessa (2000) attributes to intuitive knowledge and defined intuition as the sometimes fragmented knowledge which children bring with them in a learning situation and are mostly capable of expressing in ways which are contrary to what is formally acknowledged as correct. In general, diSessa (1988) supports the existence of hidden knowledge, of knowledge 'we aren't used to recognising' and describes intuitive physics as 'a fragmented collection of ideas, loosely connected and reinforcing, having none of the commitment or systematicity that one attributes to theories' (p.50). DiSessa (2000) provides a detailed description of these fragmented ideas which he calls p-prims (phenomenological primitives)

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<sup>37</sup> One may claim that this discussion does not concern the children that followed S1, since they were children that successfully constructed a square without the need to experiment by following a very clear and linear procedure. Nevertheless, as was described in Chapter 7 there was a certain variation among these children (Table 22, p.121). Even though this variation was very distinct it raises interesting issues in relation to the nature of the children's understandings. For example, is there a difference in the understandings of children who needed to physically compare sticks from those that did not? Did the children that did not select the four equal sticks in advance 'knew' from the beginning that they would need four equal sticks or is it something that they realised parallel to constructing the square? In the same way that there is no evidence that these children were not thinking in words there is no evidence that they were indeed thinking in words. This issue will be addressed again in Chapter 10 (§10.4).

and which constitute intuition, by outlining and explaining their properties and characteristics. In the following paragraphs, I will investigate whether the properties which diSessa attributes to intuitive knowledge (p-prims) can be attributed to the knowledge the children exhibited during their attempt to construct a square.

The first property which diSessa (2000) attributes to intuitive knowledge is that it constitutes 'little' pieces, fragments of knowledge. In Figure 29 (p.176) each strategy was analysed in terms of understandings and was connected with specific aspects of the shape's structure. Thus, through the strategies, the children exhibited an understanding not of the square as a whole but as a shape which has specific structural characteristics. The children were in a position to recognise and isolate specific aspects of a square's structure and base their attempt to construct a square on that recognition.

Furthermore, diSessa (1998) distinguishes intuitive knowledge from theories because of the lack of systematicity and commitment. This brings us back to the variation that was identified among the construction routes the children followed. We saw how children that followed the same strategy ended up with different products and how children that followed the same strategies and ended up with the same product followed different construction routes. This allows me, as diSessa (2000) did, to distinguish intuitive thinking from logical thinking.

DiSessa (2000) attaches great importance to the relationship between intuitive knowledge and words. He strongly supports the point of view that it is very difficult to express intuitive knowledge into words and thus, that intuitive knowledge is 'problematically related to language'. On the contrary, intuitive knowledge is 'encoded visually and kinesthetically' (p.96). This property of intuitive knowledge was highly emphasised in the data of this study. Even though the children were, to a great extent, incapable of describing the structure of a square in words, they were quite capable of constructing the shape or other shapes which shared some of the properties of a square. This incapability of expressing their understandings conventionally in words was not only apparent in the DT which preceded the CT but also in the RT which followed the construction experience. Of course, the children's incapability during the RT had to do with their ability to use conventional language and not their ability to refer to the shape's structure. As we saw from the examples of dialogues between the interviewer and the children (Chapter 8) they showed an amazing ability of using inventive, original, constructive ways of communicating and explaining their understandings.

The use of conventional language was limited during the DT. Only 5% of the responses collected during the DT made use of conventional language (Graph 4, p.105) and none of these responses referred to the fact that a square has right or 90° angles. Furthermore, as far as the RT is concerned, only in the case of the choice of four equal sticks, which was the subject of the first part of the RT, were the children capable of using conventional language. Also, in many cases, the use of conventional language was the result of a long and laborious procedure. See for example the case of Anna who had followed S1 and successfully constructed a square in her first attempt without the need to experiment (Chapter 7) but found it very difficult to give a conventional answer in relation to the choice of four equal sticks in the RT (Chapter 8).

In relation to the spatial arrangement of sticks, which was the subject of the second part of the RT, the subjects did not use conventional language at all. This is interesting if we consider the fact that 81% of the subjects of the study (42/52) constructed a shape with right angles in their first attempt to construct a square (Graph 11, p.119).

In comparing intuitive knowledge with logic, diSessa (2000) also expresses the point of view that whereas intuition is 'rich', 'flexible' and 'diverse', logic is 'sparse', 'stiff' and 'homogeneous'. Thus, whereas logic is 'closed', intuition is 'generative'. Whereas you cannot easily change logic in light of a new situation, with intuitive knowledge one can wander and change gradually in the light of experiencing new situations. How can this apply to this study's findings? We have watched the children experiment and try twice and sometimes three times to construct a square during the CT. Based on their intuitions, the children followed specific strategies but were able to adapt, experiment and change in order to end up with a square. So there were cases of children who went from a shape with its opposite sides equal (two smaller sides and two bigger sides) in their first attempt to a shape with four equal sides in their second attempt, or from a shape with no right angles and no equal sides in their first attempt to a shape with four right angles and its opposite sides equal (two smaller sides and two bigger sides) in their second attempt, and finally to a shape with right angles and equal sides in their third attempt. Many such routes are illustrated in Chapter 7 (Table 28, p.145).

On the other hand, according to diSessa (2000), intuitive knowledge can also be rather unstable. This property of intuition reminds us of many of the construction routes the children followed, as described in Chapter 7. First of all we had the case of Andys. Andys was one of the children that had to try three times before successfully constructing a square in the CT. In his first attempt, Andys had



constructed a rectangle with the use of two sets of equal sticks (Figure 15c, p.126). After being encouraged by the interviewer to try again, Andys selected two equal sticks and placed them parallel to each other. As was commented in Chapter 7, 'even though the distance between the two parallel sides was a sign that Andys was 'doing well' after experimenting with different sticks, he picked a third stick which was longer than the other two and placed it as shown in Figure 21b (p.136). Instead of searching for a stick that would fit, Andys adjusted the distance between the two sticks he had used at the beginning so that the new stick would fit. He then found a fourth stick that was equal to the third stick he had used and closed the shape.'

The cases of Christoforos (Figure 11e, p.122), Magdalene (Figure 12b, p.123) and Anna (Figure 18, p.129) were also cases of children who began their constructions by making all the right choices (chose sticks that were equal and constructed right angles) but ended up with shapes that were not squares. This attribute of unstableness is perhaps the reason for this sense that the children were moving backwards and forwards between 'knowing' and 'not knowing' to which I referred earlier.

One last attribute that diSessa (2000) assigns to intuitive knowledge is that it is frequently effective and sometimes even correct. This does not contradict the high percentage of children that effectively constructed a square from their first attempt.

This evidence can therefore support the point of view that through their involvement in the CT, the children exhibited a rich intuitive structural understanding of squares, and reinforce the definition of intuition as the **fragmented knowledge which children bring with them in a learning situation and are mostly able of expressing in ways which are contrary to what is formally acknowledged as correct**. It is important to reflect for a minute on the phrase 'bring with them in a learning situation' included in the aforementioned definition. From where do children acquire this knowledge which they bring in a learning situation?

One general, obvious and widely supported point of view is that children acquire intuitive knowledge through past experiences. But it is important to be a little bit more specific in relation to how children acquire intuitive knowledge of shapes through past experiences (which, in young children, are mostly shape recognition experiences<sup>38</sup>). This question may be answered indirectly based on the nature of

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<sup>38</sup> One example of a shape recognition experience is the DT we included in the task sequence before the CT. In the DT the children were asked to observe a set of shapes, recognise the square and distinguish squares from other shapes. As

the intuitive knowledge that was identified within the findings. It seems that children acquire visual images from shape recognition tasks. The question is what is the nature of these visual images?

One possible answer is that it is a holistic picture of the shape. This is the answer given by most existing studies which support the point of view that at first, children see a shape as a whole and pay no attention to its properties. But the findings in relation to the children's involvement in the CT (Chapter 7) showed that children do pay attention to shape properties. In fact, the findings of this study can support the idea that children '**abstract**' from the concept square specific aspects of its structure. This abstraction is part of visual recognition. When recognising a shape, or when involved in a shape recognition experience, children are actually involved in an abstracting procedure where they isolate specific aspects of a shape's structure. Based on the strategy the children followed during their first attempt to construct a square, the children were exhibiting specific abstracted aspects of the shape's structure. So children abstract specific aspects of a shape's structure based on previous learning experiences through which they absorb visual messages of what a square is. We will come back to this when we try to answer the study's third research question (how do children use existing knowledge in the process of meaning construction?). But now we have to return to the first research question (What do children know of squares?)

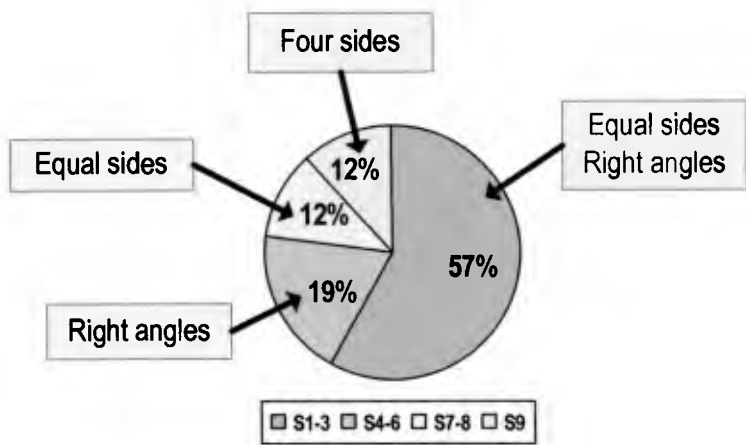
#### **9.2.4. So, what do children know of squares?**

I concluded in the previous section that children acquire intuitive structural knowledge of shapes mainly through past experiences of shape recognition and that this intuitive knowledge concerns specific aspects of the shape's structure. If we go back to Figure 28 we can track the intuitive knowledge on which the children based their attempt to construct a square by following each strategy. Based on this analysis, the strategies were categorised into four groups. In Chapter 7 (Graph 9 p.117), we saw the results in relation to each strategy separately. In Graph 23 we have the results in relation to each category of strategies separately. Each category of strategies is connected with the corresponding properties of a square based on the analysis of Figure 28.

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one can see in the interview script (Appendix C) followed the interviewers would show the subjects a square and name it in those cases where the children had some difficulty recognising the square among other shapes.

**Graph 23: Percentage of children who followed each type of strategy during their first attempt to construct a square and corresponding structural knowledge exhibited**



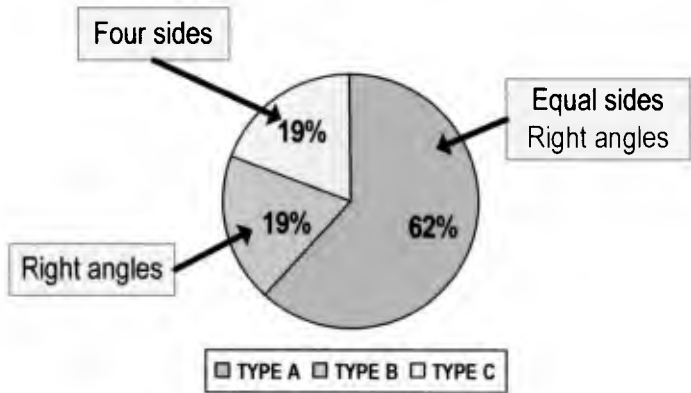
By combining the information of Graph 23 with the analysis of Figure 28 (p.175) one can see that based on the strategy followed, all of the children exhibited some intuitive structural understanding of squares through the CT. The majority of children showed an understanding of the two properties which, when combined, distinguish the square from other quadrilaterals. To be more precise, 57% of the children (30/52) exhibited a combined understanding of the fact that a square has four equal sides and four right angles. The percentage of children who exhibited an understanding of only one of these two characteristics is much lower - 19% (10/52) exhibited an understanding of the fact that a square has right angles and 12% (6/52) exhibited an understanding of the fact that a square has four equal sides. Finally, 12% (6/52) of the children simply exhibited an understanding of the fact that a square has four sides through the strategy followed.

In the previous section I concluded that intuitive knowledge is rich, flexible and thus generative, on the one hand, and sometimes unstable on the other. This description of intuitive knowledge explains why the original intuitions the children exhibited through the strategy followed did not have a linear static relationship with the final product of the construction attempt. Thus, the intuitive knowledge expressed through the strategy followed was transformed during the construction procedure. As a result, in many cases the intuitive knowledge expressed through the product of the attempt was somehow different to the intuitive knowledge expressed through the strategy followed at the beginning of the attempt. This proves that children were in a process of meaning construction during the CT. At the end of the previous section of this chapter I supported the point of view that, through shape experiences, children are involved in an abstracting procedure. Similarly, we can establish

that, through their involvement in the CT, the children were involved in a process of mathematical abstraction and support Noss & Hoyles' (2006) definition of mathematical abstraction as the process where the learner 'builds upon layers of intuition'. I will address this subject again in §9.3 but it is important, at this point, to see what intuitive knowledge the children exhibited through the product of their attempt.

Graph 24 reproduces the results in relation to the products of the children's first attempt to construct a square as these were presented in Chapter 7 (Graph 11, p.119). Each type of product is connected, in Graph 24, with specific properties of a square. If we compare these results with the results in relation to the strategies the children followed (Graph 23) there seems to be, at a first look, some connection. But we must have in mind that it is only the percentage rates that are similar and not the children. The children who represent the 57% that followed S1-3 (Graph 23) are not exactly the same children that represent the 62% who ended up with a Type A product (Graph 24). Similarly, the 19% that followed S4-6 (Graph 23) are not exactly the same children that represent the 19% who ended up with a Type B product (Graph 24). And whereas there were four types of strategies there are only three types of products.

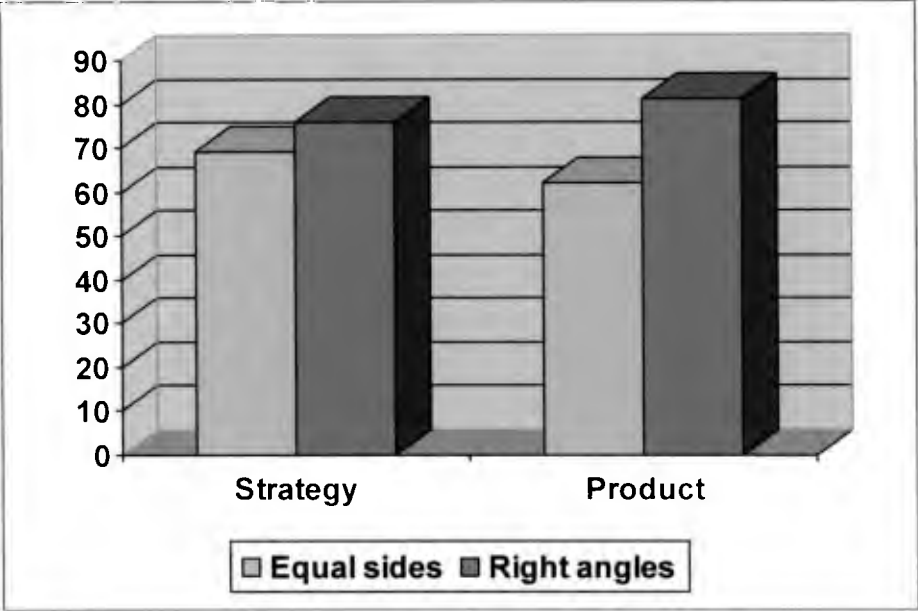
**Graph 24: Percentage of children who ended up with each type of product in their first attempt to construct a square and corresponding structural knowledge exhibited**



What is worth noting at this point is the difference which seems to exist in relation to the understanding of equal sides and the understanding of right angles if we compare the results in relation to the strategies (Graph 23, p.191) with the results in relation to the products (Graph 24). So as one can see in Graph 25, whereas through the choice of strategy 69% of the children exhibited an understanding of the fact that a square has equal sides and 76% exhibited an understanding of the

fact that a square has right angles, through the choice of product, 62% of the children exhibited an understanding of the fact that a square has equal sides and 81% of the children showed an understanding of the fact that a square has right angles.

**Graph 25: Comparison between the results in relation to the knowledge exhibited through the strategy followed and the results in relation to the knowledge exhibited by the product of the children's first attempt to construct a square**



All values are given in percentage rates.

In both cases (strategies and products), the percentage of children that exhibited an understanding of the fact that a square has right angles is higher than that of children that exhibited an understanding of the fact that a square has equal sides. But what is more important is that whereas the percentage of children which exhibited an understanding of the fact that a square has four equal sides through the strategy followed was higher (69%) than the corresponding percentage of children which exhibited an understanding of the fact that a square has four equal sides through the product constructed (62%), the situation is reversed in the case of the percentage of the children which exhibited an understanding of the fact that a square has right angles. The percentage of children which exhibited an understanding of the fact that a square has right angles through the strategy followed was lower (76%) than the corresponding percentage of children which exhibited an understanding of the fact that a square has right angles through the product constructed (81%). Therefore, through the construction route, some children realised that a square has right angles whereas some children 'failed' to stick to their original intuition that a square has equal sides.

Furthermore, if we go back to Figure 31 (p.181) 83% of the children that followed S7-8 and thus simply exhibited an understanding of the fact that a square has equal sides finally constructed a Type A product in the end of their attempt, thus exhibiting an enriched intuitive understanding of the fact that a square combines two characteristics: equal sides and right angles. Based on this evidence, we can support the point of view that the intuitive understanding of the fact that a square has right angles is much stronger and frequent in young children than the understanding of the fact that a square has four equal sides. This is probably why whereas we had 19% of the children constructing a Type B product (rectangle with two sets of equal sticks/more than four sticks/gaps/extensions) we had no children constructing rhombuses (with or without gaps and extensions).

One might argue that it is discussable whether the children who constructed a square with gaps, extensions or the use of more than four sticks 'knew' that a square has four equal sides and thus there is a need for four equal sticks for its construction. But the reason why some children were satisfied with a square with some 'flaw' must have a different interpretation than to say that the children did not know that a square has four equal sides. This can be supported by the way the majority of these children transformed their original construction in their second attempt to construct a square. If we go back to Chapter 7, we will see that eight out of the children that had to try for a second time successfully constructed a square by 'Fix'ing it (Table 23, p.131). This means that eight out of the eleven children that constructed a square with a 'flaw' (P2-4, Figure19) in their first attempt successfully constructed a square in their second attempt with a simple move; by correcting the mistake. Examples of such procedures are provided in Figure 20 (p.134). This fixing procedure shows that it was just a matter of realising the accuracy which sometimes characterises mathematical knowledge. Furthermore, if we look at the results in relation to the products of the children's second attempt to construct a square (Table 24, p.132) we can see that all eleven children that had constructed a Type A product in their first attempt (CT(A)), constructed a 'perfect' square (P1) in their second attempt (CT(B)). It is as if in their first attempt, the children looked beyond and overlooked these flaws as though they didn't exist.

### **9.3. Addressing Research Question 2 (RQ2): How is this knowledge (children's knowledge about squares) expressed?**

The second research aim of the study had to do with the ways in which children express their understandings of squares. In the review of the literature I criticised the tendency among existing research to assess children's understandings through their verbal utterances. As a result, children's important, rich intuitive knowledge of squares has been eliminated. In the previous section, after defining intuitive knowledge, I uncovered children's rich intuitions through their attempt to construct a square. We have therefore seen how the children expressed their understandings of squares through construction. Furthermore, in an attempt to define intuitive knowledge based on the study's findings, we saw that the ways in which this knowledge is expressed is part of its nature. We supported intuitive knowledge as the knowledge which is mostly expressed in ways 'which are contrary to what is normally acknowledged as correct'. So this question concerning the ways in which children express their understandings has, to some extent, been addressed. In addition, this study's research design allows the comparison between the ways in which children express shape understanding through construction and after a construction experience and the ways in which they express shape understanding within a setting restricted to traditional means of expression.

Before the CT, the subjects were interviewed through the DT, to assess their ability to recognise and describe a set of squares. The setting of the DT was restricted to a simple recognition and classification task and some open-ended questions, where the children were expected to respond verbally<sup>39</sup>. The findings of the children's involvement in the DT will enable the evaluation of the findings of the CT in comparison to those of existing research. As we saw in Chapter 6, the children in the DT showed a limited structural understanding of squares. Only 18 out of the 52 subjects, (34%) of the study, responded in ways that exposed some structural understanding (Graph 7, p.114). The majority of responses that were reported during the DT did not implicitly indicate structural understanding of squares. The majority of non-structural<sup>40</sup> responses were 'NO' responses (Graph 1, p.103). There were, in fact, four children that did not reply to any of the interviewer's question during the DT (Graph 7, p.114). Thus, the picture we draw in relation to children's understandings is highly connected with the setting in which the 'evaluation' act takes place. The picture of children's

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<sup>39</sup> Even though the DT was restricted to verbal means of expression, we saw in the findings (Chapter 6) that some children used construction and manipulatives to respond to the interviewer's open-ended questions.

<sup>40</sup> With the characterisation 'non-structural' we are not referring to responses which indicate that children had no structural awareness but that they responded in a way that did not refer to the shape's structure.

understandings that was sketched through their involvement in the DT was dramatically different to the picture sketched through the children's involvement in the CT.

In terms of the RT, where the children were given the opportunity to justify their choices during their attempt to construct a square and reflect on the construction process of the CT, the findings turned out to be quite interesting. A careful examination of the findings, as these were described in Chapter 8, leads us to the realisation that the ways children expressed themselves about squares did not change much after the construction activity. It was the context that changed.

As in the case of the DT, the percentage of children who answered in a conventional manner was low in the RT. In the DT, only 5% of the children responded with the use of conventional language to the interviewer's question (Graph 1, p.103). Similarly, in the first part of the RT only 23% of the children replied in a conventional manner to the interviewer's opening question – 'Why did you take these sticks in order to make your square?' (Graph 16, p.150). Even though this percentage is quite higher than the corresponding percentage of children that used conventional language during the DT, it is amazingly low if we bear in mind that all of these children (with the exception of two) had just constructed a square with the use of four equal sticks. In fact, 40% of these children had constructed a square quite easily from their first attempt (Graph 8, p.116). As we saw from the examples of dialogues in Chapter 8, even children that had constructed a square from their first attempt by following S1 (made all the right choices with no experimentation required) found it difficult to give a conventional answer during the RT. One such case was the case of Anna (Dialogue 8, p.159).

In the second part of the RT none of the children used conventional language. On the contrary, the categories of answers identified were quite similar to the categories identified during the DT. The children used alternative means of expression like construction, unconventional language, manipulatives, self-evident responses and similes to respond to the interviewers' questions (Graph 17, Graph 20) in relation to the spatial arrangement of sticks for the construction of a square.

We cannot, therefore, claim that through the construction experience the children transformed their intuitive knowledge into more conventional ways of thinking (e.g. logical thinking, verbal thinking). They were still mainly thinking in unconventional ways and their knowledge was still mostly intuitive. Nevertheless, there was a major difference between the context of the children's responses during the DT and during the RT. Whereas in the DT reference to the shape's structure was limited (Graph



2, p.103) in the case of the RT all dialogues and the children's responses evolved around the structure of the shape. The children's involvement in the CT equipped them with ways of expressing themselves in relation to the shape's structure. The children were not using conventional but constructional language. They were using the construction as a tool to express themselves about the shape's structure.

Furthermore, in the dialogues of the first part of the RT we saw the children exhibiting an amazing ability to communicate. They were able to invent and find many ways (verbal and non-verbal) to explain their understandings in order to assist the adult to understand.

#### **9.4. Addressing Research Question 3 (RQ3): How is existing knowledge used in the process of constructing squares?**

I supported earlier the point of view that children 'abstract' from the concept 'square' specific aspects of its structure: 'When recognising a shape, children are actually involved in an abstracting procedure where they isolate specific aspects of a shape's structure'. By following the children's route throughout their attempt to construct a square one can see that this abstracting procedure is continuous and evolving. We already saw how the children's intuitions, as these were expressed through the products of their first attempt to construct a square, were in most cases different to the intuitions expressed through the strategies the children followed at the beginning of their attempt. If we take a closer look at the children that had to try twice and even three times to construct a square, we can reinforce the idea that this abstraction procedure is continuous and see how it evolves.

If we go back to the construction routes the children followed during their attempt(s) to construct a square we can sketch the paths the children followed and thus the ways in which they used existing knowledge in the process of meaning construction. The children began their attempt to construct a square by exposing specific intuitions (through the strategy they followed). Through their attempt to construct a square, these intuitions evolved, were transformed and enriched. Based on the ways the children acted throughout the course of the CT and the ways in which they expressed their understandings in the RT, one can see that the children's knowledge did not change through the procedure from intuitive knowledge into conventional knowledge. Thus, based on the study's findings, I can support the definition of '**mathematical abstraction as a process which builds upon layers of intuitions and meanings**' (Noss & Hoyles, 1996).

The most insightful data in relation to how children were involved in a process of mathematical abstraction was the data from the children's second attempt to construct a square. In Chapter 7 (§6.2.2), I described the routes that children followed in order to pass from a Type B product -quadrilateral characterised by right angles- to a Type A product -quadrilateral characterised by right angles and equal sides- (Figure 21, p.136), from a Type C product -characterised by four sides- to a Type B product -quadrilateral characterised by right angles- (Figure 25, p.140) or from a Type C product -characterised by four sides- to a Type A product -quadrilateral characterised by right angles and equal sides- (Figure 24, p.139) thus adding layers to their original intuitions.

Visual recognition played an important role in this process of mathematical abstraction. The process of mathematical abstraction was evolving parallel to the process of trying to make the shape look (more) like the group of shapes (squares) which resulted from the classification task, which was included in the DT and preceded the CT. If we look at the interview script (Appendix C) we will see that the interviewers were instructed to encourage the children to look at the group of squares (which was the result of the DT) in order to make a square or in order to reflect on their original construction in case it was not a square. This helped the children towards more abstractions. Some of these abstractions were more 'vague' than others.

We had the cases of children like Loukas, for example. Loukas had constructed a rectangle in his first attempt to construct a square. After being encouraged by the interviewer to try again, he followed the route illustrated in Figure 22a (p.137). He ended up with a shape that looked more like a square than his original construction in the sense that the distance between the two vertical parallel sticks was more similar to the distance between the two horizontal parallel sticks. But the distance was still not equal<sup>41</sup>. So Loukas acquired a vague intuition in relation to the distance between the sets of parallel lines contrary to other children such as Chara (Figure 21c, p.136) who acquired a more clear intuition of the fact that it is not only the parallel sides that have to be equal but the adjacent sides as well. Similar to Loukas' was the case of Marina (Figure 26a, p.141) In an effort to make her original construction (quadrilateral with no equal sides and no right angles) look more like a square, Marina tried to close the angles. But the angles of her second attempt were still not right angles.

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<sup>41</sup> In many cases the children's actions allowed me to think of squares in ways I have never thought and look at square properties from a different perspective; one such case was that of Loukas. Through his actions, Loukas gave a new perspective in relation to the square properties concerning the equality of the sides. Another way of expressing the equality of the sides is that the distance between the two sets of parallel sides has to be equal.

## **9.5. Concluding remark**

This chapter addressed the research questions of the study based on the findings as these were described in the previous chapters. In our attempt to sketch children's understandings of squares we saw what structural understandings children have of squares, how these understandings are expressed and how they are used in the process of constructing squares. In the following chapter, an attempt is made to generalise these findings from a discussion on squares to a discussion on shapes, and from a discussion on shapes to a discussion on thinking and learning.

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# CHAPTER 10

## CONCLUSION

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### 10.1. Introduction: Towards resolving the riddle

I began this thesis by describing my initial reaction to van Hiele's (1986) paradoxical claim that 'thinking without words is not thinking'. Since this claim contradicted all of my experiences with babies and young children, my first reaction was to reject van Hiele (a reaction I fortunately soon overcame). My experiences with babies and young children drove me towards the empirical realisation that thinking need not always depend on words and that wordless thinking might play an important role in human development. So, I set out on a journey to challenge the widespread acceptance of words as the only vessel of thought. A trip to resolve the riddle of the existence of thinking without words that is not ..... thinking. This trip began with the initial realisation that thinking without words is not verbal thinking (Chapter 1). This statement which, at first sight, sounds rather naive was initially nothing more than a harmless attempt to step away from the paradoxical stance that 'thinking without words is not thinking'. But this naive claim actually had the power to open a door that was blocked by the sometimes explicit and sometimes implicit rejection of thinking without words as a 'real' form of thinking.

The decision to return and study van Hiele in depth, despite his opening claim with which he seemed to diminish the importance of wordless thinking, led me towards an additional realisation. However unorthodox it may sound, an in-depth study of van Hiele led to the surprising realisation that van Hiele's theory was full of key ideas which could open the same door that was blocked by the rejection of non-verbal thinking as a 'real' form of thinking. This was due to the many paradoxes which characterised his theory; paradoxes that were created because of the ways in which his theory clashed with his original claim (that 'thinking without words is not thinking'). So by studying van Hiele in depth I discovered a different van Hiele; a van Hiele that was neglected by van Hiele scholars and had a lot to offer to a more constructive approach towards geometric thinking; A van Hiele who could actually offer many open windows to the investigation of thinking without words. As a result, this study soon became a journey of investigating what thinking without words might be rather than what it is not.

In Chapter 2 I had the opportunity to unfold some of the hidden aspects of the van Hiele theory and supported the point of view that van Hiele was misleadingly placed among those who chose to concentrate on children's misconceptions and whose attempt was to place children within a predetermined hierarchy. This stance adopted by van Hiele scholars was more compatible with van Hiele's original claim that 'thinking without words is not thinking' than with other valuable claims made by van Hiele and that were actually neglected; claims which contradicted his original claim and created the van Hiele paradoxes discussed in Chapter 2.

This discussion led to the realisation that most van Hiele-based studies, and thus substantial pieces of research concerning geometric thinking, were characterised by a one-sided interpretation of van Hiele and consequently by a number of epistemological and methodological gaps. Thus, the conclusion of Chapter 2 was that the tendency of existing research (a) to evaluate children (mainly) through their verbal utterances, (b) to fit children into predetermined hierarchical level systems based on what they did wrong and (c) to evaluate children's understandings independently of the setting in which they had the opportunity to construct and express these understandings, led to a restricted view of what children know of shapes.

With these in mind, I set out to design and conduct a study in an attempt to better describe and analyse young children's understandings of shapes. To be more precise, the aspiration of the study was to investigate what knowledge young children have about the structure of simple shapes, how this knowledge is expressed and how it is used in the process of constructing shapes. The emphasis that was placed on the use of construction as a methodological tool originated in the personal experiences that were described in Chapter 1 and was supported by substantial pieces of literature, as thoroughly analysed in Chapters 2 and 3. For reasons that were explained in Chapter 1, the study's research questions were addressed through a more focused investigation of squares. The aims and design, the process of data collection and data analysis, the findings and the discussion which was raised by these findings were described in detail in Chapters 3-9.

The search for a methodology to overcome the restrictions identified within most existing research concerning young children and shapes led to the Description, Construction, Reflection framework (DCR) which characterised the task sequence and was used for data collection. The 52 subjects of the study were involved in this DCR task sequence which consisted of three phases. In Phase A (Description Task), the children were involved in classification and shape recognition activities, in Phase B (Construction Task) they were given wooden sticks of various

lengths and were asked to construct squares and in Phase C (Reflection Task) they were involved in a process of reflecting with the interviewers on the construction process of Phase B.

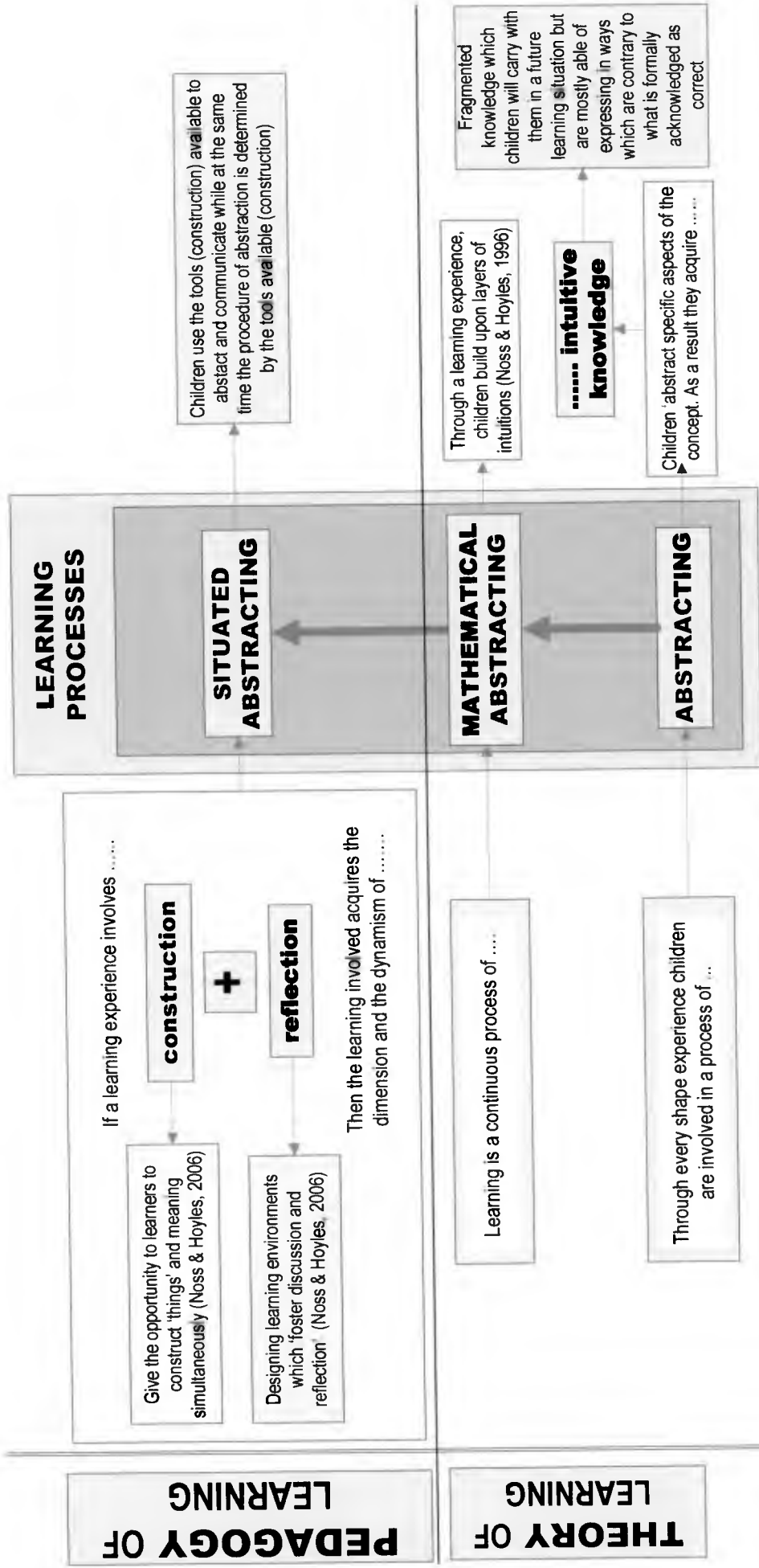
The DCR framework which began to evolve during the methodology design (Chapter 3) and was completed through the piloting procedure (Chapter 4) constituted the base on which the coding system for data analysis was developed (Chapter 5) and the findings of the study were organised (Chapters 6-8). The following section of this study provides a synopsis of the findings in relation to the ways in which the DCR framework allowed the investigation of young children's knowledge of shapes.

## **10.2. Investigating shape knowledge through Description, Construction and Reflection**

The findings of this study can be arranged in two levels. On a first level, the findings can tell us something about young children's existing knowledge about squares and consequently contribute to the attempt of understanding how children learn and think of shapes. This may be considered as the contribution of the study to a 'Theory of Learning'. On a second level, the findings can tell us something about the role of the setting in which children construct and express specific understandings. In other words, based on the findings, one can describe how the setting influences the knowledge which is constructed and expressed. Consequently, the study's findings can add to the broader attempt to design learning environments that might foster a more dynamic understanding of shapes and contribute to a 'Pedagogy of Learning'.

These two levels of findings ('Theory of Learning'/'Pedagogy of Learning') are analysed in Figure 32. As illustrated in Figure 32, the study provided evidence of specific 'learning processes' in relation to young children and shapes. These processes evolve around the idea of abstraction. Through the children's first attempt to construct a square, I was able to detect the existing knowledge which the children brought into the CT. The fact that the children exhibited specific understandings of the shape's structure, and not of the shape as a whole, supports the point of view that the children were involved in an 'abstracting' process (Figure 32); in a process through which they abstracted specific aspects of the shape's structure. The children's attempt to construct a square was based on this ability to abstract. Thus, through the first choices in their attempt to construct the shape, different children expressed different aspects of a square's structure. For example, a number of children concentrated on constructing a shape with four right angles and thus ended up with a rectangle in their first attempt to construct a square.

Figure 32: Synopsis of the study's findings: From a theory of learning to a pedagogy of learning.



This figure is separated in two horizontal parts which illustrate the two levels of this study's findings (Theory of Learning/Pedagogy of Learning). By following the arrows from left to right in each level one can summarize the study's findings.

So the study's findings support the point of view that through shape experiences children are involved in a process of abstracting specific aspects of the shapes' structure. As a result of this 'abstracting' process, the children acquire 'intuitive knowledge' (Figure 32) which, in Chapter 9, was described in detail and defined as the 'fragmented knowledge which children bring with them in a learning situation but are mostly able of expressing in ways which are contrary to what is formally acknowledged as correct'.

A complete picture of the children's involvement in the CT beyond their initial choices can support learning as a continuous process of 'mathematical abstracting' (Figure 32) through which the children 'build upon layers of intuition' (Noss & Hoyles, 1996). We saw how the children transformed their constructions through their attempt to construct a square by adding additional layers of understandings about the shape's structure. We also saw that after the successful attempt of constructing a square, children's understandings did not necessarily change from intuitive to 'logic'. Through their involvement in the RT the children continued expressing themselves mostly in ways which are contrary to what is formally acknowledged as correct.

These findings raise an interesting question: Why was it that this study, contrary to other studies, allowed the exposition of rich intuitive understandings on behalf of the children? One might argue that the children involved in this study had different understandings to the children involved in other studies. But this study was designed in such a way as to avoid this kind of interpretations. The DCR framework did not only allow the exposition of rich intuitive understandings on behalf of the children. At the same time, it allowed (a) a comparison of children's understandings as these were expressed in different settings and (b) subsequent data to be evaluated in comparison to those of existing research. This is why the interviews did not begin immediately with the CT. The first task was the Description Task (DT), where (as in other studies) the children were involved in recognition, classification and description tasks.

Through their involvement in the DT the children seemed, to a great extent, incapable of expressing structural understandings. Even though the data collected from the children's involvement in the DT was somehow compatible with that of studies which supported the existence of a visual level of thinking as the level where children view shapes as a whole, this study's interpretations were somehow different. The conclusion to draw from the children's involvement in the DT was that 'within a setting restricted to simple classification, recognition and description tasks, children exhibited poor structural understandings of shapes' (Chapter 6). This interpretation avoided statements that would label children's understandings and categorise



children into level systems. In light of the findings of the children's involvement in the CT and the RT, this turned out to be a sensible stance. The same group of children that exhibited a pure structural understanding of squares and used mainly appearance-based descriptions during the DT exhibited a rich intuitive understanding about the structure of squares during the CT and the RT.

This discussion leads towards the process of 'situated abstracting' (Figure 32): the idea that the knowledge constructed and expressed within a setting is shaped by the tools available. The study's findings therefore support the point of view that if a learning experience involves the element of 'construction' and 'reflection' then the learning involved acquires the dimension and dynamism of 'situated abstraction' (Figure 59). In other words, if we 'give the opportunity to learners to construct 'things' and meaning simultaneously' (Noss & Hoyles) and 'design learning environments which 'foster discussion and reflection' then the children are employed in a situation where they can 'use the tools (construction) available to abstract and communicate their intuitive understandings while at the same time the procedure of abstraction is determined by the tools available' (Figure 32). This is of great importance not only for methodological but for pedagogical purposes as well. Thus, thanks to the DCR framework, this study provided evidence not only of how children learn and what knowledge they acquire through shape experiences, but also of how the learning environment can influence what is learned and thus the knowledge acquired.

Even though this study began as a critical review of van Hiele, it later turned out to be a critical review of van Hiele-based research and an attempt to reinforce neglected, dynamic aspects of van Hiele. In reaching the end of this journey, we have seen how this study supports the criticism expressed in the first chapters of this study in relation to van Hiele-based research. Now, in order to complete the circle, there is a need to re-visit van Hiele in light of the study's findings.

### **10.3. Re-visiting van Hiele**

The widespread acceptance of van Hiele's visual level 0 as the level where children see shapes as a whole led to the widespread agreement that shape recognition - the visual awareness of shapes - excludes structural understandings. This acceptance implies the possibility that children may be able to recognise shapes without any awareness of the shape's structure. Despite their tendency to use visual descriptions during the DT, through their involvement in the CT the children that took part in this study exposed an understanding of the shape's structure and not of the shape as a whole. Should this lead us towards rejecting van Hiele's visual level?

In Chapter 2, I suggested a shift from the idea of levels towards the idea of epistemological pluralism; the powerful idea of the existence of multiple ways of thinking and expressing mathematics. Could we, therefore, talk about the existence of a visual way of thinking rather than a visual level? The study's findings allowed us to define visual thinking as a way of thinking intuitively; as the way children abstract, construct and express their intuitive understandings. So was van Hiele referring to a visual level or a visual way of thinking?

It is important, at this point, to repeat once more the way in which intuitive knowledge was defined in this study. Intuitive knowledge was defined as the fragmented knowledge which children bring with them in a learning situation, but are mostly able to express in ways which are contrary to what is formally acknowledged as correct. In light of this definition, it is quite interesting that in 'Structure and Insight' van Hiele (1986) strongly supports the point of view that what is often described as 'intuitive' has its origin in 'looking' at something closely', in carefully observing visual structures. In pinpointing the difficulty of expressing these structures in words, he identifies the reason for the traditional rejection of this kind of thinking. In the idea of 'looking at something carefully' and 'observing visual structures' we recognise the abstracting process identified in this study's data and which leads to structural intuitive understandings. In addition, van Hiele emphasised one of the main characteristics of intuition which has to do with the ways in which it is expressed. This was also one of the main findings of this study.

In Chapter 1, I referred to Margaret Donaldson's valuable insights in 'Children's Minds' (1978). Margaret Donaldson pinpoints the difficulty adults have in distinguishing between 'what is said' by the children and 'what is meant' and reversed the way Piaget defined egocentricity. She supported the point of view that egocentricity (the failure to see things from the perspective of the other) is not a characteristic only of young children and emphasised the inability of adults to interpret children's words because of their failure to decentre. Similarly, Sfard et Lavie (2005) identified the 'grown-ups' naïve insistence on communicating on their own terms' and emphasised the need to move away from focusing 'exclusively on how children think and how this thinking is different to that of grown-ups' towards an investigation of 'the difficulty of each of the parties with interpreting the other on this other party's own terms' (p.291). How does this connect with this study's findings and van Hiele? In 'Structure and Insight' (1986) van Hiele explains:

When I began my career as a teacher of mathematics, I very soon realised that it was a difficult profession. There were parts of the subject matter that I could explain and explain, and still the pupils would not understand. [.....]. **It always seemed as if we were speaking a different language.** (p.39)

This realisation was what led van Hiele (1986), as he himself explains, to developing the model of different levels of thinking. The development of a model consisting of different levels of thinking influenced further research and created a research culture which was characterised by the epistemological and methodological gaps identified in described in Chapter 2. But whereas van Hiele's realisation (that it seemed as if he and his students were speaking a different language) is presented here as the 'root of all evil', it actually raises a very significant issue.

It acknowledges the same difficulties in communication identified by Donaldson (1978) and Sfard et Lavie (2005). But, more importantly, the realisation of this difficulty in communication raises the need for finding common means of expression, languages which can bridge the gap. The search for such languages has been a main focus of computer-based research. But in this study, such a language was identified without the use of computers. Construction, the simple use of wooden sticks, became the language which the children could 'speak' and the adults could 'hear'. Thus, it allowed children to expose their understandings and me (as the researcher) to identify and investigate them.

In the previous section of this chapter, 'situated abstraction' was identified as the process where children 'use the construction tools available in a setting to abstract and communicate while at the same time the procedure of abstraction is determined by the tools available' (Figure 32). Thus, the point of view that what is learned is highly connected with the setting in which the cognitive act takes place was supported. The idea that understanding is closely connected with the setting was not strange to van Hiele. In Chapter 2, it was emphasised that van Hiele connected geometric understanding with specific learning processes. In differentiating himself from Piaget, van Hiele states that Piaget's stages are age-related whereas his level system is linked with a specific teaching-learning programme. Thus, van Hiele himself implies that different teaching procedures, different learning tools might lead to different understandings.

In concluding, I cannot diminish the fact that the roots of what has been accomplished in this study may be identified in van Hiele's theory as this unfolded over twenty years ago. During the course of this study, there was an increasing realisation that the links between the epistemological and methodological origins of this study and the van Hiele theory were so many

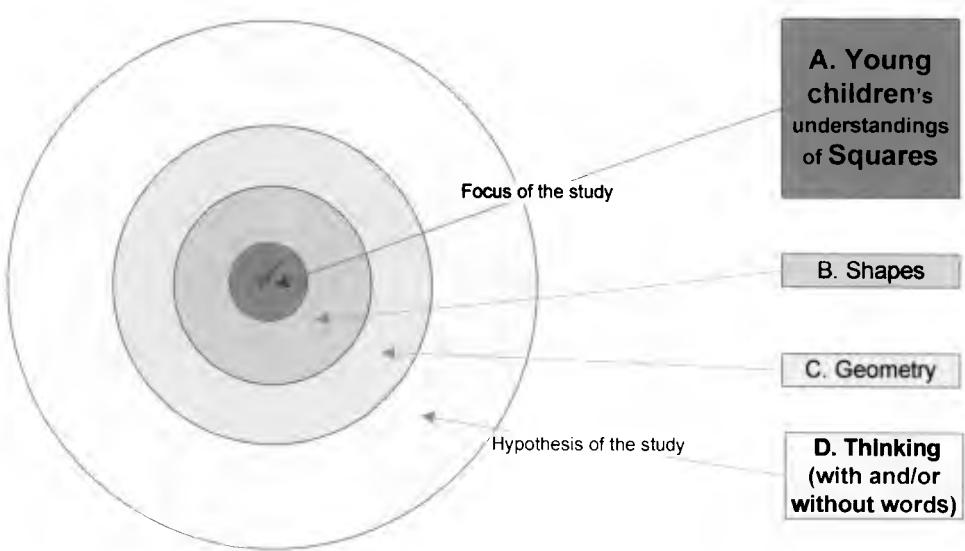
more than the gaps. Now, before one final word, there is a need to identify this study's limitations and make suggestions for further research.

**10.4. Limitations of the study and suggestions for further research**

As this thesis is coming to an end, I have the general feeling that it has not resolved the issue of alternative means of expression in an exhaustive manner. Rather, it has raised the issue in a way that opened a blocked door to a world that needs exploration, a world with pathways that may lead to fascinating trips with constructive destinations.

In Chapter 1, this study was presented as an attempt to 'investigate the hypothesis that, children might think in alternative ways and challenge the idea that thinking depends on language (thinking in words)'. This was accomplished through a more focused investigation of squares. As was illustrated in Figure 1 (p.10), whereas this study was formulated as an investigation of young children's understandings of squares, it addressed the broader issue of understanding 'thinking' which might not depend on words. Therefore, although this study has managed to investigate the hypothesis that children might think in alternative ways and challenge the idea that thinking depends on language, it was restricted to an investigation of squares.

**Figure 33: Reflecting on the initial hypothesis and focus of the study**



As illustrated in Figure 33, the study managed, efficiently and sufficiently, to meet its focus, but has only covered a small piece in relation to children's understandings of shapes, which constitutes a part of the broader domain of geometry. Furthermore, it has only 'touched' upon the issue of thinking with (and without) words. Many suggestions are therefore raised in relation to further research.

The first suggestion for further research involves conducting a similar investigation of other shapes and other geometric concepts. This would lead to a more generalised picture of children's understandings of shapes and specific pedagogical suggestions for designing a learning environment in which children may abstract, construct and express geometric thinking.

Furthermore, by addressing the issue of visual thinking, of thinking without words, this study opened a door for more research into this direction. There are many possibilities of how research in this direction could be shaped. One such possibility has to do with the aspect of age. It is interesting to conduct similar investigations by expanding the age range of the subjects upwards and downwards. On the one hand, it is interesting to find ways of detecting structural understandings of shapes in even younger children and, on the other, it is interesting to investigate whether visual thinking is a way of thinking of much older children and adults as well. And, if so, it would be interesting to investigate in what ways visual thinking of older children and adults is different to that of younger children.

The chances that visual thinking is a characteristic of all humans (regardless of their age) are quite high. As was mentioned in Chapter 2, many studies that investigated geometric thinking (Mayberry, 1983; Burger & Shaughnessy, 1986; Gutierrez et al, 1991, Gutierrez & Jaime, 1998) were led to the conclusion that the different levels of thinking were not strictly related to age and that many older students expressed understandings the same way as younger children. The shift from thinking about levels to thinking about different ways of thinking, in combination with the findings of existing studies, leads to the hypothesis that visual thinking is a characteristic of all humans (regardless of their age); a hypothesis that needs to be addressed. Such investigations may offer valuable insights into all grades of mathematics education, adult education and teacher training programmes.

It would also be useful to further investigate the conditions under which shape recognition and classification tasks shape the abstraction process. The majority of shape experiences which young children have of shapes are shape recognition and classification tasks. Children are constantly exposed to situations where a certain name is attributed to a specific shape. Thus, it would be interesting to see how specific shape recognition and classification tasks might influence the intuitive knowledge acquired and expressed. In this study, the children expressed their intuitive structural understandings during their attempt to construct a square (CT), after being involved in the Description Task (DT) consisting of simple shape recognition and classification activities. The design of the study did not allow the investigation of how the children's involvement

in the DT might have shaped the abstracted knowledge which the children exhibited during the CT. A study where different groups of children would be involved in construction tasks immediately after being involved in different shape recognition and classification tasks could shed some light in this direction.

Furthermore, in some cases presented in this study there was no way to tell whether the children were thinking with or without words (for example in the cases of children which followed S1). This raises interesting questions in relation to the ways in which verbal and non-verbal ways of thinking might interact during a learning situation. It might be that visual thinking or wordless thinking does not necessarily imply an incapability to think in words. So it might be that children may think without words despite their capability to think in words. This is another hypothesis that needs to be investigated

These are some of the ideas for further research which emerged from the limitations as well as the possibilities of this study.

### **10.5. Closing remark**

The study's findings challenge the widespread view within much of the existing literature, namely that children's limited verbal descriptions of shapes indicate little understanding of the shape's structural properties. In the process of the tasks, the children articulated, through the 'language' provided by the setting, rich intuitive knowledge about the structure of squares and were, at the same time, able to situate their abstractions in the context of construction. In uncovering young children's rich intuitive knowledge about squares (and thus shapes), this study opens a promising way for the domain of early childhood mathematics education.

At the end of this fascinating trip my luggage is full with wonderful memories and excitement for the new trips to come. Everything is so vivid; From the smallest detail to the most powerful idea; From children's awareness of the rhombus as a crooked square to the promising and powerful idea that, provided sufficiently sensitive techniques are employed, it is possible for children to express structural knowledge in diverse and often unconventional ways. But above all, this trip has given me the satisfaction of allowing young children to reveal their rich and amazing capabilities.

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# Appendix A

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General guidelines for the student teachers

EDU229: School Experience I  
Spring Term 2002-2003

Course taught by Chrystalla Papademetri


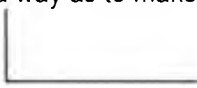
PARTICIPATING IN A STUDY  
ABOUT YOUNG CHILDREN'S UNDERSTANDINGS OF SHAPES

GENERAL GUIDELINES

**Interviewer/observer role:** You will work in pairs. Each of you will interact with four children while your pair observes and completes the observation form provided.

**Interview scripts:** For the interviews, you will follow the interview script provided. You must study the interview script carefully. In order to use it effectively you **shouldn't depend on it during the interviews** as this would hinder your flexibility and adjustment to the children's responses. You should study the interview script well before the interviews and familiarise yourselves with the procedures. A way of achieving this is to get involved into role-play with your pair, going through all possible scenarios arising from the script, before the interviews.

**Observer:** While one of you is 'interviewing' the child, the other has to observe and complete the observation form. The observer has to complete only the two columns that concern the child's behaviour (verbal and non-verbal). Do not complete the column which refers to the 'interviewer', or the 'comment' column. In the observation form you should include in detail as many of the children's choices as you can. Where necessary, use drawings in order to transcribe the child's behaviour in more detail and more quickly.

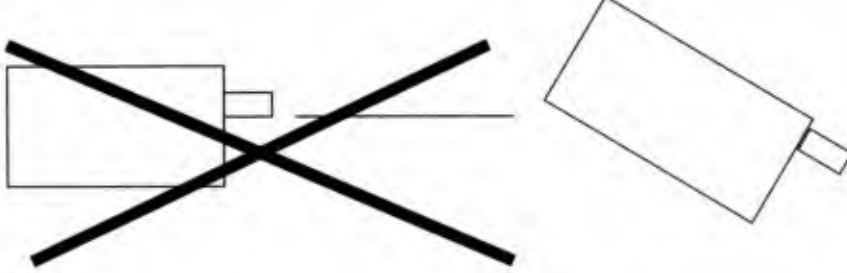
Avoid descriptions like. ...	Provide detailed descriptions like ...
The child takes two sticks ...	The child takes two equal sticks ...
He makes a shape	He makes this shape 
He takes two sticks and places them down. ...	He takes two unequal sticks and places them in such a way as to make a right angle ... 

**Selection of children:** The children should be selected randomly from the class's list of pupils. For this purpose, you will need to provide me with this list.

**Videotaping:** All interviews must be videotaped. Some of the schools will have video cameras that you can borrow. In case you will use the school's camera you might need to coordinate with the other student teachers that will be conducting interviews at the same school as you and who might intend to use the school camera also. That is why it is important to plan ahead on when you will be conducting the interviews and notify the other pairs accordingly. You can also borrow a video camera from the University supervisor. In order to do so you will need to reserve it in advance.



The videotaping must be done carefully so that it captures the children's choices clearly.



The camera must be positioned and tried out before the interviews. You should also put visible limits (e.g. ribbons) to set the boundaries of the area inside which the child is supposed to work so that his/her actions are within the camera's field of view. The observer must constantly check that the camera is indeed working properly during the interview.

On every videotape you must write

- The names of the children interviewed in the order in which they were interviewed
- The name of the interviewer for each interview
- The name of the observer for each interview

**The interviewer:** As an interviewer you must be very careful especially in relation to the ways you will communicate and interact with the children (instructions, use of language, encouragement, non-verbal communication, time control) and the way in which you will create the atmosphere in which the interviews will take place. It must be an atmosphere which will make the child feel comfortable, happy and safe. The child must feel as if playing with their teacher.

**Activities/materials used:** The activities and materials to be used were designed after a series of piloting phases so that they are attractive, easy to manage and so as to allow both verbal and non-verbal communication. The activities must be conducted in the order prescribed in the interview scripts. The materials must be well organised and ready in advance so as to avoid interruptions during the interviews.

**Time and place:** The interviews must be conducted during freeplay as part of the everyday activities. The place chosen must be comfortable and familiar to the children. Choose a place that is away from noise and movement that might disturb the children and interfere with the quality of the videotaping. Prefer places where the interviews can be conducted on the floor (e.g. gym with carpet), if available. The place must be decided in advance and be prepared in order to make the children feel welcome and comfortable. You might need to coordinate with the other student teachers that will be conducting their interviews in the same school since they will most probably want to use the same place as you.

**Evaluation:** All pairs must submit the eight observation forms duly completed along with the videotapes of the interviews. In addition to that, each student teacher must describe in her reflective journal her experience in relation to (a) the communication skills involved in the interviews and (b) how children exposed their understandings in relation to shapes. You will be evaluated in relation to (a) the quality of your interaction with the children during the interviews, (b) the quality of the completion of the observation form and (c) the quality, professionalism and consistency of your involvement in the research project.

# Appendix B

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The observation form

OBSERVATION FORM

1. Preschool: \_\_\_\_\_

2. Date: \_\_\_\_\_

3. Time of commencement: \_\_\_\_\_

Time of ending: \_\_\_\_\_

4. Name of interviewer: \_\_\_\_\_

5. Name of observer: \_\_\_\_\_

6. Personal information about the child:

6.1. Name: \_\_\_\_\_

6.2. Sex:            Boy ☐            Girl ☐

6.3. Date of birth    \_\_/\_\_/\_\_

Age:            ☐            years and            ☐            months old

INTERVIEWER	CHILD		COMMENTS
	NON-VERBAL BEHAVIOUR	VERBAL BEHAVIOUR	

# Appendix C


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The interview script

INTERVIEW SCRIPT			
Materials Introduced	Instructive interaction/ Inquiring interaction	Scaffolding	
		In case a child ...	Then you should ...
A. DESCRIPTION TASK: The children are asked whether they recognise any of the shapes, whatever they know which is the square and finally they are asked to classify the shapes into two groups, placing all squares in one group and all shapes that are not squares in another group. After the completion of the classification task the children are asked to say what they know about squares.			
Cutout shapes (of variable colour, size and type).	Shape recognition ‘Look at what I have brought for you. Do you know what these are? Do you know the name of any of these?’ ‘Do you see any other shapes?’ ‘Do you know its name?’ ‘Do you know which one of these shapes is the square?’	... cannot recognise the square among the other shapes	... show the child a square and name it. Then ask the child to show you more squares.
	Shape classification ‘Now I would like you to separate all these shapes into two groups. Put all the squares in one group and all the shapes that are not squares in another group. OK?’	... includes non-squares in the group of squares ... includes squares in the group of non-squares	... inform the child of the number of shapes misplaced: ‘... (define number) shapes that are not squares have slipped your attention and got in the group of squares. Can you find them and put them in the right group?’ If the child doesn’t respond correctly help him/her by pointing to the misplaced shapes yourselves. ... inform the child of the number of shapes misplaced: ‘... (define number) shapes that are squares have slipped your attention and got in the group of the shapes that are not squares. Can you find them and put them in the right group?’ If the child doesn’t respond correctly help him/her by pointing to the misplaced shapes yourselves.
	Do not forget to support and encourage the children ‘Well done, go on!’ ‘You must be careful so that a shape doesn’t slip your attention and get into the wrong group!’ ‘You are separating them very well!’ ‘All shapes must be placed into the two groups. None of them must be left out!’		
	Shape description: ‘How did you know all these were squares?’ ‘Why do you say all these are squares?’ ‘Is there anything else you know about squares?’  Note: Accept all answers as correct.		

B. CONSTRUCTION TASK: The children are asked to choose sticks from a pile of sticks provided in order to construct a square			
<p><b>Wooden sticks</b> (of variable length including many sets of four-six equal sticks)</p>	<p>'Now we will put aside the group of shapes which are not squares and leave all the squares here in front of us so that we can see them. Can you choose some of these sticks and make a square?'</p> <p><b>Note: Do not use the expression 'make the same square as this one'. The right expression is 'make a square that looks like these squares'.</b></p>	<p>... constructs a shape that is not a square</p>	<p>... ask the child if s/he likes his/her construction and look carefully at the squares from the classification task. Ask the child to choose other sticks or change some of the sticks used for his/her construction. ('Would you like ...'). Encourage and support the child to be careful and think about his/her choices in order to make a nice square like the ones in front of him/her. If the interaction is not successful and you feel you have tried everything, construct a square yourself ('would you like it if I made a square for you?') Then ask the child to try again.</p>
	<p><u>Do not forget to support and encourage the children:</u></p> <p>'Look at the sticks carefully and think which ones you are going to take in order to make your square!'</p> <p>'Good! I like what you are doing.'</p> <p>'You must be careful so that the square you make is nice like the ones in front of you.'</p> <p>'Look carefully at these squares.'</p> <p>'Think.' 'Take your time.'</p>		

# C. REFLECTION TASK The children are asked to reflect on the choices they made in order to construct a square

All materials used in previous activities	<p>Interact with the child in relation to the choice of sticks for the construction of a square.</p> <p>'Nice! Why did you take these sticks in order to make your square?'</p> <p><b>Note:</b> In the directions given for scaffolding you will need to move backwards and forwards according to the child's answers as the discussion develops. The aim is to conclude the discussion with the child verbalising that there is a need for four equal sticks in order to construct a square.</p>	... answers that s/he selected these four specific sticks instead of others because they are big/small	... ask : 'How many big/small sticks do we need to make a square?' ... 'Do you mean that with four small/big sticks I can make a square?' ... Let's try. 'Pick up four sticks that are relatively small or relatively big but that are not equal describing at the same time what you are doing in a manner that you are re-voicing the child's answer. 'I took four small /big sticks. Can I make a square with these sticks?' If the answer is yes ask the child to try. If the answer is no ask the child 'why?' The discussion goes on until the child tells you that you need four equal sticks in order to construct a square.
		... answers that s/he used the specific sticks because they are four	... ask: 'So if I take four sticks can I make a square?' if the answer is no ask why and proceed based on the child's answer. If the answer is yes pick up four unequal sticks while describing your actions aloud. 'Let's take four sticks and try. One, two, three, four (count aloud while selecting the sticks). With these four sticks can I make a square?' If the answer is no, ask 'why?' and if the answer is yes ask the child to try and construct a square.
		... s/he cannot tell you why s/he used those specific sticks	... show the child four unequal sticks and ask if s/he can make a square with these sticks. If the answer is no ask why. If the answer is yes, ask the child to try. If there is still no reply take four equal sticks in one hand and four unequal sticks in the other and ask: 'With which sticks can we make a square? These or these?' If the child points to the equal sticks ask why? If the child points to the unequal sticks tell him/her to try and construct a square.
			
		Interact with the child in relation to the spatial arrangement of the sticks for the construction of a square	
<p>'So in order to make a square we need four equal sticks! How must I put these sticks in order to make a square?'</p> <p><b>Note:</b> Accept all answers as correct.</p> <p>Ask the child to reflect in relation to the construction of a rhombus</p> <p>Select four equal sticks while describing what you are doing aloud. 'Let's take four equal sticks. So you are saying I can make a square with these sticks. Right? I will try. Do you think I will succeed?' Instead of making a square make a rhombus. 'Is this a square?'</p>		... answers yes	... ask him/her to compare it with a square that was constructed during the previous procedure or construct one and ask which one of the two is a square. If s/he points to the rhombus ask him to compare the two constructions with the cutout squares from the starting classification task and think about his answer again. 'Try and make sure that his answer that the rhombus is a square is not due to the fact that the child might feel uncomfortable to say that the adult's construction was wrong.
		... answers no	... ask why and after the child has replied ask him/her to transform the rhombus into a square unless s/he has already volunteered to do so.



# Appendix D

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Reading assignment for the student teachers

**EDU229: School Experience I**  
**Spring term 2002-2003**

Course taught by Chrystalla Papademetri

**PARTICIPATING IN A STUDY ABOUT  
YOUNG CHILDREN'S UNDERSTANDINGS OF SHAPES**

Assignment: Read carefully the following research studies, discuss them with your pair and answer the questions below:

- Burger, F. W. (1985). Geometry, **Arithmetic Teacher**, *February*, p. 52-56.  
Shaughnessy, J. M. & Burger, F. W. (1985). Spadework prior to deduction in geometry, **Mathematics Teacher**, *September*, p. 419-428.  
Burger, F. W. & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry, **Journal for research in Mathematics Education**, 17(1), p. 31-48.

For a better understanding of the research studies first read the following article:

Van Hiele, P. (1999). Developing Geometric Thinking through activities that begin with play, **Teaching Children Mathematics**, *February*, p. 310-316.

Questions:

1. What is the main aim of these research studies and what are the research questions they are trying to answer?
2. Describe all the activities that were used in these studies by the researchers in order to answer the research questions?
3. What are the main conclusions of these studies?
4. What are the teaching implications of these studies?
5. How do you evaluate the involvement of very young children in these studies? Do you identify any problems in the way these researchers have chosen to uncover very young children's understandings?

Further reading:

- Clements, H. D. & Battista, T. M. (1992). Geometry and Spatial Reasoning, In A. D. Grouws (ed), **Handbook of Research on Mathematics Teaching and Learning**, MacMillan Publishing Company: New York.  
Clements, H. D. et al (1999). Young Children's Concepts of Shape, **Journal for research in Mathematics Education**, 30(2), p. 192-212.  
Clements, H. D. et al (2001). Composition of Geometric Figures, **PME25**, 2-273-280.

