A Phenomenographic study of Students' Experiences of Dimension

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Thesis submitted in fulfilment of the regulations of the Institute of Education, for the degree of Doctor of Philosophy

Institute of Education

University of London

June 2011

Abstract

This study explored the experiences of dimension among young school children. Dimension is a powerful mathematical construct that is rarely taught or researched explicitly and not normally construed as something that is 'experienced'. The existing literature showed that the notion of dimension could be investigated through various perspectives: everyday life, scientific discourse, psychology, school curriculum and teaching. From this exploration, I developed an orientation of what I might consider to be an experience of dimension. A phenomenographic approach was followed, and meanings of dimension were generated from 24 students during four situations. Data were collected using clinical interviews accompanied with the design of tasks using the software Elica, physical objects, the film Flatland, and the software Google SketchUp in each of the situations respectively.

The meanings generated from the first three situations were compared and grouped into theme categories. The aim of the fourth situation was to design an environment in which we might witness experiences of dimension not observed before, by building on preceding research on how modelling can foster the utility of mathematical concepts. SketchUp and its dimensional tools helped the students to form situated experiences about mathematical ideas relating to vectors and capacity. Dimensional experience was categorised as *Dimension as Action*, *Dimension as Material*, *Dimension as Vector* and *Dimension as Capacity.*

An analysis of the relationship between the categories pointed to the duality of the passive or the dynamic way of experiencing dimension as well as looking *within* and *between* dimensions. These characteristics of the dimensional experiences informed

the notion of dimension in general as incorporating a dual nature, as an object but also as a process. Conclusively, an examination of the four situations gave an insight into what makes a window expressive both for the student as a means of 'seeing' the phenomenon of dimension and for the researcher to 'see' how the student experiences that phenomenon.

Acknowledgements

In undertaking this study, and in bringing it to this state of fulfillment, I owe a profound debt to many people who have contributed in so many ways:

Professor Dave Pratt, my supervisor, whose encouragement, guidance and support from the initial to the final level enabled me to make this thesis possible.

Dr Candia Morgan, first as my upgrade examiner and later as my internal reader whose feedback played a significant role in the development of the thesis.

Dr Esmé Glauert, my second upgrade examiner, for being very supportive.

The staff and the student colleagues at the Institute of Education, for providing a stimulating and fun environment in which to learn and grow.

The headteachers and the staff at the St Cyprian's school in London and Christakion primary school in Limassol who made me feel welcome by accommodating the conduct of the study in every possible way. I especially thank the children that participated in the study and their parents for giving their consent.

Friendship was also valuable during my efforts to complete this thesis. I would like to thank my friends, both here in London and back in Cyprus, for the many discussions we had, which research-related or not, were always occasion for new discoveries.

Last, I am grateful to my family, whose constant encouragement and insistence on positive thinking brought me through the low points and made me believe in my ideas.

Declaration

I hereby declare that, except where explicit attribution is made, the work presented in this thesis is entirely my own.

Signature: Manu Kai

Date: 10^{th} June 2011

Word count (exclusive of appendices and bibliography): 92 828 words

Table of contents

List of figures

List of tables

Chapter 1: Introduction

1.1 A focus on dimension

Children already have an idea of geometry and dimension before entering the primary school. Estimating how much water will fit in a glass, cutting enough paper to cover the presents, and drawing their house and family are just some of the many situations in which children experience dimension. This knowledge is carried to a new environment when they start schooling.

While trying to connect prior experiences with school geometry, many difficulties arise which if they are not taken into consideration, can cause many problems and create gaps in the overall knowledge the child will gain from school. According to Williams and Shuard (1994) by the age of 4 until the age of 7/8, the child's thinking is dominated by its perceptions:

> [...] children may be more deceived because they tend to identify what they see with similar but not identical things seen before. It is not possible to reverse a perception as one can an action because it depends on earlier experiences as well as on visual image one is actually seeing (Williams and Shuard, 1994, p. 28)

This prior knowledge is easily underestimated and yet can be the basis on which new knowledge is built. However, the geometry curriculum is not always designed to support the building of such connections. For example, the targets in the UK's National Numeracy Strategy (NNS) are presented below in italics (DfEE, 1999, p.102-111). These are four out of the five targets for the section Shape and Space, with transformations as a fifth. There is evidence to support that dimension is presented as an action of counting or measuring:

Classify 3-D and 2-D shapes according to their properties: The classification of shapes requires counting the number of sides and angles for the 2D shapes and the number of faces and edges for the 3D ones. A major element for the categorisation of shapes lies in actions taken out.

Make shapes and patterns with increasing accuracy: A big part of this target is about visualising 3D shapes from 2D drawings; but in order to construct nets or even visualise how a net might be transformed into a 3D shape when folded, requires the ability to count the number of the sides that are needed. Another significant exercise, which is suggested in the curriculum, is for students to work out the least number of unit cubes needed to turn a 3D shape made of cubes into a cuboids. Again, this task implies that students have to carry out actions for counting the cubes.

Recognise positions and directions, and use coordinates: For locating a point on a grid of squares students need to make use of the lines which are numbered. For example, to locate (3,5) they are expected to start from the origin (0,0) and move three lines across and five lines up, an action of counting or measuring.

Make turns: estimate, draw and measure angles: Last but not least, the outcomes of this target have the need of (a) estimating in degrees the size of each of a set of angles (b) using a protractor to measure given angles to the nearest degree (c) using the protractor to draw angles to the nearest degree (d) checking that the sum of the three angles of a triangle is 180° and many more measuring actions.

The above targets suggest that measuring and counting would be sufficient to support students in connecting their pre-existing ideas of shape and space to these abstractions (i.e. coordinate plane, 0D-1D-2D shapes), yet my own experiences would suggest that such actions are too limited to introduce the child to the richness of dimension and more generally geometry.

As a student at school I also had that experience in my general mathematics education. Although I was a very good student at maths, I never appreciated the source of relevance of ideas such as calculus and vectors, which in a sense remained abstract for me. Where could they be applied? What was the purpose of doing all these apparently meaningless calculations, according to my naïve mind at the time? What is more, I had noticed that although students became experts in arithmetic, algebra and proof of 2D geometry, they failed in topics regarding 3D geometry. This was a phenomenon I could never understand: We are born in a 3D world and it is the only world we experience in real time, yet we seem to endure deep difficulties in 3D geometry.

I wondered about the sophisticated mathematical notion of dimension, which in some sense distinguished between a world that we live in and yet find hard to formalise, and an artificial world, which in some respects seems somehow easier to mathematise. As Banchoff (1990) pointed out, the sophisticated nature of dimension is integrated within coordinate geometry, topology, vectors, projective geometry, statistics such as rates and averages (in the way that data could be interpreted geometrically), graphs and calculus. He also suggested that if the teaching of dimension is approached more intuitively, students are more likely to understand these more advanced notions:

> Students who explore models of pyramids with sets of blocks and stacks of cards throughout their early school years are certainly more likely to understand and appreciate the formal proofs presented for such theorems in calculus classes: students who have never thought about properties of volumes until they arise in calculus will not get nearly as much out of their experience (Banchoff, 1990, p. 19).

Banchoff's ideas reminded me of Papert's (1980b) 'gears' of childhood. According to Papert from a really young age he developed an interest in cars and more specifically he was fascinated in how the gearbox worked and the role of the differential. The gears served as models, and when at school he was taught about differential equations, everything was making more sense to him than to any other student:

> I found particular pleasure in such systems as the differential gear, which does not follow a simple linear chain of causality since the motion in the transmission shaft can be distributed in many different ways to the two wheels depending on what resistance they encounter [...]. I saw multiplication tables as gears, and my first brush with equations in two variables (e.g., $3x + 4y = 10$) immediately evoked the differential. By the time I had made a mental gear model of the relation between x and y, figuring how many teeth each gear needed, the equation had become a comfortable friend. (Papert, 1980b, foreword)

Following the lead of Banchoff and Papert, perhaps situations involving dimension could be designed in order to help students to build intuitions of dimension and other related mathematical ideas. However, dimension is rarely taught or researched explicitly, and not normally construed as something that is 'experienced'; therefore, more explicit research on experiences of dimension was needed in order to inform the design of the appropriate situations.

Thus, the aim of this study was to explore children's experiences of geometric dimension, the way these experiences are structured and the factors that influence the formation of those experiences. Dimension is experienced both in school, as a learning practice, and outside of school, as an everyday experience. This study intended to explore students' experiences of dimension, and in trying to link dimension with learning, the work of Marton et al. (2004) and their notion of 'space of learning' offered me an entry point.

1.2 The space of learning

Learning is either considered as a cognitive construct by psychological theoreticians or as a social structure by the socio-culturalists (Piaget, 1978; Vygotski 1978). It is of course not possible to gain direct access to the child's mind but it is possible to observe their experiences. We might interpret experience as a realistic articulation of thinking. While experience is evident, thinking is an inference. This study considers experience to be personal, influenced by the setting in which it takes place, and my aim is to map out and characterise these experiences. As I move from direct observation to characterising experience, the description becomes less direct and more inferential.

Marton et al. (2004) talked about the 'enacted object of learning' or 'space of learning', which illustrates what is possible for the students to learn of a specific phenomenon. It is a type of learning that the researcher infers as a creative description after exploring students' experiences of a specific phenomenon in a given setting:

> The enacted object of learning is the researcher's description of whether, to what extent, and in what forms the necessary conditions of a particular object of learning appear in a certain setting (p. 5).

Considering both the previous experiences students have on a specific phenomenon, such as everyday experiences, and the experiences students acquire after interacting with a specific setting, Marton et al. (2004) talked of a dialectical relationship between the two:

> Our previous experiences affect the way in which we perceive a situation, but the way in which we perceive the situation also affects what experiences we see as relevant in that particular situation (p. 5)

According to Marton et al. (2004) in order to actually create a space of learning of a specific phenomenon, the students need to experience the following patterns of variation:

> **Contrast:** [...] in order to experience something, a person must experience something else to compare it with.
>
> **Generalisation:** In order to fully understand what "three" is, we must also experience varying appearances of "three", for example three apples, three monkeys, three toy cars, three books, and so on.
>
> **Separation:** In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant.
>
> **Fusion:** If there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously. (p. 16)

To sum up, considering how experience can be researched, I would argue that Marton (1996) properly distinguished between personal observable experience and learning as an inferred description. Consequently, in this study, the role of the researcher becomes one of mapping experience and building characterisation of this dimensional experience, clearly delimited by the aim to understand the space of learning dimension.

1.3 Structure of thesis

This study is organised into thirteen further chapters:

Chapter 2: *Dimension as a construct of experience* is the exploration of the literature on dimension, looking at dimensional experience in three contexts: everyday experience, formal definitions and the aims of school geometry.

Chapter 3: *An approach to researching dimensional experience* states the aims of this study by exploring why the phenomenographic approach to study students' dimensional experience was chosen to best explore the aims of the study, pointing to the significance of the situation in which the experience takes place.

Chapters 4-9 describe Phase 1 of the study. Phase 1 took into consideration the role of the setting by using it in order to design 'windows' (Noss and Hoyles, 1996) on students' dimensional experience. Chapter 4 gives information on the method chosen as well as the sample of the study. Chapters 5, 6 and 7 describe the three situations as windows on student experience and provide detailed analysis of that data. Chapter 8 creates the initial categories of describing dimensional experience and discusses how they emerged from the data. Finally, Chapter 9 sets out how *situation* became an organising idea to inform the Phase 2 of the study.

Chapters 10-13 describe Phase 2 of the study. More specifically, Chapter 10 discusses the theoretical framework needed to inform Phase 2 in order to design a fourth situation to act as a window into students' experience. Subsequently, Chapter 11 describes in detail the design of the fourth situation, referring especially to the importance of 'dimensional tools'. Chapter 12 presents the data analysis of the fourth situation and challenges the nature of the initial categories of description. Last but not least, Chapter 13 illustrates how the outcome space, the space of learning dimension, was created first by exploring the relationships between the categories of description, and second, by investigating the four situations in terms of how they acted as expressive windows into students' dimensional experience.

Finally, Chapter 14 offers a retrospective analysis of how my initial thoughts and ideas changed as I moved through to the various phases of the study.

Chapter 2: Dimension as a construct of experience

2.1 Overview

As the main purpose of this study is to understand the space of learning dimension, this chapter is an exploration of the existing literature relating to this specific notion. Section 2.2 is an investigation of the nature of dimension by drawing attention to the everyday experiences of dimension and how dimension is defined through Sciences. Subsequently, section 2.3 takes a psychological stance by examining dimension through the developmental theories, intuitions, visualisation abilities and prototypical thinking situations. Section 2.4 explores how dimension is presented at school by pointing out the ways it is portrayed in the aims of school geometry, the curriculum, the mathematics teaching and the role of educational technology. The aim of this chapter is to develop an orientation towards dimension, which will constitute the focus of this study.

2.2 The nature of dimension

This section is an exploration of what is meant by 'dimension'. The nature of dimension is interpreted in different ways across a variety of settings. First, I shall consider how dimension is articulated as part of everyday experience and then within scientific discourse, giving some formal definitions.

2.2.1 Everyday experience of Dimension

Everyday geometry is the geometry we experience either consciously or unconsciously in applications on a daily basis. We as humans can only perceive up to the third dimension while we have knowledge of our travel through the fourth. We cannot, however, see anything past the fourth. Human vision is founded upon the retina, a 2D surface area which detects light entering the eye. In this sense, human vision involves a 2D *projection* of the 3D world we are looking at (see Figure 1).

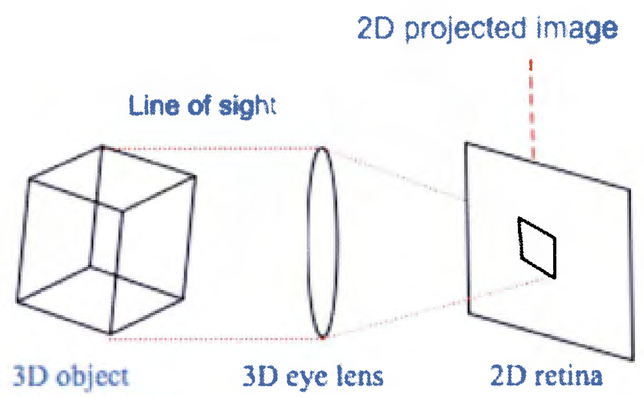


Figure 1: Human Vision

In spite of this, we are quite able to grasp the concept of 3D. The brain regenerates a 3D image of the world around us from the 2D projection on the retina. It does this by using indirect information in the 2D images such as light and shade, parallax, and previous experience (Pizlo, 2008; Todd, 2004; Von Helmholtz and Southall, 2004). Our sense then of 3D is itself a construction and one that is occasionally subject to deception through illusions or through physical handicaps such as the loss of one eye.

When the visual perception is limited or absent, physical perceptions such as touch and body motion receive more weight in identifying objects (Wijntjes et al., 2009). While working with blind learners, Healy and Fernandes (2011) argued:

> Whereas vision is synthetic and global, touch permits a gradual analysis, from parts to the whole. The activity of seeing with one's hands is hence a different cognitive practice from seeing with one's eyes and we might therefore expect that the same properties of the 'seen' object will not necessarily be highlighted in both cases (p. 3).

They gave the example of a blind learner, André, who used his hands for identifying shapes and he defined the shape of the pyramid as 'a pile of decreasing squares', a definition that is not usually given by the curriculum (Healy and Fernandes, 2009; Healy and Fernandes, 2011).

Whereas the brain is designed to unconsciously construct 3D images from the 2D retinal projections (Von Helmholtz and Southall, 2004), geometric knowledge as perhaps developed in a formal mathematics course is a construction that demands a conscious engagement with the subject matter. In the example above, André was aware that he was defining the pyramid; however, this is not always the case in our everyday experiences.

Even though in general we do not recognise our own tacit geometrical knowledge encountered in everyday life, we can infer to a significant amount of information without being explicitly aware of it (Gravemeijer, 1998). We might suppose that everyday experience can nevertheless facilitate that formal construction by providing an intuitive foundation. In this sense, we might regard certain everyday experiences as possessing geometric potential.

Geometric potential is experienced directly through the natural world or indirectly through the designed artefacts. Geometry, and dimension in particular, are principally experienced in the natural world through three-dimensional objects and their manipulation. For instance, the turning of the handle for opening the door is a form of rotation, and the opening and closing of the door creates an angle. Furthermore, dimension in geometry is used for measuring the size of an object by drawing on numbers such as the height, the width and the depth. For example, when people buy a house they pay for the dimensions of the house measured by the square footage. Nevertheless, everyday experience of two-dimensional geometry is also apparent in the form of shadows and in the touch of surfaces, which take the form of geometrical shapes such as oval, square, and rectangle. As Gravemeijer (1998) argued "geometric sense implies a relational framework for, among other things, vision lines, shadow lines, mental images, side views, top views, and maps" (p. 48).

On the other hand, dimensional geometry can be experienced through designed artefacts. Designers use dimensional geometry to create artefacts for teaching geometry. For instance, the teacher will use geometry in order to create materials and manipulatives that students can use in their learning. Moreover, the software designer uses geometry to create dynamic geometry environments for the children. The above are pedagogic artefacts designed to engage students in thinking about geometry.

However, there are other artefacts designed for other purposes and to do so make use of geometry and its applications. For instance, some artefacts are designed or programmed in order to make people's life easier and more practical. These artefacts are used willfully in industries and businesses. For example, in the car customisation business, geometry is important for figuring the rim size to put on a car. Box designers make boxes in that way that helps them ship large amounts for a cheaper price. Paint manufacturers need to figure out how much paint the can should cover. Although in the professional world the skills of geometry have often been delegated to machines, people still need those skills in order to program the machines.

Considering all the above, geometry and the notion of dimension appears around us in different forms and is used every day, even by young children. All the above examples illustrate dimension as part of an object describing its size (length, width, height), shape (square, circle, cube etc) and manipulation (creation of boxes, design of rim, playing with cubes). But is dimension only expressed in terms of n-dimensional objects? The objects in the examples mentioned above belong to a specific world: 3-dimensional space. However, there are not only n-dimensional objects but also n-dimensional spaces. Therefore, this section urged the need for exploring the more general (or even formal) ways of talking about dimension. The next section presents some formal definitions of dimension, and by 'formal' I mean generally acceptable definitions from the field of Science.

2.2.2 Formal Definitions of Dimension

The word 'dimension' comes from the Latin word 'dimensus' and means 'measured out'. Exploring the everyday experience, dimension was considered as a parameter or measurement required to define the characteristics of an object such as length, width, and height or size and shape. Dimensions can also be other physical parameters such as the mass and electric charge of an object, or even, in a context where cost is relevant, an economic parameter such as its price. In the physical sciences and in engineering, the dimension of a physical quantity is the expression of the class of physical unit that such a quantity is measured against. The dimension of speed, for example, is length divided by time. In the SI system, the dimension of a derived quantity is given by the product of powers of the dimensions of the seven base quantities. Whereas we may talk about dimensions more than two, especially for non-spatial applications of the specific concept, when we come to visual we perceive only up to the fourth dimension. Thus, higher dimensions then become metaphorical.

Considering the restriction of visual perception, an inductive approach to visualise dimension is by moving objects into space. By imaginatively dragging a 0-dimensional object (a point) in some direction we get a 1-dimensional object (a line). By dragging a 1-dimensional object in a non-parallel direction, we get a 2-dimensional object (a parallelogram). In general if we drag an n-dimensional object in a direction not spanned by that object we generate an n+1 –dimensional object. In this inductive approach, dimension is expressed as a quality of object, and more specifically as a way of generating and creating objects by using other lower dimensional objects.

In mathematics, no definition of dimension adequately captures the concept in all situations where we would like to make use of it. All mathematical definitions on dimension, however, are inspired by the notion of the dimension of Euclidean n-space E^n. The point E^0 is 0-dimensional. The line E^1 is 1-dimensional. The plane E^2 is 2-dimensional. And in general E^n is n-dimensional. Figure 2 consists of examples of objects in various dimensions presented as 2-dimensional renderings.

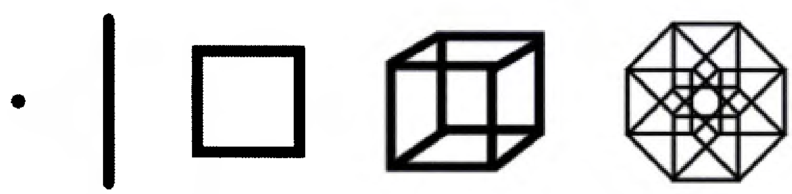


Figure 2: 2-dimensional renderings (i.e. flat drawings) of a 0-dimensional point, a 1-dimensional line segment, a 2-dimensional square, a 3-dimensional cube, and a 4-dimensional tesseract

A tesseract is an example of a four-dimensional object. Whereas outside of mathematics the use of the term ‘dimension’ is as in: “A tesseract *has four* *dimensions"* mathematicians usually express this as "The tesseract *has dimension 4"*, or "The dimension of the tesseract *is* 4". According to Euclidean geometry the objects (i.e. point, line, square, cube) behave as spaces, and dimension is an attribute of space. In contrast to the previous definitions of expressing dimension as a quality of object, the Euclidean geometry introduces dimension as a quality of space, and this quality varies according to the approach taken to determine it.

For example, one approach is by estimating the form of space that is occupied by a shape within that space. For example, three-dimensional polyhedrons are spatial enclosures made out of connected two-dimensional faces, and the four-dimensional polychorons are enclosures of four-dimensional space made out of three-dimensional cells. Consequently, dimension is the attribute of the space filled in by a shape, expressing in that way a quality of containment.

Talking of dimension as an attribute of space filled in by a shape, it is worth mentioning that dimensions are not only presented as integers, but also as fractals. Fractal dimension (D) is a statistical quantity that gives an indication of how completely a fractal appears to fill space, as one zooms down to finer and finer scales. A commonly used definition is:

$$D = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \frac{1}{\epsilon}}$$

For instance, the fractal dimension of Sierpinski triangle (Figure 3) is given by,

$$D = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \left(\frac{1}{\epsilon}\right)} = \lim_{k \to \infty} \frac{\log 3^k}{\log 2^k} = \frac{\log 3}{\log 2} \approx 1.585$$

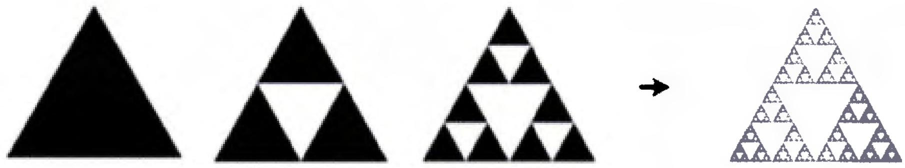


Figure 3: Sierpinski triangle

A different approach of expressing dimension as a quality of space is also articulated by using coordinates to locate a given point in space (Cartesian approach). In Science, dimensions are the number of independent coordinates needed to specify any point in a given space. Roughly speaking, the dimension of a space is the minimum number of coordinates needed to specify every point within it. Usually these coordinates are labelled as (x,y,z) for representing three dimensions. The Cartesian approach is also used in Geography where dimensions are used for locating a place on a 2-dimensional map or even locating a person on the surface of the Earth. For example, locating a point on a plane (e.g. a city on a map of the Earth) requires two parameters — *latitude and longitude*. The corresponding space has therefore two *dimensions*, its *dimension* is two, and this space is said to be *2-dimensional* (2D). In contrast, locating the exact position of an aircraft in flight (relative to the Earth) requires another dimension (altitude or elevation above the sea level) hence the position of the aircraft can be rendered in a three-dimensional space (3D).

Similar to the Cartesian, another approach for locating an object in space is through the Polar coordinate angle. A point on a circle, in a plane, can be specified by two coordinates (Cartesian coordinate system), but it can also be specified by a single coordinate (Polar coordinate angle). For instance, consider the function of the circle in Cartesian coordinate system which is $\mathbf{X^2 + Y^2 = 1}$ and the same circle in polar coordinate angle which can be represented as $r = 1$. Even though the circle might be regarded as 1-dimensional, as demonstrated by the polar equation, the Cartesian equation illustrates how often a 2-dimensional space (or more) is used to represent the circle algebraically or graphically.

Up to this point, two different expressions of dimension have been presented: first, dimension as a description of a shape (i.e. n-dimensional shape) and second, as an attribute of space (i.e. n-dimensional space). However, it was also noted that a space could be a potential domain for an object, for instance a plane can 'house' a line. This distinction between spaces and objects is rather blurred. A line, which might be regarded as an object, can be a domain for a point and so then the line might be regarded as a 1D space. Similar to the example of the circle given before, a line by itself can be considered a 1-dimensional space if we think about a point moving on it, because there is only one direction in which the point can move (regarding back and forth as positive and negative movements in the same direction). The same line can exist in a plane and be expressed by two coordinates (x,y) and possibly thought of as 2-dimensional. Consequently, these two perspectives can be merged into one general construct, in which objects and spaces become synonymous; dimension is then a quality of the space/object.

Exploring dimension further, I noted many theories that were developed to examine the number of dimensions that exist in outer space. Although at ancient times, it was believed that outer space has three dimensions, many great scientists have added more dimensions to it. Time was added as a fourth dimension by Poincaré and Einstein's special relativity theory, while theories such as string theory and M-theory predicted that the space in general has in fact 10 and 11 dimensions respectively, but that the universe, when measured along these additional dimensions, is subatomic in size.

In spite of the number of dimensions that exist in outer space, mathematicians have formulated numerous definitions of dimension for different types of spaces. One approach is the dimension of a vector space V, which is the cardinality (i.e. the number of vectors) of a basis of V. It is sometimes called Hamel dimension or algebraic dimension to distinguish it from other types of dimension. All bases of a vector space have equal cardinality and so the dimension of a vector space is uniquely defined. The dimension of the vector space V over the field F can be written as $\dim_F(V)$ or as [V : F], read "dimension of V over F". When F can be inferred from context, often just $\dim(V)$ is written. I will refer here to the mathematical concept of Hilbert space, which generalises the notion of Euclidean space in a way that extends methods of vector algebra from the two-dimensional plane and three-dimensional space to infinite-dimensional spaces. In more formal terms, a Hilbert space is an inner product space — an abstract vector space in which distances and angles can be measured — which is 'complete', meaning that if a sequence of vectors approaches a limit, then that limit is guaranteed to be in the space as well.

It is worth mentioning here the notion of linear independence. A family of vectors is linearly independent if none of them can be written as a linear combination of a subset of different vectors. The example that follows helps to clarify the concept of linear independence and its relation to the notion of dimension: A person describing the location of a place said: "It is 8 miles north and 3 miles west of here". This is sufficient information to describe the location because the geographic coordinate system may be considered a 2-dimensional vector space. The person might also add "The place is 8.54 miles northwest of here" (Figure 4). Although the last statement is true, it is not necessary. In this example the "8 miles north" vector and the "3 miles west" vector are linearly independent. That is to say, the north vector cannot be described in terms of the east vector, and vice versa. The third "8.54 miles northwest" vector is a linear combination of the other two vectors, and it makes the set of vectors linearly dependent, that is, one of the three vectors is unnecessary. Also note that if altitude is not ignored, it becomes necessary to add a third vector to the linearly independent set. In general, *n* linearly independent vectors are required to describe any location in *n*-dimensional space.

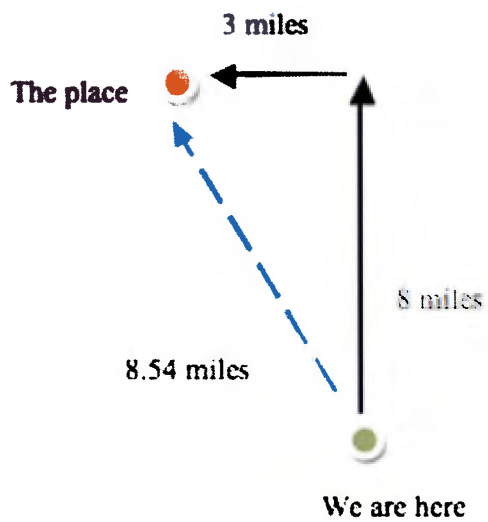


Figure 4: A person describing the position in space.

A more dynamic way of seeing dimension is as degrees of freedom. Two dimensions and three dimensions offer two and three degrees of freedom respectively. The space we experience has three linearly independent degrees of movement freedom, for example, left/right, forward/backwards and, up/down. Imagine a person walking down the street and is free to go right or left, backwards or forward. Then that person approaches a building and climbs a set of stairs, and that is where the third degree of freedom or dimension appears (up or down).

A good example showing the difference in degrees of freedom between various dimensional worlds is Abbott's book "Flatland" (1884). According to the book, if from our vintage point in a 3-D world, we look on a square in a two-dimensional flat world, we are able to see the entire object at a single glance. Actually we could place our finger inside the object without touching the sides. The inhabitants of Flatland cannot do that. They need to enter their house through the door. Similarly, a four dimensional being should have the ability to visualise an entire cube at one glance and could easily pass through a three-dimensional wall. Of course, it is quite difficult or even extraordinary for us, as three-dimensional beings to visualise this, but mathematics often requires us to generalise from the limitations of our own experience.

The two previous sections consisted of an analysis expressing the multiple meanings of dimension. It was noted that some of them refer to dimension as a quality of an object while others as a quality of space. In my search for an orientation for my study, I was not looking so much for an accurate mathematical definition; on the contrary, I was seeking for an intuitively meaningful statement that could guide the process of differentiating between what might be considered as an experience of dimension and what might not. Such a statement was articulated in the previous sections. However, a more concise statement was needed for acting as a leading heuristic to my thought processes.

Since what in one context might be regarded as object, in another context might be regarded as space, I recognise in such a short statement the ambiguity or flexibility when dimension is referred to as a quality of space/object. The quality that dimension portrays is the one of freedom and capacity. In other words, the greater the dimension of a space/object the more are the degrees of freedom for movement within that space/object. At the same time, there seems to be a sense in which space/objects with greater dimension have the capacity to incorporate space/objects of lower dimension.

So far, the individuals' everyday experience as well as some formal definitions contributed in starting the construction of an orientation on dimension identifying what an experience of dimension might be and what might not. As this study explores the experiences of dimension among students, an orientation of dimensional experience would be incomplete if dimension was not explored through the psychological learning theories and school. Thus, the following paragraphs build on this orientation by exploring the notion of dimension from the psychological perspective.

2.3 **Psychological perspective: Cognition**

This section presents how dimension can be investigated from a psychological perspective. First, the cognitive development of dimension as described by Piaget is presented. Subsequently, the next three sections' focus is on how geometrical (and dimensional) cognitive structures are built, giving great attention to the significance of intuitions, visualisation abilities and prototypical thinking.

2.3.1 Cognitive development

Many educators tried to explain children's cognitive development. In Chapter 7 of their book *The child's conception of Geometry*, Piaget et al. (1960) discussed how people measure in one, two and three dimensions. They argued that in one dimension people measure the distance from the origin to the point in question. In two and three dimensions however, they claimed that:

> (a) Sooner or later children measure in two or three dimensions by means of paired measurements along the two axes of a right angle.
>
> (b) Children cannot locate a point in two or three dimensional space without first evolving a coordinate system. (p. 153)

Subsequently, Piaget et al. (1960) proposed some age levels regarding measurement pointing out that two-dimensional measurement age levels tend to emerge closely with the three-dimensional ones (Figure 5):

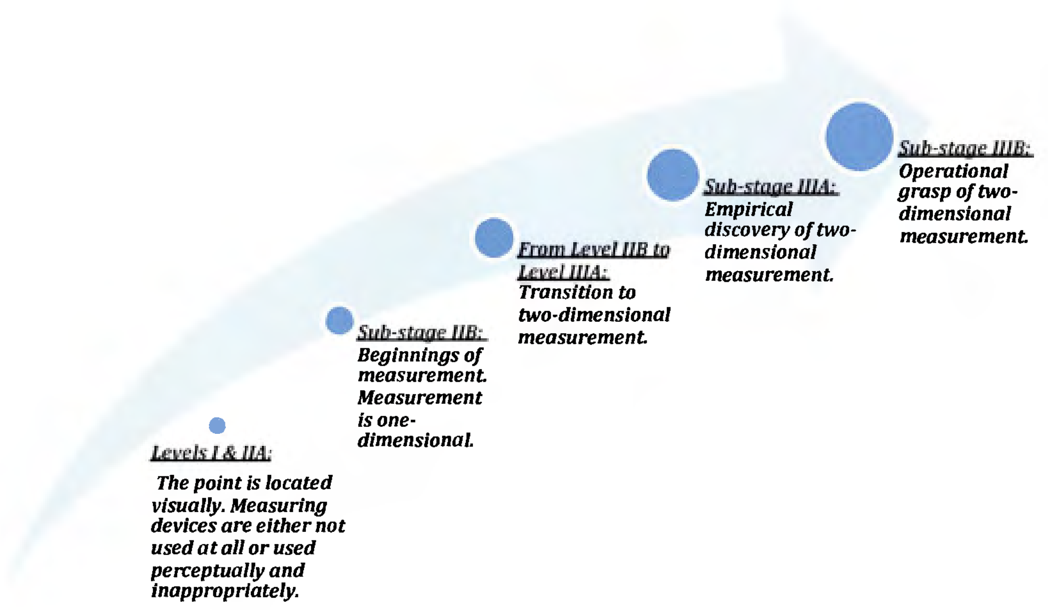


Figure 5: An illustration of Piaget's levels of measurement's development

Comparing the age levels on 2-dimensional measurement to the measurement in three dimensions, Piaget et al. (1960) argued that although there is a slight time lag in the reaching of levels in one-dimensional and two-dimensional measurement, this time lag is not present between the two-dimensional and three-dimensional measurement. This was justified by the view that one-dimensional measurement consists of additive logical groupings while two- or three-dimensional measurement is dependent on multiplicative logical groupings such as position in two or three directions.

In contrast, from an intuitive point of view, the problem of fixing a point in a volume seems much harder than that of fixing a point on an area. Sandberg et al. (1996) talked about locating a point in space visually and compared their views with Piaget's, stating that their own research had different theoretical foundations:

> For Piaget, the emergence of a cognitive awareness of multidimensionality and the ability to break space down into dimensions for separate dimensional analyses are tied to the emergence of a cognitive spatial coordinate system and the physical application of that coordinate system. [...] However, to accurately locate a point in space "by visual estimate" for a two-dimensional location implies that some ability to metrically handle dimensional relations must be present, even if the child is not conscious of it and does not employ overt measurements. Because our experiments do not require subjects to measure using implements, they can detect metric encoding of space in two dimensions at a much earlier age. The results are not so much contradictory to Piagetian predictions as they are based on differing theoretical criteria. (p. 732)

These kinds of theories can used for developing an orientation of what type of experiences individuals might have on dimension according to their cognitive development. They are useful for offering an insight into the type of experiences that children are expected to acquire in different stages of their age. These experiences can refer to the identification, creation of relationships and classification of shapes according to dimension, or to the location of a point on a dimensional domain and the measuring in different dimensions.

In a thesis on geometric thinking, it is almost required behaviour to attend to van Hiele's theory of stages in learning geometric concepts. However, a phenomenographic study such as this, must seek to map out the many ways in which dimension might be experienced, and not to pre-judge how those experiences might be compared in some developmental hierarchy. The developmental stages of these theories have a static structure. Dimension, however, is not an independent objective existence that children acquire. On the contrary it is a notion that evolves and constructed by the child. Referring to Piaget's 'conception of a number', Papert (1988) added:

> Children don't conceive number, they make it. And they don't make it all at once or out of nothing. There is a long process of building intellectual structures that change and interact and combine (p. 4).

Likewise, dimension is made through a process of building structures, which are modified, changed and combined by the child. Thus, this study cannot ignore other factors that influence these processes. By looking at the process of building structures, the notions of intuitive knowledge, visualisation and prototypical thinking are explored in the next sections.

2.3.2 Intuitive Knowledge

It was argued before that "some ability to metrically handle dimensional relations must be present, even if the child is not conscious of it and does not employ overt measurements" (Sandberg, Huttenlocher and Newcombe, 1996, p. 732). Likewise, the exploration of everyday experiences of dimension in this chapter also showed that people are familiar with experiences of dimension without reasoning since their birth, by living in a 3-dimensional world and manipulating objects. Therefore, these types of experiences are explored by referring to intuitions. Fischbein (1987) pointed out that 'intuitive knowledge' is a type of cognition. He added that intuitive knowledge is self-evident and … :

> By contrast, the statement: 'the sum of the angles of a triangle is equal to two right angles' is not self-evident, is not accepted intuitively. (p. 13)

As Tall (2004) argued, the intuitions that individuals have are of great importance in order to move from the embodied world to the world of processes and concepts.

Consequently, the kinds of intuitions children have regarding dimension are examined in order to understand the development of their experiences. Fischbein (1987), for instance, classified intuitions into primary intuitions and secondary intuitions according to their origins:

> Primary intuitions are those which develop on the basis of normal everyday experience (which is of course subject to cultural variation).
>
> Secondary intuitions, by contrast, are those which are acquired, not through natural experience, but through some educational intervention. Often these are inconsistent with the corresponding primary intuitions relating to the same concepts. (p. 70-71)

Primary intuitions consist of ground intuitions which are general and common and, individual intuitions formed by particular but natural circumstances (Fischbein, 1987, p. 64). Fischbein also added that primary intuitions "may be either pre-operational or operational" (p. 65). Considering Fischbein's characterisation, individuals are more likely to have both types of intuitions (primary and secondary) regarding dimension. For instance, the degrees of freedom of moving in 3D space are introduced as a primary intuition as they develop on the basis of normal everyday experience. In the same way, an experience of moving on 2D space can be considered as primary as children draw on 2D surfaces from an early age. However, children's experiences on vectors and coordinate geometry developed at school can be considered as secondary intuitions as they imply the teacher's intervention.

Adding to the above, diSessa (1988), although referring to Physics, talked about the nature of intuitions as "a fragmented collection of ideas, loosely connected and reinforcing, having none of the commitment or systematicity that one attributes to theories" (p. 50). However, he argued that intuitions are all we have to use for developing students' understanding and that can happen by exposing and confronting intuitive theory with evidence and argumentation:

> So we should begin with experiences that have roughly the same character as those that generate and support intuitive physics as we find it. This is the idea of microworlds, constructing artificial realities and intersect enough with students' ideas that they can immediately begin to manipulate them, but whose "deep structure", if you like, leads inevitably beyond those initial perceptions and conceptions. (p. 62)

The above illustrated the significance of intuitions as an initial stage in the development of conceptions. The intuitions students have on dimension form part of their experiences. Thus in studying experience, this study needed to pay particular attention to the heuristics people use to make sense of dimension and the way that intuitions of dimension can be generated or supported. A second component of the micro-perspective of this psychological analysis is visualisation, which is discussed in the next paragraphs.

2.3.3 Visualisation

Similar to intuitions, visualisation is a component of Psychology that can be used for describing the development of cognition in a micro-perspective. This section examines the term visualisation with regards to mathematics, giving examples from dimensional geometry.

To begin with, Presmeg (1997) considered visualisation in mathematics to be "the processes involved in constructing and transforming visual mental images, as well as those used in drawing figures or diagrams or constructing or manipulating them on computer screens" (p. 304). Although some students are 'visualisers' and some are 'non-visualisers' (Presmeg, 1986), visualisation in mathematics is considered to be essential and helpful in order for the students to construct mental images.

Similar to Presmeg's (1997) characterisation of visualisation, Gutierrez (1996a) defined visualisation in mathematics as "the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems or prove properties" (p. 7). He argued that there are four basic elements of visualisation which I present below:

a) 'mental image' as "any kind of cognitive representation of a mathematical concept or property by means of visual or spatial elements" (p. 7). Gutierrez (1996a) considers this to be the basic element of visualisation and he added that only a few types of 'mental images' are necessary in mathematics such as concrete, kinaesthetic and dynamic images.

b) 'external representation' as any kind of verbal or graphical representation of concepts or properties including pictures, drawings, diagrams, etc, that helps to create or transform mental images and to carry out visual reasoning.

c) 'process' of visualisation as a mental or physical action where mental images are involved. Gutierrez (1996a) defined two processes performed in visualisation: 'visual interpretation of information' to create mental images, and 'interpretation of mental images' to generate information.

d) 'abilities' of visualisation as the performance of the necessary processes with specific mental images for a given problem. I present the main visual abilities mentioned below together with some examples from 2D and 3D geometry which were created for showing a better view of the notions:

• 'Figure-ground perception': The ability to identify a specific figure by isolating it out of a complex background (Figure 6).

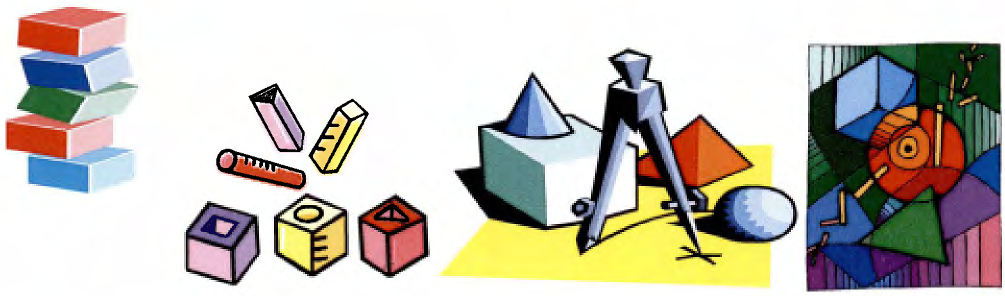


Figure 6: Figure-ground perception here might refer to the ability to identify the cube in each of the four contexts

• 'Perceptual constancy': The ability to recognise that some properties of an object (real or in a mental image) are independent of size, colour, texture, or position, and to remain unconfused when an object or picture is perceived in different orientations (Figure 7).

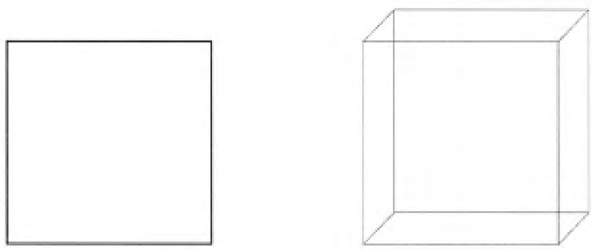


Figure 7: Each angle of a square is 90°

• 'Mental rotation': The ability to produce dynamic mental images and to visualise a configuration in movement (Figure 8).

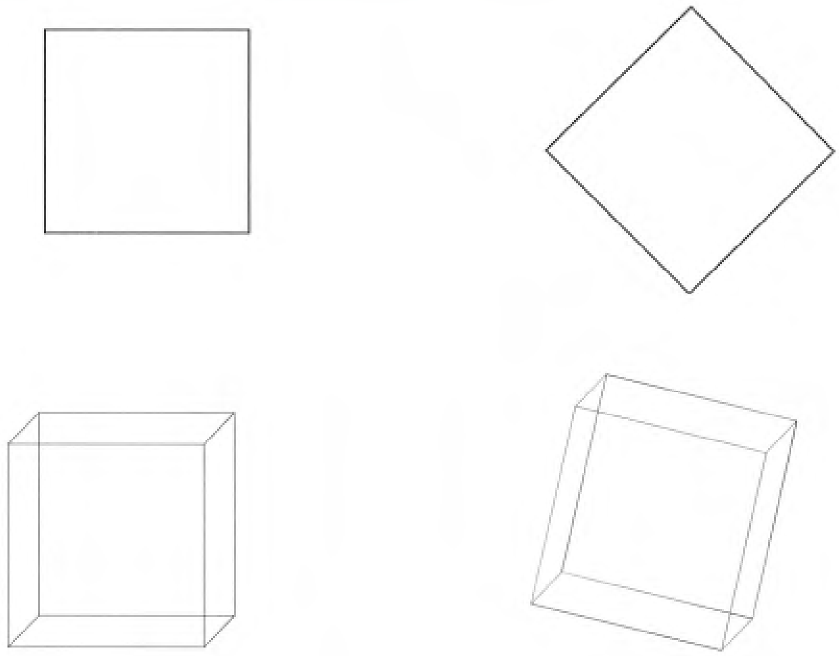


Figure 8: Rotation of a square and rotation of a cube.

• 'Perception of spatial positions': The ability to relate an object, picture, or mental image to oneself (Figure 9).

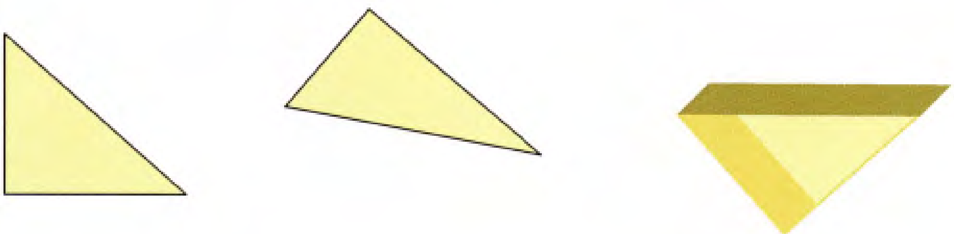


Figure 9: Identification of the same triangle in different positions

•'Perception of spatial relationships': The ability to relate several objects, pictures, and/or mental images to each other, or simultaneously to oneself.

i.e. The cube can be created by folding 6 squares (Figure 10)

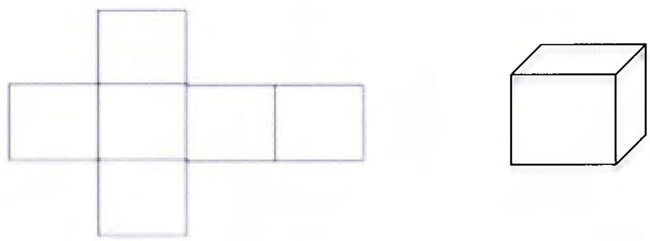


Figure 10: Net of a cube

•'Visual discrimination': The ability to compare several objects, pictures, and/or mental images to identify similarities and differences among them (Figure 11).

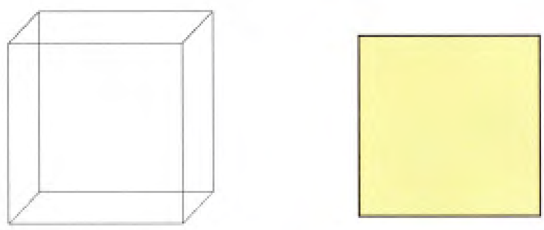


Figure 11: What are the differences between a square and a cube?

Gutierrez (1996a) pointed out that little research is conducted regarding visualisation in 3-dimensional geometry in particular; however, what is available offers an insight into students' reasoning in relation to their visual and spatial ability. For instance, in their longitudinal study of children's reasoning about space and geometry, Lehrer et al. (1998) examined the development in children's conceptions about: (a) two- and three-dimensional Euclidean forms, including angle (b) the measurement of length and area and (c) related skills such as mental manipulation of images, drawing and graphing (p. 138). Lehrer et al. (1998) conducted experiments on the representation of 3D objects on paper, and they concluded that in drawing 3D objects, children tend to:

- relate figures to known figures (e.g., "looks like a pushed-in rectangle") and,
- view figures dynamically (morphing via non rigid transformations by 'pushing' or 'pulling' at vertices)

Moreover, they pointed out that children tend to "view figures as 'slanty' or 'pointy', to count sides or vertices, or to describe figures in terms of other properties such as parallelism remained stable" (Lehrer, Jenkins and Osana, 1998, p. 144). Similarly, Morgan (2005), while exploring the notion of definition within mathematics, argued that students and teachers construct "a multi-faceted notion of dimension" (p. 104), including (among others): a description of 2D as 'flat' and 3D as 'solid', an identification of dimension with regards to 'thickness', and considering 3D as having something extra compared to 2D.

Adding to the above, Raghubir's (1999) study was considered although he referred to marketing trying to explain how people perceive volume in boxes at the supermarket. Raghubir (1999) observed that "heuristic processors of real-world, three-dimensional information are likely to simplify the volume judgment task in terms of one or two dimensions, which can lead to systematic biases in volume perceptions" (p.314). In other words, when trying to estimate the volume, people's judgements are likely to be based on one or two dimensions only. More specifically, Raghubir (1999) showed that the volume perception is mostly influenced by the 'height' on its own or in a combination with 'width'. Consequently, taller shapes are perceived larger than shorter ones.

Similarly, Piaget (Piaget, 1968; Piaget, Inhelder and Szeminska, 1960) recognised that primary school children appeared to use only the height of the container when making volume judgments. These children also thought that the volume had been reduced when the liquid was poured into a wider glass. This predominant use of a single dimension, height, to make three-dimensional judgments was termed by Piaget as 'centration hypothesis'.

To sum up, visualisation is considered to be an important ability, which helps students to identify objects in various environments but also to create relationships between the objects according to their properties. Even though little research has been conducted regarding visualisation in 3-dimensional geometry (Gutierrez, 1996a), what is available tends to show the limitation in young students' visualisation skills, such as making judgements ignoring some vital dimensions. Therefore, this study needs to further examine how visualisation is incorporated into students' dimensional experiences. A result of such limitations of visualisation and perception is prototypical thinking, which is the third component considered in the examination of dimensional experience.

2.3.4 Prototypical thinking

Similar to the intuitions and the visualisation abilities, prototypical thinking was considered as a characteristic of experience. While examining the psychological perspective of dimension and geometry, Ogden (1937) talked about a special type of geometry called 'naive geometry' which includes "those favoured forms which, in perception, are outstanding, and, if you please, self-evident, because they are conditional upon a geometrical system" (p. 199). For instance, a line is part of naive geometry because it is self-evident being a basic shape in the geometrical system. Those self-evident shapes are divided into two categories: those bounded by curved lines such as circles and ellipses, and those bounded by straight lines such as triangles, rectangles and polygons. Any shape of any of the two categories, when it is discovered by the child becomes self-evident and outstanding compared to the undiscovered or more complicated shapes.

> Thus a child, who usually sees rectangular objects like table-tops in perspective, will discover the true shape of the top only after he has seen or felt it as a rectangle. Once made, a discovery of this kind is and remains decisive. The one rectangular shape is favored, and becomes the essential meaning of the table to which all variant perceptions of it are afterwards referred, as to a norm. So strong is this influence that in drawing a rectangular object seen in perspective both the child and the artist underestimate obtuse and overestimate acute angles, thus bringing them nearer the right angles which they are. (Ogden, 1937, p. 199-200)

Ogden (1937) probably explained what is called by a prototype or a naive thought of an object, which is usually created from specific examples of the object that appears to the child. Or, another explanation is something similar to what Lehrer et al. (1998) mentioned before: that children tend to relate new objects to previously recognised ones. In the next paragraphs, the notion of prototypical thinking, which is described as having limited explanatory power, is examined.

Üstün and Ubuz (2004) pointed out that during the mental processes of recalling and manipulating a concept, some special examples, particularly figures in the case of geometry, are brought into play, consciously and unconsciously affecting the meaning and usage. These special examples are often called *prototypes*:

> The prototype is a result of our visual-perceptual limitations that affect the identification ability of individuals, and individuals use the prototypical example as a model in their judgements of other instances (Hershkowitz; Shwarz and Hershkowitz, cited in Üstün and Ubuz, 2004, p. 2).

As mentioned above, the prototypes can be the result of the limitation in visualisation and their development is unavoidable as it begins from the child's first experiences of the real world. In the case of geometry, prototypes can act as a source for developing geometrical notions and they can be modified or replaced by instruction.

There are many examples of students' prototypical thinking. In their book *Developing thinking in Geometry*, Johnston-Wilder and Mason (2005) describe two situations during their research illustrating examples of students' prototypical thinking. The first one referred to a class studying the concept of parallel lines. The authors asked one boy to show some examples of parallel lines in the classroom. The boy pointed out many examples such as window and doorframes, some panelling on the ceiling and the drawers of a desk. At first, the authors were impressed and believed the boy had a good understanding of the concept. After a while, they asked him to point out examples of parallel lines on a Tanzanian national flag that was on the classroom's display (Figure 12). The boy showed the edges of the flag only, unable to identify the two diagonal lines as parallel.

Figure 12: Tanzanian national flag

This example shows the naïve thinking the student had about parallel lines. It is the instruction from the teacher that can help the student to extend or generalise his notion of parallel lines. The second example of prototypical thinking, given by Johnston-Wilder and Mason (2005), is the one with two girls who were learning to name basic shapes. The first girl had to name the shape, which was presented from her point of view like in Figure 13:

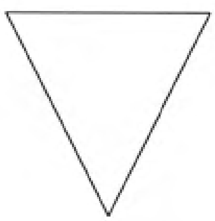


Figure 13: Shape presented to children

The girl answered that she was not sure of the name of the shape from her view, but she was sure that for the girl sitting opposite to her it was a triangle. Considering the nature of prototypes, many prototypes are created for dimension as well. For instance, Gravemeijer (1998) gave the example of a square that students may not regard as a rhombus as well (Figure 14):

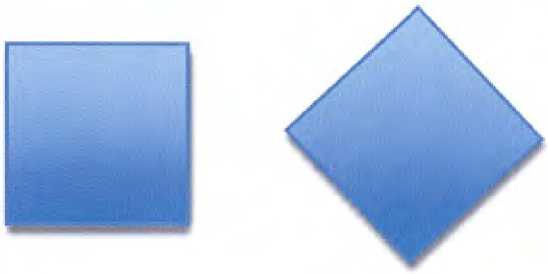


Figure 14: Students may not recognise the square as a rhombus unless it is tilted

Students also create prototypes in relation to 3D geometry. For instance, students are taught three-dimensional objects as drawings on paper and as French (2004, p. 21) pointed out, students face a difficulty "of interpreting two-dimensional diagrams of three dimensional objects". He gave an example of a cube that can be equally seen as a two dimensional picture of a hexagon (Figure 15).

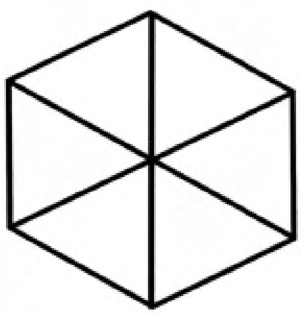


Figure 15: A cube or a hexagon? Considering the above as a 2D drawing, its perimeter is a perfect regular hexagon

As argued before, these prototypes can be the source for developing more advanced mathematical ideas. For instance, the ability of interpreting two-dimensional diagrams of three-dimensional objects is necessary for coping with various representations in everyday life. For instance, doctors often need to determine from the two-dimensional pictures of X-rays and MRI scans the position of a three-dimensional organ. Moreover, this ability is needed in order to interpret road maps to plan trips and for walking or driving directions. As for the latter, even though there is a development of 3D road maps, like Google Earth for instance, these maps are still represented on a 2D surface such as the computer screen. Concerning the above, in case of dimension, the child unavoidably creates prototypes through his/her experiences of the real world. Thus, the question raised now that this study needs to further elaborate is *how the individuals' primitive knowledge emerges in their experiences of dimension.*

So far, in building the orientation of dimensional experience, the psychological theories illustrated that students' accounts of experience can be interpreted according to intuitions, visualisation and prototypical thinking. However, this is a study about students, and thus, the orientation would be incomplete if this study did not consider how dimension is represented at schools. Consequently, the next paragraphs complete the exploration by investigating this aspect of the orientation.

2.4 **Dimension and School**

Dimension as a notion is mostly promoted in schools as being part of geometrical thinking. Therefore, our angle on dimension will be oriented through an exploration of how geometrical thinking is presented in schools revealing the way the notion of dimension is exposed to children. Dimension is examined through the aims of school geometry, the teaching of geometry, and the particular learning environments such as dynamic geometry environments.

2.4.1 Dimension in the aims of school geometry

Not many educators explained how the aims of school geometry progress in the primary school, and thus this study used general aims of school geometry as well as objectives referring to secondary education in order to make sense of what could be significant for a primary school curriculum. To begin with, the aim of school geometry as expressed by Johnston-Wilder and Mason (2005, p. 4) is "learning to 'see', that is, to discern, geometrical objects and relationships, and to become aware of relationships as properties that objects may or may not satisfy" (p. 4). This is an object-oriented approach, which ignores the importance of 'the quality of space' where dimension lies according to our definition constructed in the previous sections of this chapter. However, Battista and Clements (1988, p. 11) argued that geometry at the primary level should be "the study of objects, motions, and relationships in a spatial environment" (p. 11) which corresponds better to our definition: on the one hand, it is the quality of space filled up by a shape, and on the other, it is the degrees of freedom of movement in space. Consequently, an experience of dimension might relate to:

- the identification and distinction between 2D and 3D (and other dimensions) space/objects
- the articulation of dimension as a property of space/object (in any level of formality)
- the creation of relationships between/within geometrical objects/spaces
- the identification of the degrees of movement within the space/object

The third expression above refers to two types of relationships: between geometrical objects, and within a spatial environment. A third relationship, addressed by many educators, can be the relationship between reality and abstraction. Geometry has a dual nature: the theory and the reality-linked part of mathematics and consequently, the notion of dimension can be described by these two ways:

> Geometry has a dual character of being "both abstract and concrete". Although geometry deals in abstractions as much as any other branch of mathematics – points, lines, and planes are just as much things of the imagination as are polynomials – it is natural to relate *visually* to geometric objects (Goldenberg, Cuoco and Mark, 1998, p. 21)

The next step is to explore how this dual nature can be used effectively in schools, which is the first context in which abstraction is introduced. Chazan and Yerushalmy (1998) pointed out that students consider the relationship between Euclidean geometry and sensory experience, and that at school "there are choices to be made in deciding how to represent geometry authentically" (p. 68). Adding to this, Fujita and Jones (2002) argued that this dual nature of geometry could be used for the linking between theory and everyday experience even though there is a danger that this dual nature might be experienced as a gap between reality and abstraction. They gave the example of the Japanese curriculum in which the early stages of geometry in schools include practical activities such as the drawing and measurement of geometrical figures, and in a later stage of schooling are then devoted to deductive geometry. Fujita and Jones (2002) pointed out that in this sequence the relationship between practical and deductive geometry is not clear. On the contrary, they suggested that geometrical intuition should be linked to geometrical theory in order to improve geometrical pedagogy. Gathering all the above, our orientation of dimensional experience was enriched in order to include the relationship between reality and abstraction as well as the importance of geometrical intuition:

- the creation of relationships between reality and abstraction
- the development of geometrical intuition

Building on this, Armitage (2003) suggested that the nature of school geometry should involve a description of the real world, stating any assumptions about space clearly as happens in Physics: "Some understanding of *the relation between geometry and the physical world*- for example in mechanics and special relativity" (p. 521). He also proposed three other aims of school geometry that have not been discussed so far: familiarity with different kinds of space; the notion of a group of transformations; (in particular) the structure of affine and Euclidean space in two and three dimensions and an appreciation of the pleasure to be derived from re-discovering geometrical propositions and proving them; (p. 521).

So far in this section, the defining of the experiences of dimension was restricted to a specific spatial environment. However, the richness of various dimensional spaces is a component that cannot be ignored having in mind the previous discussion on the formal definitions of dimension. Secondly, transformations give a dynamic perspective to our orientation, by recognising what stays invariant and what changes in a shape after a translation, a reflection or a simple rotation. Thirdly, the notion of proof is introduced which is a significant component of geometry. Thus, the characterisation of what might be considered as an experience of dimension is enriched by:

- the familiarity with different kinds of space
- the identification of what stays invariant and what changes in a set of transformations of spaces/objects
- the development of the ability of reasoning and proof in geometrical contexts

Adding to the above, French (2004) also recognised the significance of reasoning as one of his three main purposes for including geometry in the curriculum. He suggested that although other subjects at school can help to the development of the skills of reasoning as well, geometry can develop some ways of thinking that are important to the mathematician: the constant urge to pose and solve problems, to seek patterns, to note links and connections, and above all, to prove conjectures. French (2004) was in favour of geometry rather than algebra as geometric problems and their arguments are often simple and are not as routine and procedural as they appear in the algebraic approach.

In spite of (i) developing the skills of reasoning, the other two main reasons for geometry inclusion in the curriculum supported by French (2004) were (ii) to extend spatial awareness, and (iii) to stimulate, challenge and inform. As for reason (ii), French (2004) argued that spatial awareness is concerned with our ability to perceive and manipulate geometrical objects and having a sense of shape and space it is useful in a wide range of everyday situations such as erecting shelves at home or reading a map, or in more professional tasks like those of the builder, the architect, navigator and graphic designer. Also, by explaining the third aim which was "to stimulate, challenge and inform", French (2004) referred to geometry as having both an aesthetic and intellectual appeal and it is also part of developing an informed background to so many aspects of our world, such as the symmetry of so many objects around us and the elliptical orbits of planets and that by nature can be used in challenging but also informing students. Reflecting on all the above aims, dimensional experience can be also characterised through:

- the development of spatial awareness
- the development of an informed background of many aspects of the world relating to dimension that might be used to stimulate and challenge the students.

Last, I refer to Usiskin (1982), a school curriculum developer, who also talked of the importance of geometry as a study of the spatial aspects of the physical world. As well as pointing to the relationship between reality and abstraction, he also added representation and visualisation as basic components of geometry. Usiskin's (1982) aspects of geometry include among others: (i) visualisation, drawing, and construction of figures; (ii) use as a vehicle for representing non-visual mathematical concepts and relationships; and (iii) representation as a formal mathematical system. Considering these, this study proposes the additional expressions of dimensional experience:

- the development of the ability to visualise, draw and construct figures
- the representation of dimension-related concepts whose origin is not visual or physical
- the use of mathematical language for describing spaces/objects

Finally, Table 1 presents a summary of the overall orientation of this study towards experiences of dimension:

Table 1: Orientation of this study towards dimension

Orientation of this study towards dimension

- **the identification, distinction and creation of relationships between 2D and 3D (and other dimensions) space/objects**
- **the articulation of dimension as a property of space/object (in any level of formality)**
- **the development of geometrical intuition and spatial awareness**
- **the development of an informed background of many aspects of the world relating to dimension that might be used to stimulate and challenge the students**
- **the identification of what stays invariant and what changes in a set of transformations**
- **the development of the ability of reasoning and proof in geometrical contexts**
- **the development of the ability to visualise, draw and construct figures**
- **the representation of dimension-related concepts whose origin is not visual or physical**
- **the use of mathematical language for describing objects/spaces**
- **the creation of relationships between reality and abstraction regarding objects/spaces**

Although this categorisation of the aims of school geometry, and dimension in particular, can be very helpful and supportive in designing a national curriculum, in England there was no evidence of a framework concerning the aims of school geometry in primary school. On the contrary, evidence showed that there was a study for the progression of geometry in secondary education. According to the Royal Society/JMC 11-19 report (2001), it was recommended that the geometry curriculum was chosen and taught in such a way as to achieve the following objectives:

- to develop spatial awareness, geometrical intuition and the ability to visualise;
- to provide a breadth of geometrical experiences in 2 and 3 dimensions;
- to develop knowledge and understanding of and the ability to use geometrical properties and theorems;
- to encourage the development and use of conjecture, deductive reasoning and proof;
- to develop skills of applying geometry through modelling and problem solving in real world contexts;
- to develop useful ICT skills in specifically geometrical contexts;
- to engender a positive attitude to mathematics; and
- to develop an awareness of the historical and cultural heritage of geometry in society, and of the contemporary applications of geometry. (p. 7)

Even though this report aimed at secondary education, most of the above targets fit to the orientation of expressions set before about geometry and the notion of dimension in primary school. Nevertheless, this report can reinforce our categorisation with additional elements, which were unintentionally ignored. To begin with, the issues of spatial awareness, geometrical intuition, visualisation, relationship between reality and abstraction, proof, and reasoning are also acknowledged by the classification in Table 1. As for the ICT objective, this is explored in the next section. However, at this point, I refer to the last objective of the Royal Society/JMC 11-19 report (2001) which draws upon contemporary issues of geometry in culture and society. Considering the notion of dimension in particular, there are many contemporary issues as well as theories in history and culture that can be effectively used in motivating and challenging students towards this area of mathematics. For example, the history development of high-dimensional geometry, the significant study of fractal geometry or even the Einstein's general theory of relativity which considers time as a dimension could be used to stimulate and challenge the students. Nevertheless, these issues can be seen as incorporated within the 'creation of relationships between reality and abstraction' and 'the development of an informed background of many aspects of the world relating to dimension' (Table 1), and thus, they did not form an independent section of the orientation.

The orientation of what might be considered as an experience of dimension was finalised by exploring the way dimension is presented through the aims of school geometry. The next section moves a step forward by examining how these aims are implemented in the school curriculum giving information on how students experience dimension through their learning at school.

2.4.2 How dimension is taught

Much research has been conducted on the teaching and learning of geometry (Clements et al., 1999; Fujita and Jones, 2002; Gutiérrez, 1996b; Kaufmann, Schmalstieg and Wagner, 2000). However, this research refers either to general facts of geometry and shapes or refers specifically to 2-dimensional geometry or 3-dimensional geometry. My aim is to carry out further research that embraces both of these two significant areas of geometry identifying implications for teaching and learning the notion of dimension in general.

First, as it was argued in the previous section, the geometry curriculum should promote the relationship between reality and abstraction. In spite of this, there is evidence that the mathematics curriculum as it appears now at schools treats geometry as an independent concept from reality, and as a consequence, independent from students' prior-experiences and knowledge. I still support what Glenn (1979, p. 21) argued some decades ago that "the child comes to school with a good practical working knowledge of this world, but instead of building on it we tend to force all subsequent learning into the two-dimensional abstraction". Indeed due to the fact that the geometry taught refers mostly on shapes and their properties, this makes it even more difficult for a child to link the 2-dimensional abstraction of a shape to his/her 3-dimensional world. As a result, students face a difficulty in building the new knowledge on the already existing one they gained from their everyday experiences; and even if they do, it is most of the times based on incomprehensible examples, such as a piece of paper (rectangle) without thickness.

Building on that, Frobisher et al. (2007) talked about the difficulty of some children to relate the representation to the real object. They argued that the teaching of spatial knowledge involves two types of 'inter-related experiences': the 3D real world and, the representations of the real world. According to Frobisher (2007), the problem with any representation is similar to what was noted before; that some features or attributes of the object represented might be changed or lost in the process, or even on occasions some features or attributes might be present in the representation which are not part of the original object:

> When showing children a picture of a triangular prism, for example, a dimension is lost as the picture is flat; the prism is a solid, and hidden edges are missing in the representation.
>
> When showing children a triangle using triangular pattern blocks, depth (a third dimension) is part of the block, yet a triangle has only two dimensions. (p. 27)

A second issue arising is the significant load on 2-dimensional geometry compared to geometries of other dimensions such as the 3-dimensional one. It is globally accepted that both three-dimensional and two-dimensional geometry are of great importance to children; however, more emphasis is given to the teaching and learning of 2-

dimensional geometry. There are two possible reasons for that: is it because it is the *easiest way* we can make sense of more formal systems in a later stage or is it because it is the *only way* to do that?

Exploring the literature further, Cooke (2007) argued that even though we live in a three-dimensional world, 3-dimensional geometry is harder to be learned than the 2-dimensional one. He referred to the complexity of the representation of solids as well as the fact that there are more aspects to be dealt with in 3-dimensional geometry than in the 2-dimensional case:

> In two dimensions, polygons have edges and vertices. In three dimensions, polyhedral have faces, edges and vertices. In three dimensions, the equivalent of a circle is a sphere. Joining points on a circle needs straight lines, but on the surface of a sphere it involves curves – in fact, circles with differing centres and radii. Again, in three dimensions there are more ways in which shapes can be symmetrical than in two dimensions (p. 121).

Similarly, Ghali (1999) tried to explain why although we might have a three-dimensional problem, it would have been better to first find a solution as if it were a two dimensional one. Although Ghali (1999) had computer graphics in mind, his thoughts might be useful for the notion of dimension in a more general sense. For example, he suggested that studying problems in the plane might provide an intuitive basis for studying the problems in 3D space, and also "an implementation for the planar problem acts as a prototype for the problem in space" (p. 60). Moreover, he added that finding a solution for many visibility and shadow problems in the plane can be a requirement for a solution in space (Ghali & Steward cited in Ghali, 1999). Consequently, the teaching of two dimensions before the teaching of the three might happen for the exact reasons mentioned above. Probably, each dimension is a requirement and part of the next higher dimension.

Looking from a different perspective, Bell (1975) supported that teaching volume as a concept, even if it is more complex and requires more time and practical experiences than does teaching area, should not follow area in the curriculum. Bell (1975) believed that the concept of area is more abstract and difficult for the student to visualise:

> The idea of an area or surface is, in some respect, more abstract than that of the volume of a solid object or the total space enclosed in a box. Solid objects or how much a jug would hold are related to the real observational world of the child and it can be argued that the area of a surface is not often met with in the everyday life of the young pupil. (p. 113)

Even though, for estimating the volume we need to split it up to three dimensions (length x width x height) compared to the two dimensions that the area could be split up (length x width), research showed that young students do not consider volume consisting of three dimensions, but as the total space occupied by the object (Bell, Hughes and Rogers, 1975). Therefore, the teaching of volume in the early years of school does not have to include calculations; on the contrary, it can be based on practical experience.

Another issue relating to the teaching of dimension is the connection between the two types of geometry, 2D and 3D, identifying their similarities and differences and thus, the significant role of dimension. As is made clear in the NNS training materials (DfEE, 1999), the priority at Key Stage 1 is to develop pupils' facility with the language associated with shape and space through practical exploration. At Key Stage 2, the training materials highlight the following three aspects of shape and space (DfEE, 1999c: Ch.7): 2D and 3D shapes and their properties; position and direction; and, transformations. Even though some of these objectives relate to both 2D and 3D geometry, their teaching is usually separate. For instance, 2D and 3D shapes and their properties are often taught independently, and therefore it is not easy for the students to make the connections between the two types of geometry. Moreover, the aims are also object-oriented; focusing on the shapes as objects and ignoring the quality of space they fill.

Another issue arising is the time spent in the curriculum for the teaching and learning of dimension. This curriculum does not offer much time for the teaching of geometry, and thus the notion of dimension. In the school curriculum of England and Wales, the teaching and learning of geometry is integrated in the section of 'Shape and Space', an area neglected by the teachers who seek for extra time to teach the rest of the curriculum areas such as number (Jones and Mooney, 2003). Talking of teachers, it is worth mentioning here the attitudes of teachers towards geometry, which also influences the way the notion is taught (Barrantes and Blanco, 2006). Geometry has been a rather frightening area of Mathematics that teachers do not have confidence in teaching (Jones, 2002).

In spite of the limited attention given to geometry, there are many examples of how dimension could be integrated in primary school geometry curriculum. Here, I present an illustrative exercise that is commonly used at primary schools (Figure 16). In this exercise students are asked to identify which hexominoes are nets of a cube and which are not. Such exercises could promote the relationship between 2-dimensional and 3-dimensional geometry, but unfortunately they are not introduced as an objective until Year 4 (DfEE, 1999). This specific exercise could reflect most of the aims of teaching and learning geometry, and subsequently dimension. It could challenge students to visualise how six 2-dimensional squares can form a three-dimensional cube. Thus, it could promote the extension of spatial awareness, both two-dimensional and three-dimensional. Secondly, through the justification of their answers, students could develop their skills of reasoning by forming conjectures that they might try to prove at a later stage, by seeking patterns and by identifying links and connections between the tasks. In other words, it aims to explore students' ability to visualise cross-dimensional geometrical transformations.

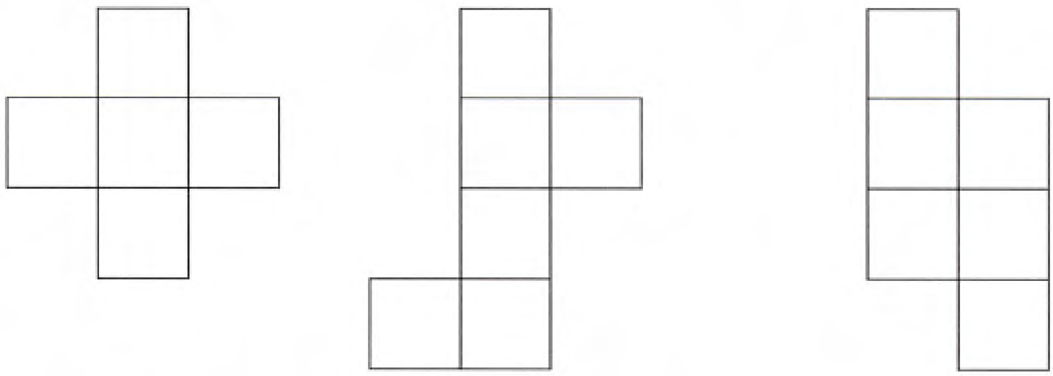


Figure 16: Hexominoes

There are many such examples in the curriculum that can be used for teaching geometry in primary school. In trying to create a strong coordination between teaching and learning about space for young children, Lehrer et al. (1998) worked with a small group of primary-grade teachers to develop a primary level geometry based on children's everyday experiences related to:

> (a) perception and use of form (e.g. noticing patterns or building with blocks), leading to the mathematics of dimension, classification, transformation;
>
> (b) wayfinding (e.g. navigating in the neighbourhood), leading to the mathematics of position and direction;
>
> (c) drawing (e.g. representing aspects of the world), leading to the mathematics of maps and other systems for visualizing space; and
>
> (d) measure (e.g. questions concerning how far? how big?), leading to the mathematics of length, area and volume measure. (p. 170)

The above ideas are helpful for including dimension in the geometry curriculum in the primary grades. Furthermore, nowadays with the development of dynamic learning environments, learning can move from the traditional static paper and pencil working materials to a more modern and dynamic setting, the world of dynamic geometry. The next section is an exploration to this innovative working environment, which promises to alter the teaching, the learning and the understanding of dimensional geometry completely.

2.4.3 Dimension in particular learning environments

Looking at the ways that geometry and more specifically dimension is integrated at school, led to an exploration of how dimension is presented in particular learning environments. To start with, I refer to dynamic geometry which is active, exploratory geometry carried out with interactive computer software. Dynamic geometry environments (DGE) are computer programs, which allow one to create and then manipulate geometric constructions. DGEs, such as Cabri Geometry and Geometer's Sketchpad, were primarily constructed as plane geometry environments, though recently there have been developments in 3D.

The purposes and advantages of using dynamic geometry environments relate to both students and teachers. By using DGEs, students can construct 2-dimensional and 3-dimensional figures by combining geometrical objects such as points, angles, segments, planes and solids. They can also create expressions and relationships using numbers, variables and operations. Moreover, students can explore the properties of a figure by manipulating its variables, and examining the effects that the dynamic transformations, such as stretching and enlarging, have on the figure. Furthermore, students can make conjectures about geometric properties of objects, and then by using the software they can verify those relationships on the figure.

Teachers, on the other hand, can use the DGEs to create tasks to introduce students into a new concept or to promote the discovery of theorems rather just showing them to students. They can create tasks, in which the students can manipulate and transform figures and extract useful results. Here is an example of how students can manipulate a cube using Cabri 3D (Figure 17):

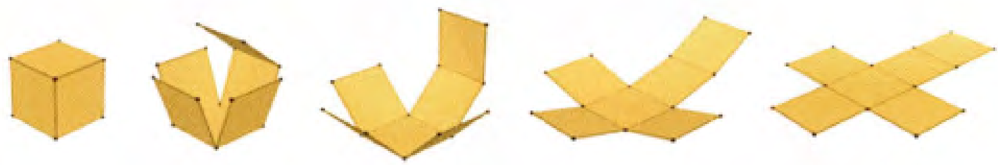


Figure 17: Cube in Cabri3D

Gutierrez (1996a) argued that computers and their software are useful in the formation of visual images because with computers students can notice things that are not so evident with their everyday contact with 3D objects. An example given is the representation of a pyramid or an octahedron that is not easily 'seen' by the physical rotation of the object (Figure 18):

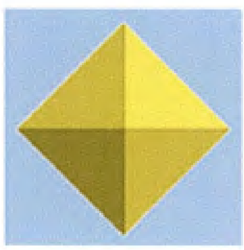


Figure 18: Pyramid or octahedron

The evolution of dynamic geometry, and especially the representation of 3D objects on the 2D screen, was developed by the use of Projective geometry, which is the branch of geometry dealing with the properties and invariants of geometric figures under projection. The 3D projection, a method of mapping three dimensional points to a two dimensional plane, was used for creating a 3D picture (Figure 19):

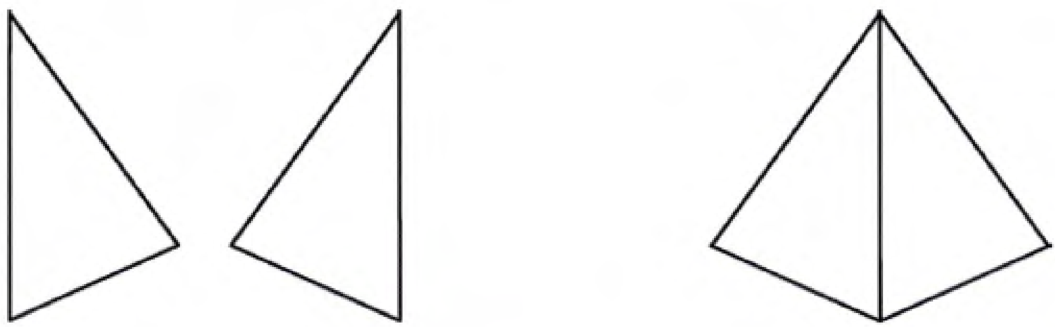


Figure 19: An example of a 3D projection

Projective geometry together with Computational geometry has managed to create such precise representations of 3D objects that look like real pictures of the real world. The example that follows shows the difficultly in distinguishing the real picture from the synthetic (Figure 20):

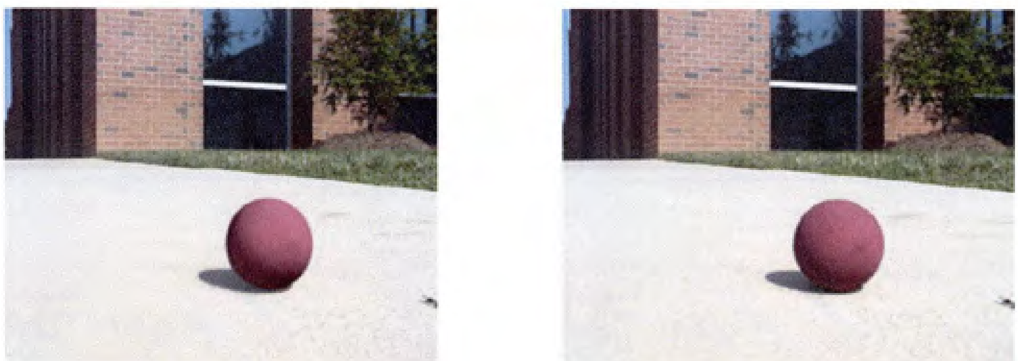


Figure 20: One of the above pictures has been created graphically. Which picture is real and which is not? The difficulty in distinguishing illustrates the effectiveness of projection

I should also mention here a modality that was first created a couple of decades ago, 'virtual reality'. Virtual reality (VR) offers students an idea of 3D objects close to reality by creating a 3D virtual environment. The purpose of VR is similar to the aims of dynamic geometry in education. It offers an environment where students can manipulate objects and realize the connections between them more easily than working with real objects. However, although the use of VR may be effective in teaching, the cost prohibits its deployment in schools and therefore, VR is considered not to be viable at this point in time for this study.

While this technological development can be an effective use in education especially for the use and manipulation of 3D objects, it can often seem as tricky and misleading. The previous sections of this chapter showed that since we only have physical manifestations of three dimensions at most, a tesseract for instance, a four dimensional object, can only be represented by a projection in 3D or 2D. In general, mathematical figures of n-dimensions will often be represented in a space with more than n-dimensions. However, when an n-dimensional figure is represented in a space of n or less dimensions, some loss of information will be inevitable in making that projection. In an n-dimensional space, a point - although 0-dimensional - is located by n co-ordinates (X1, X2, ... Xn). In contrast, a 3D solid cube can be drawn on paper, a 2D space, only through a projection, hence losing some information in the process. Although computational geometry generates digital 3D representations in a realistic way, its physical manifestation is still a 2D projection, similar to 2D drawings on paper. The 3D objects are nothing more than another 2D representation of the real objects. And even though they can be interactive with the user, they differ from the physical contact the user can have with actual real objects. Even though the software might be characterised as 3D, it cannot replace the 3D real world, and it can sometimes act as a misleading illusion. For instance, Dreyfus & Hadas (cited in Gutierrez, 1996a) pointed out that students take very seriously the solids they 'see' on screen, and for instance they may consider a right angle as acute only because "it looks like it" on screen. Therefore, the software chosen for teaching geometry should be as representative of reality as possible, in order to avoid illusions.

In addition, the use of only this type of geometry cannot replace the experience of everyday life. What is more, formal definitions of geometry cannot be found as raw materials in dynamic geometry, and also the school curriculum cannot be taught without the use of instruction; so, the role of the teacher is vital in the teaching of geometry, and thus the teaching of dimension.

In general, however, these particular learning environments can offer great experiences to students regarding dimension. They can also contribute for connecting the natural world to the formal definitions and the school curriculum. Some progress has been made in supporting those connections. Chazan and Yerushalmy (1998), for instance, argued that:

> [...] as a result of interacting with these programs, students will come to understand that statements and procedures in Euclidean geometry are meant to be more general than the static diagrams that may accompany them in a text. (p. 82)

What is more, the invention of dynamic geometry software has led to the notion of 'figure' as a bridge between unrestrained drawing and the mental geometric ideal (Laborde, 1995a). As John and Sutherland (2004, p. 104) argued "the appropriate pedagogic deployment of ICT does not necessarily reduce subject complexity but can increase it, thereby creating new avenues for thinking and engagement" (p. 104).

2.5 Summary

This chapter explored how dimension can be experienced in everyday life, through formal definitions, psychology and school according to the existing literature. Even though dimension is a notion embedded in all the above contexts, little research has focused explicitly on dimension. What is available showed on the one hand, some interesting facts about how dimension is articulated but also pointed out some omissions and misunderstandings that lie in the literature.

In seeking for an orientation of this study, first I realised that I was not searching for a rigorous definition of dimension but rather an intuitively meaningful statement that could guide the process of identifying what might be regarded as an experience of dimension and what might not. Through the exploration of everyday experiences and formal definitions, dimension seemed to be characterised as a quality of space/object, where quality can be described in terms of freedom and capacity. The exploration of the psychological theories offered an idea of what this study might look for in an experience pointing to the significance of intuitions, visualisation abilities and prototypes. The examination of how dimension is presented in school, first, by looking at the aims of school geometry, helped in forming an orientation of dimensional experience. Finally, the exploration of how the aims of school geometry are applied through teaching, gave an insight to the difficulties students face in relation to their dimensional experiences, showing the need to explore the notion of dimension further, as elaborated in the next chapter.

Chapter 3: An approach to researching dimensional experience

3.1 Overview

The analysis of the literature in the previous chapter showed that even though dimension is a notion experienced in everyday life, through formal definitions, through the psychological theories, and in school, little research has focused explicitly on dimension. Therefore, this study examined this notion further by drawing on the experiences students have on dimension. Having constructed an orientation of what might be considered dimensional experience, section 3.2 of this chapter presents the research questions developed to explore students' experiences of dimension.

This chapter also discusses how an empirical study was designed in order to examine these research questions taking into consideration the relevant literature. First, section 3.3 compares the qualitative to the quantitative research methodology for illustrating which one better suits the objectives of this study. Subsequently, section 3.4 discusses the notion of 'experience' as reflected in the research questions, relating it to the role of setting. Section 3.5 presents the phenomenographic approach, which was chosen for this study, followed by section 3.6, which consists of an exploration of the literature about experiencing and expressing meanings within a specific situation. Last, section 3.6 discusses the limitations that this approach might have had for this study and explains how the followed research process managed these limitations.

3.2 Aims of this study

In the previous chapter, dimensional experience was examined through everyday life, through formal definitions in Sciences and through the aims of school geometry and the curriculum. This exploration showed that dimension is experienced both in school and outside of school, although there was not any specific research focusing explicitly on dimension. Thus, as a first step, this study aimed to explore the range of the experiences children have on dimension in both school and outside of school.

The exploration of the literature gave an insight into the structure of these experiences by drawing on the ideas of intuitions, visualisation and prototypical thinking. Thus, as a second step, this study aimed to explore further the structure of children's dimensional experiences.

As a third step, it was important to consider the factors that influence and shape these experiences of dimension that students might have and this was the third aim that this study aimed to achieve. Consequently, the above aims were gathered into three research questions for further exploration:

(1) How is dimension experienced in school and outside of school?

(2) How are experiences of dimension structured?

(3) What factors shape the experiences of dimension?

The next sections of this chapter seek and explain the methodological framework that was used in order to find an approach for exploring the above questions.

3.3 Searching for an approach

In searching for an approach, first it was considered useful to explore educational research in the general sense, by drawing attention to the distinction between the qualitative and the quantitative approaches of research design. Educational research is considered to be the type of research conducted to investigate the behavioural patterns (Cohen, Manion and Morrison, 2000) in pupils, students, teachers and other participants in schools and other educational institutions. There are two major approaches to research methodology in Education: Quantitative research and Qualitative research. Quantitative research is the systematic scientific investigation of quantitative properties and phenomena as well as their relationships. I would argue that it investigates the 'what', 'where', and 'when' of phenomena. The researcher using this kind of research knows in advance what he/she is looking for and the study is well designed before the collection of data. Data are categorised using large samples and "quantitative researchers endeavour to achieve objectivity in their research" (Robson, 2002, p. 23). They look at precise measurements (statistics and numbers); therefore, the results are more likely to be generalisable in a statistical sense.

By contrast, qualitative research is "any type of research that provides findings not arrived at by statistical procedures or other means of quantification. It can refer to research about persons' lives, lived experiences, behaviors, emotions, and feelings as well as about organizational functioning, social movements, cultural phenomena and interactions between nations" (Strauss and Corbin, 1990, pp. 10-11). I would argue that it investigates the 'why' and 'how' of phenomena. Data are categorised using small samples. The researcher using this kind of research is generally open; the design of the study emerges as the study unfolds. The researcher interprets the results in a subjective way using words in description and categorisation, which is difficult to be representative of the wider range of population. Moreover, due to the nature of qualitative research and the methods used such as participant observations and interviewing, the researcher tends to become subjectively immersed in the subject matter.

Concerning the above, both types of research have their strengths and weaknesses although they use different methods and follow different philosophical orientations. In order to examine this study's objectives, this study had to choose the most appropriate methodology for exploring the phenomenon of dimension and more specifically how individuals experience this phenomenon. The methodology had to be an exploratory one, in order to investigate the variety of ideas and not to test specific pre-decided proposals like most quantitative studies do (Robson, 2002). Even a mixed methods approach would not have been appropriate as the research was unlikely to reveal sufficiently precise hypotheses that could be tested within the scope of available resources for the study. This study had to follow an approach, emphasising explanation by recognising phenomena ignored by most previous researchers and literature, rather than being confirmation; a form of qualitative research methodology that carries the opportunities and the limitations that qualitative research methodology in general offers.

3.4 The significance of 'experience' for choosing an approach

After deciding on a qualitative approach, I had to delimit my choices to the most appropriate research strategy of this specific methodology. There are many strategies that can be followed while a doing qualitative research, such as case study, experiment, ethnography, grounded theory and action research (Cohen, Manion and Morrison, 2000; Denscombe, 2010; Robson, 2002). For choosing the most suitable strategy, I had to consider which one would be more effective for achieving this study's aims. As aforementioned the aims of this study were to explore children's experiences of dimension. The significance of 'experience' is directly linked to this study's research questions; therefore a strategy had to be chosen that reflects 'experience' in the same way as this study's aims.

The Introduction chapter referred to the 'space of learning' and how *experience* is more evident for a researcher to explore than thinking. After forming an orientation of what this study might consider as an experience of dimension based on the existing literature on the aims of school geometry, the research questions were formed, having 'experience' as a central element. Thus, a more concrete definition of experience was needed in order to choose a suitable research strategy for exploring the specific research questions. The Longman dictionary (2009), gives three definitions of experience:

- Knowledge or skill that you gain from doing a job or activity, or the process of doing this
- Knowledge that you gain about life and the world by being in different situations and meeting different people, or the process of gaining this
- Something that happens to you or something you do, especially when this has an effect on what you feel or think

Gathering the components of the above definitions, it seems that experience is an action or an effect of an action, as it happens by *doing, being, meeting, feeling and thinking*. Although it is an action of an individual, it is gained after a job, an activity, or a process, which involves exposure in different situations and real life phenomena. As Marton and Booth (1997) argued, experience is "an internal relationship between person and world" (p. 122). In this sense, as seen in the previous chapter, the cognitive theories of children's development bring in the personal and individual role to learning. This study however, considered *the individual way of looking* at a particular situation, without underestimating the nature and the role of the situation in which the experience is created.

In relation to the nature of *situations*, it stressed the need to consider that the situation could have an even more fundamental role in how dimension is experienced and understood. An example of a study on 3D geometry that showed the importance of the situation in which the cognition takes place is the one of Stavridou et al. (2005) who studied adolescents' drawings of the same array of 3D objects in three cases:

(a) the representation of 3D objects without the presence of models (through verbal instructions),

(b) by observation of physical models, and

(c) by observation of their digital models on a computer screen.

The comparison of the three kinds of drawings of the same array of objects showed that children have difficulties in depicting the 3rd dimension in the absence of a model. On the other hand, in drawing from observation, the nature of the model seems to play an important role in the drawing outcome, with a clear superiority in performance when the model is digital. Consequently, this study's approach was informed by the view that "the more extended use of 3D models, either physical or digital, could help the better understanding of spatial relationships and evoke the use of more advanced drawing techniques for the depiction of 3D layouts" (Stavridou and Kakana, 2005, p. 53). The core problem identified by Stavridou et al (2005) was the representation of depth onto a two-dimensional drawing. The specific research showed that most of the children when drawing from observation of digital models, used more perspective elements and more advanced drawing techniques comparing to children who drew the same forms from observation of 3D models.

The above example showed that students' experiences were examined by involving two components: first *what* the students drew, which was drawings of the same array of 3D objects; and second, *how* the students made use of available resources or techniques to reach their goal. The 'what' and the 'how' of experience was identified by Marton and Booth (1997) who argued that:

> The "how" aspect is then related to one's overall understanding of learning, reading or problem solving, and the "what" aspect is made sense of in terms of one's understanding of the topic dealt with in the text or the phenomena involved in the problem (p. 56)

The experience, however, including both the 'how' and the 'what' takes place in a particular situation. And looking again to our example, it was reasonable to consider that similar to students' experience of drawings of 3D objects, the experience that children have on dimension might be influenced by the situation in which it takes place.

Exploring the term *situation* further, it conveyed notions from the literature relating to *activity*, *setting*, and *context*. In her book 'Cognition in Practice', Lave (1988) distinguished between the terms 'setting', 'arena' and 'context' while talking about individuals' experiences in supermarkets. Lave characterised the supermarket as an arena, which is "a physically, economically, politically, and socially organised space-in-time" (Lave, 1988, p. 150), a higher-order institutional framework that is out of the control of the individual. An arena, after repetitive experience by the individual, can be transformed into a 'setting' for activity. Thus, setting can be considered as "a relation between acting persons and the arenas in relation with which they act" (Lave, 1988, p. 150). Talking in Lave's terms, neither the setting nor the activity can exist independently or detached from each other, as if their relationship is dialectical. Adding to this, the commonly used term 'context' seemed to incorporate Lave's two notions of 'arena' and 'setting'.

> A setting is generated out of a person's grocery-shopping activity and at the same time generates that activity. In short, activity is dialectically constituted in relation with the setting. [...] Neither the setting nor the activity exists in realized form, except in relation with the other [...] The relationship between these newly differentiated units of analysis, 'arena' and 'setting', is reflected in common uses of the term context. On the one hand context connotes an identifiable, durable framework for activity, with properties that transcend the experience of individuals, exist prior to them, and are entirely beyond their control. On the other hand, context is experienced differently by different individuals. (Lave, 1988, p.151).

Although Lave talked about shoppers' experiences in a supermarket, her distinctions and relationships between setting and activity were thought to be of great use for this study as what distinguishes the setting from the arena is the construction of experience, which varies between individuals. As a consequence, the 'dialectical relation' between the 'activity' and the 'setting' is influenced by the individual's unique experience. Thus, it seemed that by looking at the dialectical relationship of the activity and the setting involved in a situation, the experience of an individual could be exposed.

Usually in a research process there is a particular phenomenon studied through the experiences of the individuals. For instance, in the example given before (Stavridou and Kakana, 2005), the phenomenon was the representation of 3D objects. Marton and Booth (1997) distinguished the term 'phenomenon' from 'situation' and they explained the learning situation:

> We have to be clear about one important distinction, that *a situation* is always experienced with a sociospatiotemporal location - a context, a time, and a place – whereas *a phenomenon* is experienced as abstracted from or transcending such anchorage. In the learning situation that prevails, however, the two are inextricably intertwined. That nobody can experience a phenomenon in the absence of a situation is strongly intuitive. That a situation can be experienced only in terms of that which transcends it follows our ability to make sense of the here and now only through the experiences which precede it. [...] We refer to the wholeness of what we experience to be simultaneously present as a *situation*, whereas we call entities that transcend the situation, which link it with other situations and lend meaning to it, *phenomena*. (p. 82-83)

Connecting this phenomenon-situation relationship to the present study, it seemed that we could not study how students experience the phenomenon of dimension without drawing on their experiences within the particular situation. However, attention was thought to be given first to the students' pre-existing experiences of dimension which were formed through particular situations they were part of in the past; and second, to the fact that these experiences could be modified, transformed, or developed through the situations that the specific research process could involve.

> We cannot separate our understanding of the situation and our understanding of the phenomena that lend sense to the situation. Not only is the situation understood in terms of the phenomena involved, but we are aware of the phenomena from the point of view of the particular situation. Furthermore, not only is our experience of the situation molded by the phenomena as we experience them, but our experience of the phenomena is modified, transformed, and developed through the situations we experience them in. (Marton and Booth, 1997, p. 83)

Consequently, the present study took into account the role that the situation has on the formation of experience of a particular phenomenon and it argued that it is possible that these experiences would be modified or even new experiences could be created after exposure in new situations. It also supported the possibility that in different situations students might reveal different experiences of the specific phenomenon.

The next section presents the approach chosen to be most appropriate for examining students' experience of the phenomenon of dimension.

3.5 Phenomenography as an approach to experience

The exploration of the notions of 'experience', 'phenomenon' and 'situation' as well as their relationships in a research process, gave an insight into the type of strategy that could be followed. The research approach of phenomenography was chosen as the most suitable for investigating this study's research questions, first because it embraces the above notions and relationships, and second because it was developed within studies of learning.

Phenomenography as a word is constructed by the Greek verbs of *φαίνεσθαι* (bring to light, make to appear, show), and thus the noun φαινόμενο (the apparent, the visible, the perceivable), and *γράφω* (to describe something in words or pictures). In other words, phenomenography denotes the description in words or pictures of that which manifests itself; in our case, the description in words or in pictures of the experience of dimension of the individuals.

Phenomenography was first introduced by studies of learning carried out by Ference Marton and his colleagues at the University of Göteborg, Sweden, in the early 1970s. Marton (1981) defined phenomenography as the kind of research that aims in describing, analysing and understanding first various aspects of the world and second the way people interpret and experience those aspects. Phenomenography is an approach exploring the understanding of a learning content as well as how individuals perceive learning in different situations. By 'understanding', Marton referred to the interaction between an individual and a phenomenon through experience arguing that human learning results from the alteration of this understanding (Marton, 1992). According to Marton (1994):

> "Phenomenography" is the empirical study of the limited number of qualitative different ways in which various phenomena in, and aspects of, the world around us are experienced, conceptualized, understood, perceived, and apprehended (p. 4424).

Marton (1981; 2005) argued that phenomenography describes the conceptions of our environment. Phenomenographers interpret conceptions of reality as "the 'what' of thinking, the meaning people see in and ascribe to what they perceive" (Saljo, 1988, p. 37). The conceptions of reality are portrayed as categories of description that are used for facilitating the grasp of concrete cases of human functioning. Individuals move from one category to the other depending on the occasion they are in.

The unit of phenomenographic research is *a way of experiencing something* while the object of the phenomenographic research is the *variation in ways of experiencing a specific phenomenon* (Marton and Booth, 1997). The unit involves the relationship between the person who experiences something and the phenomenon that is experienced, while the latter engages the various ways or aspects of seeing the phenomenon. Marton (1992) added that "our world can be seen and understood in only a limited number of distinctively different ways" (p. 253). Individuals express different meanings of the phenomenon because they understand it in different ways:

> '....the children give distinctively different answers because they understand the problem in distinctively different ways'. (Marton, 1992, p. 257)

Thus, the outcome of a phenomenographic research is to find this limited number of ways in the form of concrete cases that would form the categories of description.

Phenomenography is not Phenomenology although they have the same object of research: the human experience. The main difference is that phenomenologists engage in investigating their own experience as philosophers as well, while phenomenographers describe only the experiences of others (Marton and Booth, 1997). The basic aim of phenomenography is to describe learning as if it was through the eyes of the learner (Marton, 1994). Adding to this, Saljo (1988) argued that the "access to the learner's perspective on the activities of teaching and learning is essential for understanding educational phenomena" (p. 35). The fundamental idea that lies on the grounds of phenomenography is that people's actions are based on their interpretations of the situations they experience rather on the objective features of situations (Saljo, 1988). Thus, the limited number of ways that a certain phenomenon can be perceived can be either extracted as embedded in the immediate experience of the phenomenon or in a reflected thought about the same phenomenon.

Marton (1992) gave the example of geometry to explain the various ways of experiencing a phenomenon. He argued that students already develop certain ways of understanding the phenomena of shapes, lines, space and relationships even before entering to the formal study of geometry. Students experience a specific phenomenon from various perspectives, and therefore they create multiple understandings of that phenomenon. For example, a child experiences a cube differently when he/she plays with it as a toy, when he/she tries to cut a paper to wrap a present in a box and in a completely different way when he/she makes calculations of a perimeter or volume of a cube presented in a geometry class. There are no correct or wrong understandings but just different, each one representing a specific occasion or a context.

As aforementioned, the purpose of phenomenography is to map out the limited number of ways that a phenomenon is experienced. Of course, it is not possible to map out all the possible ways of experiencing a phenomenon, thus, the outcomes of phenomenographic research consist a subset of all the possible ways of experiencing in the form of categories of description. As Marton and Booth (1997) argued, there is a part-whole relationship between the ways of experiencing a phenomenon and the phenomenon itself.

Building from the *space of learning theory* (Marton, Runesson and Tsui, 2004), as discussed in the Introduction chapter, in order to understand the space of learning, the role of the researcher is to map experience and build a characterisation of that experience, and that is what phenomenography tries to do. In doing so, Lybeck et al. (1988) supported that phenomenographers have a view of learning "as a change from one way of understanding a phenomenon to another and qualitatively different way of understanding the same phenomenon" (p. 83). Consequently, teaching methods, according to Marton (1992), should help students to "arrive at new understandings of a given phenomenon" (p. 253). For creating such methods, one needs to discover first the finite ways in which individuals may understand that phenomenon. Then, through experimentation students may reconsider their initial understandings and probably reconceptualise the given phenomenon:

> I assume that the most important form of learning involves changing the way a person experiences, conceptualizes, or understands a phenomenon. It follows that the most important form of teaching is that which brings about such changes. (Marton, 1992, p.253)

Taking all the aforementioned into consideration, following a phenomenographic approach to this study implies that the researcher investigates first the finite number of understandings of dimension that students have, and then creates a learning environment in which students may examine, explore and reconstruct, modify or even build on those understandings.

Adding to the above, the two elements, which form the specialisation of phenomenographic research, are (a) the object and the aims of research and (b) the methods used for studying this object and these aims of research. Francis (1996) talked about the relationship between the aims and the methods in phenomenography and she pointed out two characteristics of this relationship:

> (a) Its insistence on attempting to capture conceptualisations which are faithful to the individual's experience of a selected learning phenomenon.
>
> (b) Its further aim of categorising conceptions of learning and exploring relations amongst them. (p. 36)

She argued that the challenge of phenomenography as an approach is first to identify whether there are categories of conceptions that are common among different individuals within a specific content, and second, to explore whether the relationship of these categories can be extended to a wider content.

Phenomenography was considered as the most appropriate approach for exploring this study's research questions, because first it provides a theoretical platform for studying educational 'phenomena' like dimension, second, it aims to describe the 'experiences' of the individuals, and third, it takes into account the role of 'situation' in the formation of these experiences. After having the theoretical foundations of phenomenography as the umbrella of this research, the next step was to look through a micro-perspective to the designing of the research process. Thus, the next section moves a step forward by looking in more depth to how research situations of dimensional experience were designed as they are considered to play an important part in the phenomenography approach.

3.6 Researching experience within a situation

Following a phenomenographic approach in examining students' experience of dimension and considering the role of the context, this study took advantage of this dialectic relationship between experience and situation and decided to use different *situations* as *windows* on dimensional experience. The term 'window' draws on Noss and Hoyles's (1996) notion of 'windows on mathematical meanings', which is based on the idea that:

> […] we can set thinking in motion, and try to study what happens; we can set ideas in turbulence and investigate how changes occur; we can introduce new notions and try to understand how the thinker connects these with what he or she already knows (p. 9)

The windows are designed for looking through the eyes of the learner, consistent with the aim of phenomenography to have an insight of his/her experiences. In order for the window to work, it has to set students' thinking (in our terms, experience) in motion and this could happen by introducing new notions and observe how students act in specific situations:

> […] in order to make sense of how people handle problems, situations, the world, we have to understand the way in which they experience the problems, the situations, the world, that they are handling or in relation to which they are acting. Accordingly, a capability for acting in a certain way reflects a capability of experiencing something in a certain way. The latter does not cause the former, but they are logically intertwined. You cannot act other than in relation to the world as you experience it. (Marton and Booth, 1997, p. 111)

This study aimed to create situations, which had the phenomenon of dimension implanted within them in order to examine students' experiences. In this way, the categories of description of students' dimensional experiences would emerge from those situations. Thus, other research on how students abstract knowledge and generalise through a given situation was taken into consideration. In the following paragraphs, the notions of *abstracting in context, webbing*, *situated abstractions* and *situated generalisations* emerging from these types of research are explored.

Rina Hershkowitz, Baruch B. Schwarz and Tommy Dreyfus have talked about abstracting in context. They defined abstraction as being a process in which students "reorganize previously constructed mathematics into a new mathematical structure" that they referred to as 'novel structure' (Hershkowtiz, Schwarz and Dreyfus, 2001, p. 195). In order to distinguish abstraction as an outcome and abstraction as a process, they created some definitions:

> Abstraction is ***an activity*** (in the sense of activity theory), a chain of actions undertaken by an individual or a group and driven by a motive that is specific to a context. "
>
> ***Context*** is a personal and social construct that includes the student's social and personal histories, conceptions, artifacts, and social interaction. "
>
> Abstraction requires ***theoretical thought***, in the sense of Davydov; it may also include elements of empirical thought. "
>
> ***A process*** of abstraction leads from initial unrefined abstract entities to a novel structure.
>
> ***The novel structure*** comes into existence through reorganization of abstract identities and through establishment of new internal links within the initial entities and external links among them. (Hershkowtiz, Schwarz and Dreyfus, 2001, p. 202)

The significance of abstracting in context is shown clearly in the work of Hershkowitz et al. (2001). They supported a socio-cultural point of view arguing that the process of abstraction depends on the activities in which the students are involved, on any tools or manipulative they come into contact with, on the students and teachers' previous experiences and all that as part of a specific social and physical setting. In their study, they aimed in discovering or creating such processes, in which the students could use mathematical abstractions.

Looking into the process of abstraction in more depth, Nemirovsky introduced the term 'situated generalisation' to describe the way students generalise through a particular situation. He defined generalisation as "the process of inferring principles from a set of instances or applying a principle to a set of instances" (Nemirovsky, 2002, p. 233). He then referred to 'situated generalizing' as "a generalizing that strives to bring up the circumstances and trace resemblances rooted in them" (p. 236) and distinguished it from formal generalisation.

According to Nemirovsky (2002), in a formal generalisation, the phenomena that are generalised, are "mapped to or outlined in terms of an autonomous formal system" (p. 237). Every symbol has a 'radical autonomy' with respect to an object (i.e. apples, boxes). On the contrary in situated generalisation "there is no radical separation between the phenomena and the realm where the generalisation takes place" (Nemirovsky, 2002, p. 237). The formal generalisation can be considered as more general by taking place independently from any context and applying to any situation while the situated generalisation is more specific by taking place within a particular context. However situated generalisation represents particular occasions of life influenced by the social context they are in:

> In situated generalising, on the other hand, particulars are expressions of the general form within forms of life; it is for someone who lives in a society that ordinarily uses Celsius scale for temperature, that 19 degrees is immediately recognized as typical of nice days, or for someone who lives in New England to distinguish in an accent whether a speaker grew up in Boston or in New York (Nemirovsky, 2002, p. 253).

A generalisation formed is unavoidably influenced by subjectivity; everyone experiences life differently, and that is how everyone experiences any rule and any generalisation. For making sense of something, people tend to situate it into their own experiences. Thus as a first step this study had to consider that each student would experience the research situation designed differently, according to his/her own prior-experiences.

Therefore, after exploring the significance of the abstraction as a process through the work of Hershkowitz's about 'abstracting in context', and after considering the individual way on generalisation through Nemirovsky's notion of 'situated generalisation', this study looked for a more concrete statement to describe the nature of these abstractions/generalisations. Noss and Hoyles (1996) talked about students' nature of abstractions while interacting with mathematical tasks and computational settings, which was considered relevant to the designing part of this study. They argued that "the idea of webbing is meant to convey the presence of a structure that learners can draw upon and reconstruct for support - in ways that they choose as appropriate for their struggle to construct meaning for some mathematics" (p. 108).

Next, the notion of situated abstraction is introduced in order "to describe how learners construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed" (Noss and Hoyles, 1996, p. 122). According to Hoyles and Noss (1992) 'situated abstractions' are generalisations that students form in order to act in specific mathematical contexts. Situated abstractions are embedded in the particular content they take place. Although they may be mathematically correct or incorrect, they are useful for making sense of students' thinking when interacting with mathematical tasks, usually in a virtual environment. The main idea regarding webbing and situated abstraction is that:

> Learners web their own knowledge and understandings by actions within the microworld, and simultaneously articulate fragments of that knowledge encapsulated in computational objects and relationships- abstracting within, not away from, the situation. (p. 125)

Although, Noss and Hoyles (1996) focused on the designing of computational environments as situations for experience, it was considered that the notion of 'situated abstractions' could be extracted from any situation as long as it is in a situated form. Pratt and Noss (2002) built on the above definition of 'situated abstraction' and argued that:

> At the same time, a situated abstraction, or, to put it more exactly, the relationships and actions based upon the situated abstraction, are expressed in a language (not necessarily verbally articulated) that remains embedded in the situation in which it was constructed, potentially constraining its validity in new contexts, with different tools and affordances (p. 459).

Relating situated abstraction to Lave's notion of activity-setting, it can be seen as an expression of an activity embedded in the situation it was first created. The characteristics of the situation strongly influence the activity and restrain it from taking place in a new situation. There was the potential risk that the students would be engaged in the research situation and they would express experiences focused on the situation ignoring the phenomenon studied. Marton and Booth (1997) discussed this view:

> The researcher may thus opt to focus mainly on ways of experiencing the situation or on ways of experiencing the phenomenon. But the learner, as well, may focus mainly on the situation in which the phenomenon is embedded or on the phenomenon as it is revealed in the situation, as reflected in the different approaches of learning [...] Thus we have to realise that the researcher might be primarily interested in exploring the variation in the learners' experience of the situation and discover that some of her learners are largely oriented toward the phenomena that are present there. Yet again the researcher might be primarily interested in exploring the variation in the learners' experience of a certain phenomenon and find that some of the learners are largely oriented toward the situation in which the phenomenon is embedded. (p. 83)

Following Marton and Booth's suggestion of researching experience and situation, this study took advantage of this relationship and it broke down the research process into two phases. Phase 1 focused on exploring the variation in the learner's experience of the situation by studying the phenomenon of dimension *within situations*. In order to do that, three different situations were created as windows on students' experiences. The experiences of dimension were extracted and analysed according to the specific situation. On the other hand, Phase 2 of this study focused on exploring the variation in the learners' experience of the phenomenon of dimension by studying the phenomenon of dimension *between situations*. This happened by exploring the affordances and the constraints of the three situations as well as the experiences of dimension extracted in relation to the orientation of dimensional experience as formed in the literature. Through this analysis, an additional situation was designed in order to embrace the experiences of the phenomenon of dimension not observed across the previous situations.

3.7 Limitations of this approach

Although phenomenography was chosen as most appropriate research approach to study students' experiences of dimension, this study identified some limitations that the specific approach has as well as some that the general qualitative approaches possess. An important key to a successful study is the validity of the research. Invalidity is a limitation that with the right approaches and the appropriate treatment of the data can be minimised. As Cohen et al. (2000) argued:

> [...] in qualitative data validity might be addressed through the honesty, depth, richness and scope of the data achieved, the participants approached, the extent of triangulation and the disinterestedness or objectivity of the researcher (p. 105).

Another component that can act as a limitation of a research is its reliability. Reliability is linked to validity and can be identified in the degree of objectivity of the data as well. Consequently, it was considered important that the data received were as objective as possible. For maximising the validity and the reliability of the study, the participants chosen did not have any benefits from the results of the study neither did the researcher have to validate any pre-determined theories. On the contrary this study was an exploration of a phenomenon and did not test any hypotheses.

A limitation of phenomenographic research is that it captures the variety of students' experiences of learning on the one hand, but "it certainly did not imply that we could use these differences to meaningfully classify all cases" (Marton and Saljo, 1984, p. 45). Another limitation is that phenomenography requires the reduction of the variation of talking about a phenomenon presented in the quotes, to a limited number of categories, usually between three and five. However, "the conceptions of reality can be expressed in a larger number of linguistic forms" (Saljo, 1988, p. 42) that cannot be represented fully by such a small number of categories.

Moreover, as aforementioned, the findings of a phenomenographic research are based on the responses of the individuals on a specific content/context/learning. Therefore, they are embedded in or dependent on the situation in which they take place, even if they express pre-conceived ideas. Thus, the question raised was: How do we know that the individuals would not have said something different if we asked the question in a different way or used a different task? As Marton (1994) argued:

> It is simply not possible to deal with an object without experiencing or conceptualising it in some way. In this sense, subject and object are not independent, but they form a unity (p. 4426).

Similarly, Saljo (1988) pointed out that "one has to accept human thinking as contextually determined. All human thought takes place in a communicative setting is in part determined by specific circumstances in terms of situation and expectations"

(p. 42). Therefore, the conceptions of reality cannot be seen as decontextualised parts of the individuals but on the contrary the individual might use specific conceptions of reality in particular settings. Dahlgren's (1984) research, for instance, revealed that by presenting different objects as an example to the individuals resulted to a different reaction and explanation from them. Therefore the significance of the research situation was taken into consideration for the design of the present study by acknowledging that individuals might have led to specific types of conceptions of reality due to the particular tasks they were offered. Instead of avoiding the role of the situation, this study took advantage of its influence by exposing students in different situations and examining the variation of experiences they expose within each one of them as well as in between them.

As pointed before, the outcome of phenomenographic research is the categories of description, which although arising from the data, are nothing else than constructions of the researcher. Thus, it would have been possible that other categories would have come out, if another researcher performed an analysis on the same data. As Saljo (1988) stated:

> In fact, to be logical, it follows from a constructivist conception of reality that the possibility of interpreting reality differently applies to the activity of describing conceptions of reality itself (p. 45).

According to Saljo (1988), one way to examine validity is through comparison with other similar studies by testing the applicability of the particular categories. With regards to the present study, there were only a limited number of studies on the phenomenon of dimension, and even the ones that examined dimension in some way (coordinates, volume, 3D geometrical shapes) followed a different research methodology from this study. Therefore, their results were only taken into consideration for construction of the orientation of dimensional experience at the beginning of this study.

In order for the interpretation of the data to be reliable, Marton (1994) talked about having a second researcher to look at the outcome space results and judge whether the categories of description formed match to each individual case. If there is a reasonable degree of agreement between the two researchers then the findings could be considered as reliable. Similarly, Hasselgren (1996) argued that in order for the findings to be reliable and valid, a second researcher should repeat the phenomenographic analysis. This researcher should be ignorant of the outcome of the first researcher and the degree of validity and reliability will be based on how similar is the outcome of the second researcher. Hasselgren (1996) also talked about the findings being artefacts or not. This could be judged based on the degree the findings can be "interesting in ways other than only in connection to the immediacy and contextual conditions of the specific situation of this meeting and its contents" (p. 78).

In contrast, Marton (1994) argued that the categories of description cannot be re-examined due to the fact that the analysis in a phenomenographic research is "not a measurement but a discovery procedure" (p. 4429). Likewise, Sandberg (1996) characterises interjudge reliability to be a common criterion for measuring the results in mainstream social science but not in a phenomenographic research. He describes this technique as unreliable for the following reasons:

> (i) First, interjudge reliability does not take into account the researcher's procedures for achieving fidelity to the individual's conceptions being investigated.
>
> (ii) Second, and most fundamental, the use of interjudge reliability based on the objectivistic epistemology gives rise to methodological and theoretical inconsistency within phenomenography. (p. 140)

In this study, it was not feasible to have a second judge as it is an individual PhD thesis; however, my supervisor as well as participants in conferences and seminars that I presented my work-in-progress at times, were able to provide some critical examination of the analytical process. What is more, as a researcher I followed Sandberg's (1996) suggestion of phenomenological reduction as a way of acquiring reliability, and tried to ensure that the steps he suggested to do were considered during the research process (p. 138-139):

(1) The first step in this phenomenological reduction requires that the researcher is oriented to the phenomenon as and how it appears throughout the research process.

Indeed I was focused on the phenomenon of dimension throughout the research process starting from the exploration of the literature, moving to the design of the phenomenographic research and leading to the analysis of the data. The ways in which dimension appeared to individuals were updated throughout the study leading to the change of perspective while moving from Phase 1 to Phase 2.

(2) The second step in the reduction requires that the researcher is oriented toward describing what constitutes the experience under investigation, rather than attempting to explain why it appears as it does.

Doing phenomenographic research requires the researcher to be as descriptive as possible, searching for a variety of ways in which dimension could be experienced but avoiding any judgments of the way individuals experience dimension. Thus, judgments of right or wrong experiences are avoided.

(3) Treating all aspects of the lived experience under investigation as equally important.

As a researcher, I have been open to any types of possible dimensional experience and challenged all of them in the same way without heading to one perspective ignoring another. All the responses considered to be relevant to dimension were quoted and tackled in the same way pointing out their similarities and differences, and leading to their distribution into categories of description.

(4) The fourth step implies a search for structural features, or the basic meaning structure, of the experience under investigation.

As a researcher, I tried to find similarities and differences between the excerpts in order to organise the experiences reflected into groups, identifying each group's basic features and thus creating the categories of description of dimensional experience. The distinction of the categories was based to Saljo's (1988) suggestion of creating an internal structure to a category system:

> (i) There may be an internal structure to a category system in the sense that what separates conceptions of a phenomenon is what is assumed to be in need of being explained.
>
> (ii) A further aspect of the internal structure of categories that depict different conceptions of a phenomenon is that learning can be described as the change from one conception within this structure to a different one. (p. 46)

(5) Explicating the variation in the conceptions identified: (a) identifying what the individuals conceive as their reality, (b) identify how the individuals conceive that reality, (c) relating the individuals' ways of conceiving to what they conceive as their reality.

The variation in the conceptions was identified through the categories of description where each category was described by its significant characteristics. The basic features of each category were then challenged and formed relationships between them and presented in a more general structure. As Saljo (1988) argued:

> The research process has as an essential ingredient – the aim of generating a picture of the variation in human conceptions of phenomena in the world. The outcome of this endeavour results in a description of categories depicting conceptions of reality. (p. 45)

3.8 **Summary**

This chapter presented the research questions of this study, as extracted from the exploration of the literature in the previous chapter. As the aims of this study referred to children's experiences of dimension, the notion of 'experience' was further explored and defined. Subsequently, an appropriate approach to study these questions was needed. A qualitative research methodology following a phenomenographic approach was considered to be the most appropriate to study dimensional experience. After a discussion about the notions of experience and situation and their connections, it was decided that this study would take into consideration the role of the setting to offer a window to the researcher on students' dimensional experience. The notion of situated abstractions was considered as the most appropriate in order to explain the form of the articulations of students' experience.

In talking of situations and the phenomena they embed, this study was divided into two phases. Phase 1 focused on exploring the variation in the learner's experience of the situation by studying the phenomenon of dimension *within situations*. Phase 2 focused on exploring the variation in the learners' experience of the phenomenon of dimension by studying the phenomenon of dimension *between situations*. Finally, the limitations of this approach were described, explaining a procedure of steps used to minimise them.

In the next chapter, the method used in this study as well as a brief description of each of the three situations designed and the participants involved are discussed leading to a detailed explanation of how the procedure of data analysis was conducted.

Phase 1

Chapter 4: Method

4.1 Overview

After choosing phenomenography as a research approach to this study pointing to both the affordances and the limitations this might have to the outcomes, and after considering notions significant for this study such as experience and situation, this chapter moves a step forward by describing the conduct of Phase 1 of the study. As aforementioned, Phase 1 is interested in exploring the variation in the learners' experience of the situation and examining whether the participants are oriented towards the phenomena that are present there (Marton and Booth, 1997). Three situations were designed in order to examine students' experiences. In all three situations, the interview was selected as the best method for this purpose. Section 4.2 discusses how the interview was chosen to be more representative for gathering the data, followed by section 4.3 which is an overview describing the three situations designed and the participants involved. Subsequently, section 4.4 discusses the ethical issues arising, leading to section 4.5, which discusses the procedure of data analysis.

4.2 Interview as a method

According to Saljo (1996) the purpose of phenomenographic research is "to situate people in communicative practices and to attempt to understand how we make sense of the world across conceptual and linguistic boundaries" (p. 24). The interview technique is an important factor in encouraging people to express their conceptions. There are three major types of interview. First is the structured interview, in which the questions to be asked, their sequence, and the detailed information to be gathered are all predetermined; second is the unstructured interview, which is an open and a very flexible situation. The third one, which is the type used in this study, is the semi-structured interview, in which the interviewer has some initial questions to be explored but also these questions are flexible and they can be modified as the interview unfolds.

This study chose to use semi-structured interviews because, on the one hand the tasks given to students were pre-designed, but on the other hand, the questions used were open to modification and reconsideration as the interview unfolded. In this study, the researcher created the interview plan that included tasks and questions regarding the exploration of the individuals' dimensional experience. Among the advantages of using interviews as a method is that the researcher has the opportunity to have direct contact with the participants of the research, and therefore, is able to explore the subject in more depth.

The semi-structured interview is the dominant method for collecting data in phenomenographic research (Dall'Alba, 1996; Francis, 1996). Marton (1994) argued that the interview should not be a structured one having many pre-designed questions, but on the contrary, it should allow space for more questions to follow depending on how the interview unfolds. Similarly, Francis (1996) stated that interviews should be planned "in an open-ended way, specifically allowing for a range of possible responses" (p. 39) and leading and directing questions should be avoided. On the contrary, when 'leads' - as she calls the leading comments to a specific content/context/learning – appear, they should be reported in the analysis informing their effects if any.

Although it is believed that the validity and the reliability of an interview is best controlled when the interview questions are highly structured and have the exact same format and sequence for all participants, this impedes open-endedness and social interaction between the interviewer and the interviewee (Cohen, Manion and Morrison, 2000). Having then a semi-structured interview, the social interaction is more promoted which I consider as more important in this specific study.

Bowden (1996) stated that interviews in phenomenography start with a planned question or a given situation. Similarly, Saljo (1988) proposed either asking people direct questions or having them solve or explain a problem or a situation. Marton (1994) proposed two types of questions that can be used in the beginning of the interview. The first one was by asking the individual the general question "What do you mean by … (the phenomenon examined)". Correspondingly, a part of this study included questions to individuals as to what they mean by dimension and spatial dimension.

The other way of starting the interview was by introducing a concrete case to the individual, such as reading a text, a problem-solving activity or a familiar situation in which they can engage. Francis (1996) proposed that other ways of helping individuals to express their thoughts could be adopted accompanying the interview, such as writing, drawing and acting. She defined the aim of the interview in phenomenographic research as "to have the interviewee thematise the phenomenon of interest and make the thinking explicit" (Francis, 1996, p. 38). Likewise, Saljo (1996) supported that by posing problems "we have better possibilities of establishing a joint definition of what is being talked about" (p. 24) because in the phenomenographic approach the communication is a two-sided affair between the researcher and the interviewee. Pramling (1996) also stated that offering children a situation they can talk about it is a good tool for helping them to express their conceptions:

> When it comes to children's ways of expressing their conceptions, other complementary tools are needed to enable them to express themselves, e.g. drawings, drama, games, problem solving, etc. (p. 85)

Indeed for undertaking research with children, and more specifically primary school children, there is a need for giving them something to talk about or to engage with, in order for the interview to be meaningful to them. Therefore, in all of the situations designed, the interview plan included at least one task with which the students would be asked to engage and thus create a situation where they would have been able to express themselves. Adding to this, Bowden (1996) suggested two types of questions that can be used: (i) problem questions in the field studied and (ii) 'what is X?' type of questions. Both of these types of questions were included in the interview plans of this study.

Although the interview was considered to be the most appropriate method for data collection for this study, there are several threats posed while doing interviews that were considered. The sources of bias are the characteristics of both the researcher and the students but also of the content of the tasks designed. For instance, the interviewer may seek answers to satisfy his/her expectations using leading questions. Moreover, there may be misconceptions of the students regarding what they are asked, and more dangerous, misconceptions of the interviewer of what the responses mean. Consequently, as the researcher of this study, I have tried to be as objective as possible in both the interview process and my interpretations of the data, aiming to diminish as much as possible the limitations of interview as a method.

After deciding on a phenomenographic approach and to the interview as a method to study the ways individuals experience dimension, the next section discusses the three situations designed for the research as well as how the participants were chosen in order to be useful to the aims of the study.

4.3 The Situations and the Participants

The unit of analysis in this study is as for most of the phenomenographic approach the meaning for dimension. By *meaning*, I refer to an articulation of experience of dimension within an educational context. In total there were 168 meanings articulated across the three research situations developed in Phase 1 and this was regarded as sufficiently large to warrant comparisons to be made across the meanings and to introduce structure according to my orientation towards dimension as emerging from the literature review.

These 168 meanings were generated through activity with 12 students. Phase 1 of this study consists of three research situations. Details about the specific method of each research situation, the sample chosen as well as the time of the interview and the interview plan are discussed in the next chapters. However, in this section, I summarise the most important features of each situation in order to offer an idea of what it is to follow:

SITUATION I: The method used was the interview accompanied by tasks based on the computer software application Elica. 69 meanings were generated through activity with eight 10-year old students, four of them from Cyprus and four from England.

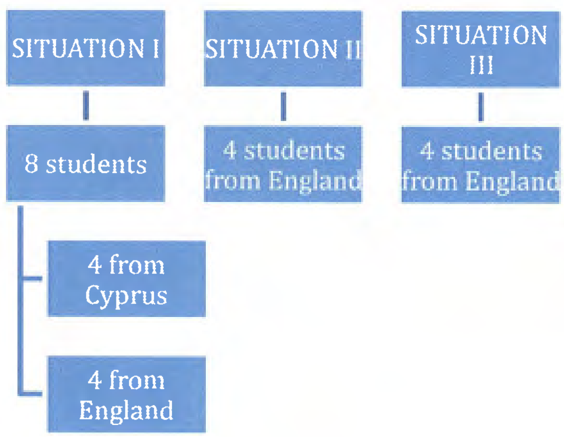
SITUATION 2: The method used was the interview. 26 meanings were generated through activity with four 10-year old students from England.

SITUATION 3: The method used was the interview accompanied by the film 'Flatland'. 73 meanings were generated through activity with the same four 10-year old students from England who participated in the second situation.

This study did not aim to represent the wider population, but on the contrary, it represents itself or instances of itself in a similar population. Therefore, the number of the participants is small. A second reason justifying the small size of the participants, is that in phenomenographic research, the individuals are not considered to be the sample; it is their meanings instead. Thus, even though the participants were just 12, the meanings they generated were 168, which was considered to be a reasonable size for the sample.

The method of sampling was the non-probability one, where some members of the wider population will definitely be excluded and others definitely included (Cohen, Manion and Morrison, 2000). Table 2 shows the sample of the population used for the study:

Table 2: Participants from which the meanings were generated



As illustrated by the table, the participants of the first two situations were from two different countries, England and Cyprus. It was decided to take data from two countries because of convenience, though it is possible that the background of the students might prove to be a factor in the way the meanings become organised. Students from Cyprus as participants did not continue after the first situation due to the difficulty of the researcher to have access to Cypriot schools. This did not have any impact on the study, as this research is not comparative between countries and also the twelve students used provided us with a sufficient range of meanings.

The students were 10 years old, the English were studying at St Cyprian's primary school in London, and the Cypriots were studying at Christakio primary school in Limassol. The sample was chosen to include 10-year olds, firstly, because of practical issues due to the fact that older students were mostly unavailable because of their SAT's examinations, and secondly, because at the age of 10 students have many experiences of dimension both inside and outside of the school environment. The students at this age had not yet experienced formal geometric teaching whereas older children were more likely to have been inculcated into the school's or the curriculum's perspectives on dimension, which may have overridden primary intuitive experiences. However, by age 10, the students were likely to be able to articulate their own thinking about dimension.

The students were from a Year 5 class; the English students are taught the English curriculum, as it is described by the country's National Curriculum and the National Numeracy Strategy, while the Cypriot students are taught the Cypriot curriculum as it is described by the National Curriculum of Cyprus. The students chosen are upper-middle ability students in mathematics. The students worked in pairs during all the situations. It can be added that working in pairs was chosen in order for the students to work with their class partner with whom they feel comfortable, and also because through discussion and argument they would easily externalise their views in language, helping the researcher understand their thoughts.

As argued in the previous chapter, the setting has a significant impact on the formation of individuals' experiences. The school could act as a setting in which activity takes place. In a micro-perspective, the task itself given to students could be the setting. Thus, in order to have a range of meanings generated, this study decided to design three situations for exploring students' experiences of dimension. Therefore, the 168 meanings were generated through three different situations, having a variety of tasks which individuals worked with.

As the phenomenographic study seeks to explain the similarities and to look for connections within the meanings, it was considered necessary to reconsider the role of the individual student and the setting, such as the task, in the generation of those meanings. In this sense, the students were not regarded as members of the sample but as factors in the analysis of the meanings.

4.4 **Ethical Issues**

> "[...] interviewing one's peer raises ethical problems that are directly related to the nature of the research technique employed" (Hitchcock and Hughes cited in Cohen, Manion and Morrison, 2000, p.66).

According to the method of data collection mentioned before, it can be assumed that many ethical issues were taken into deep consideration. Interviews were used with young 10-year old children. To start with, a letter including the purpose of the study and the conditions of the research accompanied the interview plan, was sent to the principal of the school. As soon as the principal agreed to the conduct of the research, the project outline was submitted to the teachers for further consideration. Due to the obligation to protect the anonymity of the research participants, it was wise to send a letter to parents asking for their permission to take pictures of their children's work and record them on video. It can be added here that the regulations of Cyprus education is to get permission for conducting research from the Ministry of Education as well. Therefore, before receiving permission from the school, I applied for permission from the Ministry of Education in Cyprus. Moreover, all the participants were asked whether they agree to the use of their first names during the interviews for practical reasons.

It can be noted here that before starting any interview, I explained to the children some necessary points regarding the whole process to follow, as well as some other notes: *1. There are no correct or wrong answers 2. Your answers should represent your own feelings and beliefs 3. You are free to withdraw consent and to discontinue participation in the project at any time.* As a thankful act to the principals and the teacher, copies of the final research thesis are going to be sent, hoping that they might be useful for both the teachers and students.

4.5 Procedure of Data Analysis

For collecting the data, this study made use of the software Camtasia (TechSmith, 2009) for recording the interview and the students' actions on screen for Situation I, while for Situations II and III a video camera was used. Like any other research approach, in phenomenography, after collecting the data from the interviews, the researcher has to analyse them. The analysis of this study was based on Marton's

(1994) suggestion of a procedure of analysing the data, which was considered to be based on the phenomenographic approach.

After the word-by-word transcription of the interviews, the researcher's first step was to bracket any preconceived ideas. Dahlgren (1984) argued that "the protocols have to be studied with the intention of understanding what the students are expressing, irrespective of what words or examples they may use, which may show a considerable variation even between answers belonging to the same category" (p. 29). The acceptable variation in the responses of the individuals had been limited in advance, and the researcher read, and re-read, the transcripts several times. In order for an excerpt to qualify, it had to be relevant to the orientation of dimension as created through the literature in Chapter 2 (see Table 1 in p. 61) of this thesis.

The questions that the researcher asked herself during the phase of reading were (among others): "How does the respondent construe the phenomenon? What concepts does he or she use to explain it? What types of similarities with other phenomena are introduced?" (Saljo, 1988, p. 41). Bracketing was focused on the similarities and differences of the ways in which the phenomenon of dimension appears to the participants and not to which degree the responses reflect an understanding of dimension similar to the one of the researcher. It was considered a possible option from the beginning of the analysis that the same participant might have expressed more than one way of understanding the phenomenon. Therefore, the individual is not the unit of analysis. The borders between the participants were temporarily abandoned and the individuals' transcripts gathered together to form undivided data to be analysed (Figure 21).

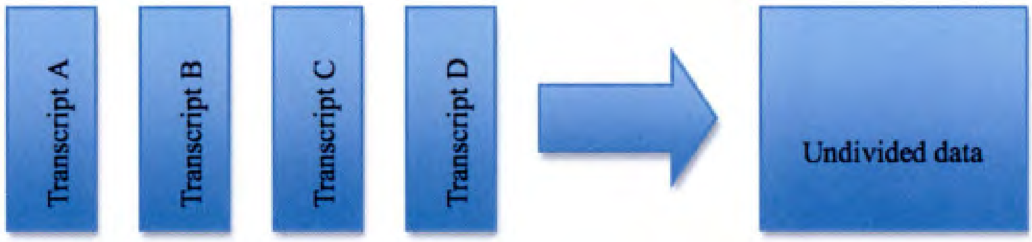


Figure 21: Transcripts as undivided data to be analysed

The bracketing of the preconceived ideas can be described in two phases. First, the researcher distinguished between what is relevant in expressing a way of experiencing dimension, and what was not. The selections made were based on the interpretations of the researcher who tried to be as open as possible to ways of experiencing or thinking about the phenomenon, having the orientation of dimensional experience formed during the exploration of the literature as the basic ingredient in this selection process. As Marton et al. (1984) pointed out:

> The meaning of a comment could occasionally lie in the words themselves but, in general, the interpretation had to be made in relation to the context within which that comment had been made (p. 41).

Thus, the phenomenon was delimited to the interpretation of the selected excerpts and the excerpts themselves were delimited according to the situation in which they took place. Considering the structure of experience as explored through the psychological theories in Chapter 2 (p. 40), the students' accounts were interpreted by looking to:

(a) The heuristics people use to make sense of dimension, which will point to intuitions (Fischbein, 1987) perhaps expressed in situated terms. The question raised was what kind of intuitions people have regarding dimension, a focus directly relevant to practicing teachers and educators.

(b) How people's accounts of experiencing dimension incorporate visualisations.

(c) How individual's primitive knowledge emerges in people's accounts.

Meanings are accounts of experience that are indicative of (a), (b) and (c) above. These accounts might sometimes be captured as phrases or short articulations including actions such as on-screen pointing or manipulations of physical apparatus. At other times, these accounts might involve relatively lengthy excerpts before the meaning unfolds.

From this first step, the 'pool of meanings' was created. The 'pool of meanings' included all the possible ways in which the participants might have experienced the phenomenon of dimension (Figure 22).

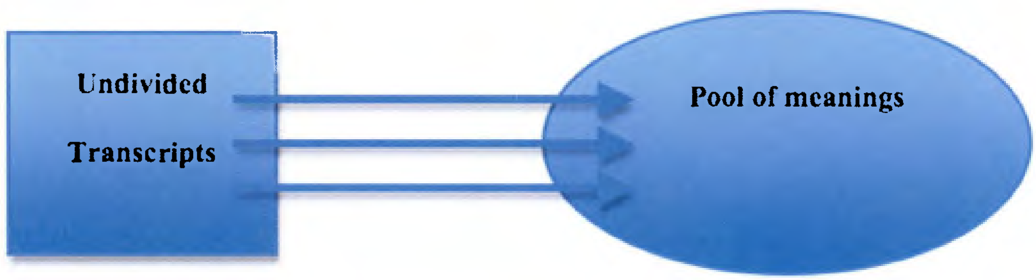


Figure 22: Creation of the pool of meanings

For instance, a random part of the 'pool of meanings' might look like those in Figure 23. I would like to mention again here that indeed there were 'entry criteria' in order for an excerpt to qualify to the pool of meanings, and these depended on the excerpts' relevance to the orientation of dimensional experience. However, there were no such pre-determined criteria for categorising the excerpts in the pool. This second stage of 'bracketing' is described in the next paragraphs.

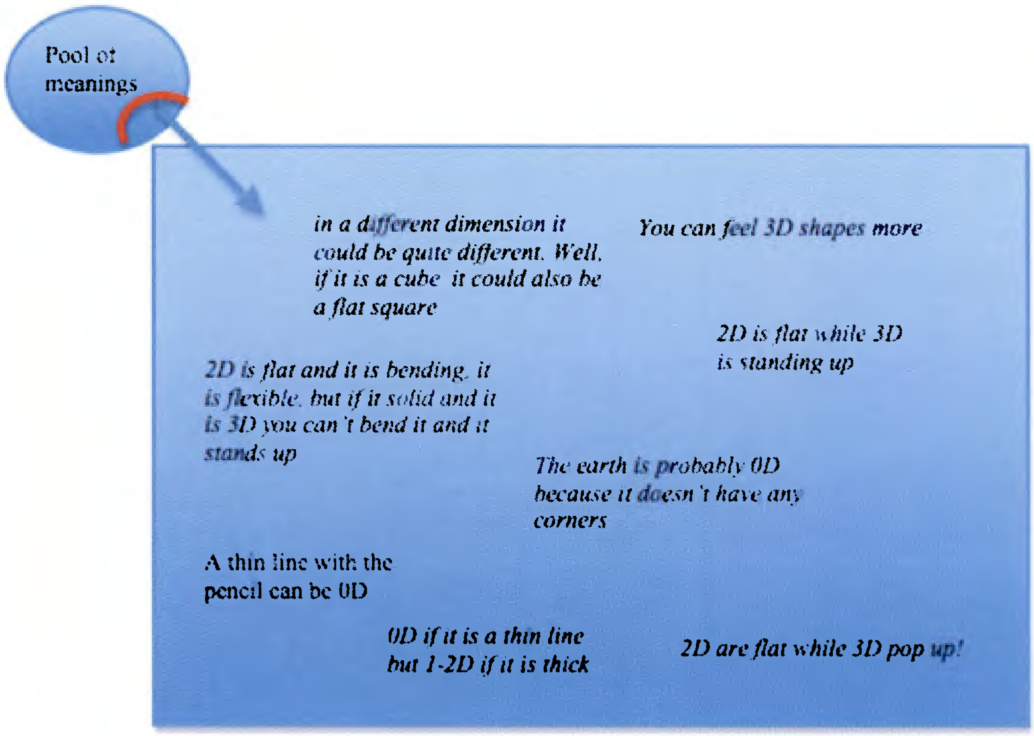


Figure 23: Example of a pool of meanings

In the second phase of the bracketing of the preconceived ideas, the researcher identified distinct ways of understanding dimension and a categorisation was formed based on:

(i) similarities that two expressions which might be different at a word level but reflect the same meaning, characterise a specific way of thinking about the phenomenon of dimension and,

(ii) two different meanings that two expressions might reflect, and thus, two different ways of thinking about the phenomenon would be characterised.

This categorisation was not based on any pre-determined criteria; on the contrary, the categories arose from the data as described above. If we use the part of 'pool of meanings' shown in Figure 23, the two expressions "2D is flat and it is bending, it is flexible, but if it is solid and it is 3D you can't bend it and it stands up" and "You can feel 3D shapes more" reflect on the same general meaning and that is to think about dimension as having materialistic attributes. On the contrary, the expression "in a different dimension it could be quite different. Well, if it is a cube, it could also be a flat square", reflects on a different way of experiencing dimension and that is seeing dimension as place.

In order for the researcher to identify and categorise the ways in which individuals think about dimension, a deep exploration of what has been said and done in the interviews was necessary. The various statements gathered had been analysed through two contexts:

(a) 'The pool of meanings' that derived from what all the participants have said about the same thing and,

(b) What the same person said about other things, and here is where the individual boundaries are reintroduced again.

in a different dimension it could be quite different. Well, if it is a cube, it could also be a flat square	*The earth is probably 0D because it doesn't have any corners*	*0D if it is a thin line but 1-2D if it is thick* *2D is flat while 3D is standing up* *2D is flat and it is bending it is flexible, but if it solid and it is 3D you can't bend it and it stands up* *2D are flat while 3D pop up'* *You can feel 3D shapes more*

Figure 25: Organised pool of meanings

After the quotes had been grouped according to the relations between them (Figure 25), the focus was shifted to the relations between the groups. In order to do that, the significant characteristics of each group were described followed by a distinction of features between the groups. Marton et al. (1984) explain the above procedure clearly:

> In very concrete terms it meant sorting the quotes into piles trying to extract a core meaning common to all the quotes in a certain pile, examining the borderline cases and eventually making explicit the criteria attributes defining each group, not the least in contrast to the other groups (p. 41).

Likewise, Dahlgren's (1984) suggestion on distinguishing the categories of description included "the reduction of unimportant dissimilarities e.g. terminology or other superficial characteristics, and the integration and generalisation of important similarities i.e. a specification of the core elements which make up the content and structure of a given category" (p. 28). Starting with a relatively large range of categories of description, after gradual refining the researcher ends up with a limited number of categories, which cannot be filtered further.

This is how the 'categories of description' were developed showing the variation in which dimension can be experienced, conceptualised and understood. Lybeck (1988) argued that in the categories of description, the answers of the questions '*What is conceptualised*' and '*How is it conceptualised*' should coexist, and he proposed that the categories of description should be in the form of "something (x) is seen as something (y)" (p. 101).

If we take the first box in Figure 25 above as an example, its basic characteristic was the focus on the domains (worlds) of dimension. The place where the object was located was considered a significant element for defining the number of dimensions the object had. Consequently, seeing "*Dimension as a State*" is a category of describing dimension. Similarly, if we take the third box, all the statements present the materialistic way of seeing dimension as an object that belongs in the real world. Thus, another category of description might be the "*Dimension as Material*".

Up to this point, there are many issues arising from the above procedure of data analysis. Here I discuss some that might need extra explanation. First, the method described above requires finding similarities and differences for sorting expressions of a phenomenon. Francis (1996), for instance, characterises this method as "a qualitative parallel to exploratory factor analysis of quantitative data" (p. 43). Second, it is true that multiple understandings of the same phenomenon might appear in one individual. As Francis (1996) argues:

> Since it is likely that a number of expressions will be reported from the same interviewee, efforts should be made to check, and if necessary explain, any contradictions or ambiguities, and to note any organising or relational theme (p. 41).

Thirdly, the categories are discovered from the data, and the data are not analysed in terms of any predetermined categorisations. The categories of description are created from the interpretation of the communication between the researcher and the individual. The categories are tools used for describing the conceptions of the individuals. They are not identical to conceptions; on the contrary as Saljo (1996) put it, the categories of description 'denote' conceptions. In phenomenography, conceptions refer "to people's ways of experiencing a specific aspect of reality" (Sandberg, 1996, p. 129). Most detailed, Bowden (1996) talked of two elements that influence the formation of the categories of description:

> Phenomenographic research produces descriptions which owe their content both to the relation between the individuals and the phenomenon (i.e. their conceptions) and also to the nature of the conversation between the researcher and each individual and its context (which includes the relation between the researcher and the phenomenon) (p. 65)

Bowden (1996) argued that we should have in mind that some of the categories created might not be of a kind that we would consider as reasonable; some understandings might occur that we have not expected. Consequently, the researcher should be as open and objective as possible to any type of responses.

Having the categories of description formed, the outcome space, which denotes the space of learning dimension, is created by challenging these categories, comparing and relating them to one another. According to Francis (1996):

> The ultimate phenomenographic aim is to explore relations between obtained categories in order to derive a meaningful structural model of the conceptions of learning exhibited by the sample of interviewees [...] This search for meaningful structure demands identification of the distinguishing features of categories and the determination of logical or other relations between them (p. 45)

The outcome space is a more detailed and precise description, which arises from the differences and similarities among the categories of description and aims in presenting a more general form of the results. As Ramsden et al. (1993) argued, outcome space involves "identifying common features of the several categories of description as a step towards a unified set of categories at a higher level of generality" (p. 308).

Taking all the above into consideration, the procedure of analysing data in this study can be summarised as follows (Figure 26):

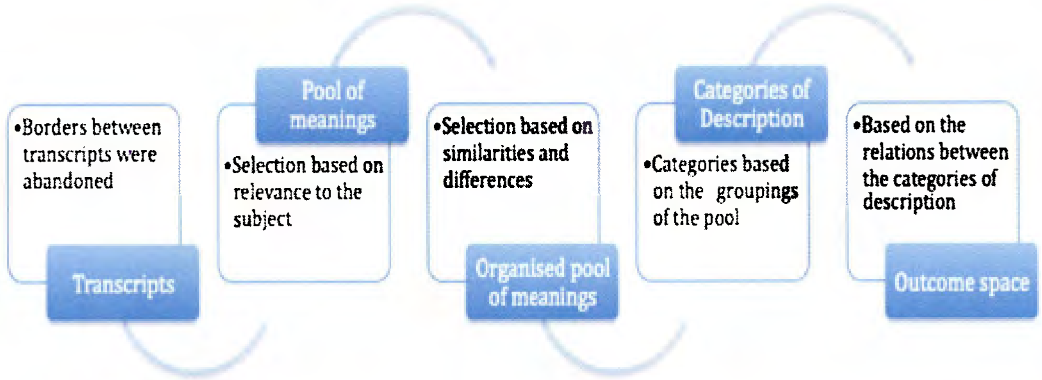


Figure 26: Procedure of data analysis

Although phenomenography follows a qualitative approach in analysing the data, it differs significantly from other qualitative data analyses. For instance, there are two main differences between a phenomenographic analysis and a standard content analysis although they are both exploratory. First, in the phenomenographic analysis, the categories in which the excerpts are sorted are predetermined, in the sense that there is a dialectical relationship between the excerpts and the categories:

> The analysis is dialectical in the sense that bringing the quotes together develops the meaning of the category, while at the same time the evolving meaning determined which of the categories are included or omitted (Marton and Saljo, 1984, p. 41).

Thus, the procedure of analysis is very lengthy with continual modifications to both the grouping of the excerpts and the precise meaning of the categories formed. After some iterational analysis, the rate of change starts of course to decrease, leading to its final stable form. The second main difference from content analysis is that in phenomenographic analysis, the researcher's attention is shifted from the individuals to the meanings rooted in the quotes whether these meanings come from the same individual or not. As Marton et al. (1984) argued:

> In this way each quote had two contexts in relation to which it had to be interpreted. First it depended on the interview from which it was taken and then on the "pool of meanings" to which it belonged. The interpretation was thus an iterative procedure which went back and forth between the two contexts for each unit of analysis. (p. 42)

4.6 Summary

This chapter discussed the method of data collection of this study which was the interview, pointing to the affordances and the limitations that this choice has to offer to the research. Subsequently, the three research situations designed for Phase 1 were described in brief pointing to the nature of the participants and the idea of *meanings* as the sample in phenomenographic research. The ethical issues arising were discussed, leading to a detailed description of the procedure of data analysis as informed by phenomenographers.

In the three subsequent chapters, each situation designed is described in more depth pointing out their specific methodological tools as well as details regarding the participants, the time of the interview and the design of the interview plan. Meanwhile, the procedure of data analysis as it took place starts to unfold.

Chapter 5: Situation I
Elica applications

5.1 Overview

The first situation designed used tasks in computer-based environments to explore students' experiences on dimension. 69 meanings were generated from eight 10-year old students, four students from the St Cyprian's primary school in London (England) and four from the Christakio primary school in Limassol (Cyprus), and according to their teachers they were characterised as being middle-ability. The eight students worked as four pairs. Interviews together with the computer-based tasks were the main methods in this study. Although each interview lasted for about an hour, the students seemed very interested and focused during the whole procedure. In the following sections, the method of data collection, the analysis and some of the findings are discussed in depth.

5.2 Method

Computer-based tasks together with the interview were the two sources of data collection. The interview questionnaire was prepared based on the principles described in the previous chapter (see Interview as a method section p. 103). Each task was designed according to the principles Ainley et al. (2006) proposed for creating purposeful tasks:

- ✓ It has an explicit end product that the pupils cared about.
- ✓ It involves making something for another audience to use.
- ✓ It is well focused, but still contained opportunities for pupils to make meaningful decisions. (p. 35-36)

The tasks of this study utilised applications designed by the DALEST project (http://www.elica.net/site/). DALEST stands for 'Developing an Active Learning Environment for Stereometry' and it is a project co-funded by the European Union under the Socrates framework (MINERVA, 2005 Selection). The University of Cyprus coordinated the project, and among the other participating institutions were the University of Southampton, the University of Lisbon, the University of Sofia, the University of Athens, the N.K.M. Netmasters and the Cyprus Mathematics Teachers Association. In particular, the DALEST project aimed to analyse the existing geometry curricula in European schools and to develop dynamic 3-dimensional software accompanied with activities for the teaching and learning of 3D geometry. Although the main objective of the project was to develop dynamic three-dimensional software suitable for teaching stereometry in middle schools, this study aims to extend its use in the primary school as well.

The DALEST applications are a set of dynamic microworlds that enable students "to construct, observe and manipulate configurations in space" but also "to promote their visualisation skills through the process of constructing dynamic visual images" (Christou et al., 2007b, p. 1). According to Christou et al. (2007a), the main purpose of the DALEST applications is "to enhance students' understanding 3-D geometrical reasoning and visualization and spatial thinking" (p. 5). The rationale of the DALEST project is based on the ideas of visualisation as defined by Presmeg (1986), Bishop (1980), Clements (1982) and Gutiérrez (1996a). This study used the Elica applications developed by the Dalest project, which is considered as *a modern object-oriented Logo implementation.* The two applications used from the DALEST project were: Cubix Editor and Math Wheel. The criteria for choosing the particular applications were that the application should (i) have the potential to act as a 'window' on students' dimensional experience, (ii) be easy for a primary school student to learn and use, and (iii) present objects as close to reality as possible without creating any illusions. In the next paragraphs, the two applications are described together with the didactical situations they can offer to students.

Cubix Editor

Description: An application on which students could create 3D structures of unit-sized cubes. Figure 27 shows what is presented on the screen of the specific application. There is a platform or a board as it was described, whose size could be modified having area 5X5, 6X6, 7X7, 8X8 or 9X9 cubes. The tasks of this study made use of a 5X5 board, which was considered to be the easiest and most representative for the specific age of students. On the board, the structures are constructed by just clicking on the desirable position. There is a variety of colours that can be used for the cubes created. The platform together with the constructions made can be dragged and rotated giving front, side and top views of the object (Figure 27).

The "Save" button as well as the disk pictured are for saving the constructions made, while the "Load" button is for loading any pre-made constructions that are saved before on the computer. Additionally, the user can also print pictures of the constructions made.

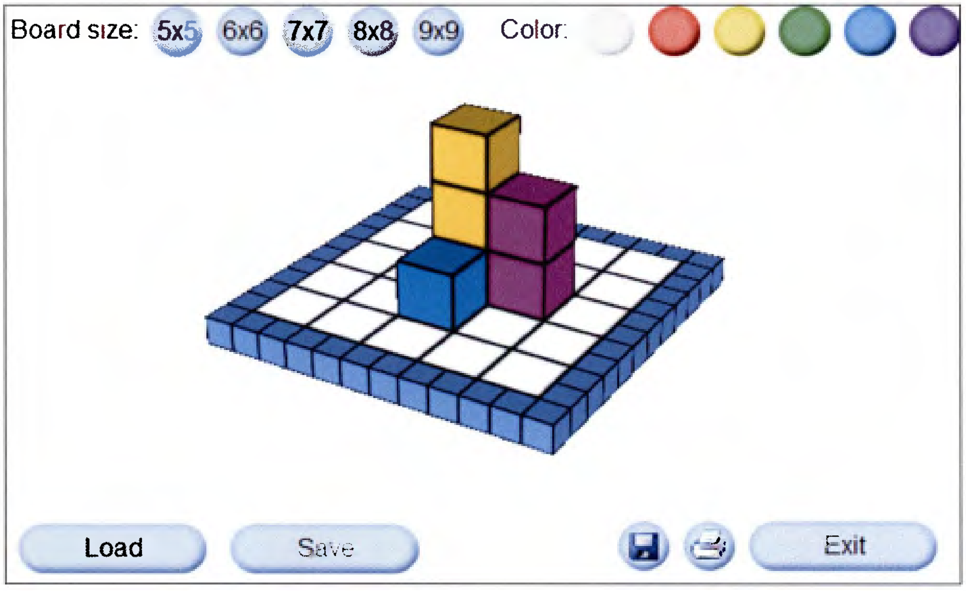


Figure 27: Sample of Cubix Editor screen

Didactical situation: There were two tasks associated with the Cubix Editor application. Task A was an introductory task that aimed to familiarise students with the application, while task B was the main task which was taken into account for the research's purposes:

A. Go to "Start", "All programs", "Elica 5.6", "Dalest" and press on the application "Cubix Editor". On the window that opens, there is a platform on which you can create cubes by clicking on it. You can change the colours of the cubes by clicking on the colours above.

   1) Create a shape using 7 blocks.

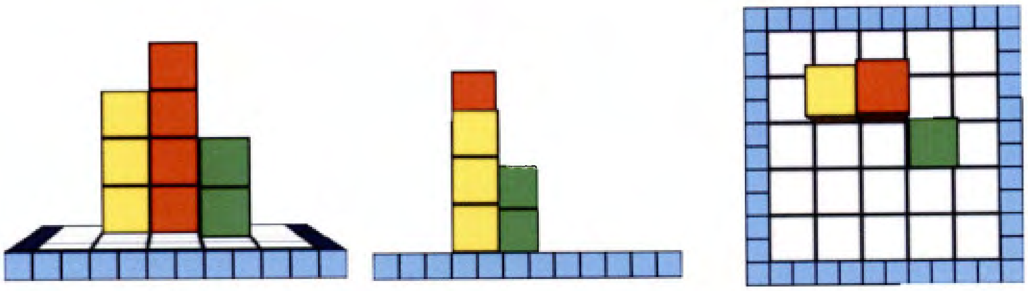
   2) Drag the platform so you can see the shape from the top, the side and the front.

   3) What are the differences on the shape if you look at it from above?

   4) What are the differences on the shape if you look at it from a side?

B. Now we are going to do the opposite process. Here are some pictures showing the front, side and top view of three 3D objects:

Object 1

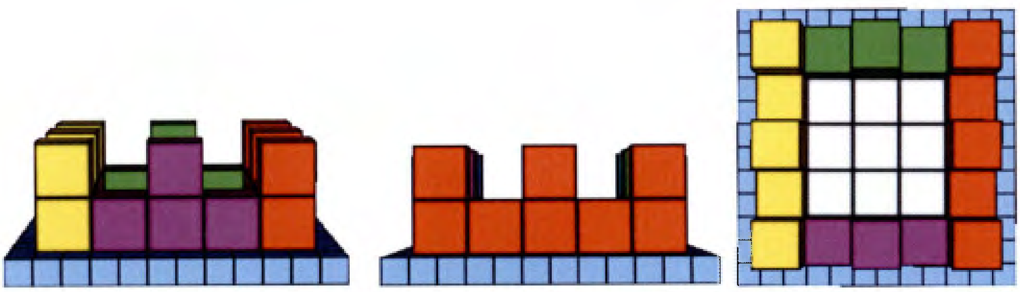


Front view | Side view | Top view

Object 2

Front view | Side view | Top View

Object 3



Front view | Side view | Top View

Figure 28: The representations of the three objects used

1.Try to imagine how the shape looks like from these pictures and draw it on the paper provided to you.

2.Create the shape on the computer using the Cubix Editor application.

3.Create the shape using plastic blocks.

More specifically, the aim of task A was to help students to discover what the application had to offer by creating cubes on the platform and dragging them in order to 'see' different views of these constructions. In task B, the choice of the objects followed an ascending order concerning the difficulty (according to the criteria below) that each type of representation brought to the task. The degree of difficulty was measured with respect to the variety of colours of the representation, the number of the cubes as well as their position in space. Students started from the easy representations and moved on to the more complicated ones. It can be added here, that the first pair of children followed the above sequence of activities on Task B. The second couple was asked to visualise the object and draw it on paper, then construct it using plastic blocks solids, having the construction on the computer as last. In this way, it was possible to reflect on whether the sequence of the tasks seemed to have an effect on students' constructions.

Aims of the task: The purpose of these tasks were to investigate the resources which students utilise as they tried (i) to visualise a 3D shape from 2D representations on paper, (ii) to reproduce the shape they visualised on paper and on the computer screen, and (iii) to compare the previous reproductions with the reproduction of using solid materials. In particular, these tasks explored the production of 3D shapes in a 2D real environment (paper), in a 2D virtual environment (computer screen) and in a 3D real environment (plastic blocks). There was a transition of dimensions from 2D (2D representations on the 2D real world) to 3D (real objects), then to 2D (2D representations on 2D virtual world) again. The task corresponded to the general aims of task design set before: First, the Cubix editor was an application that primary school students could easily use without the need of any introductory course. Second, the task involved 3D materials as well as 2D representations on paper and on the computer screen, which all appear to the everyday activities of the students, and is therefore part of their reality. Third, the task had an explicit product that students had to construct by the end of the activity. Fourth, it was designed as to be well-structured, having a sequence of activities but offering also opportunities for the students to experiment. Last but not least, it had the potential to act as a window on students' dimensional experiences about dimension by exploring how students reacted in different cross-dimensional situations.

Math Wheel

Description: An application on which students could construct 2D structures on a square sheet on screen, and these structures could be rotated on a vertical axis creating a 3D shape. The 2D shapes could be triangles, quadrangles, or circles.

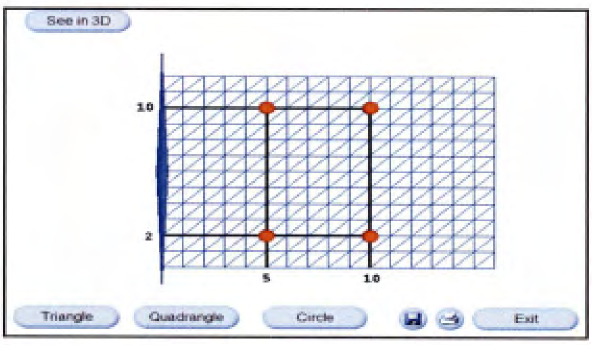


Figure 29: Sample of Math Wheel application showing the creation of a 2D shape

The user could drag the points to create angles or other, more complicated 2D shapes. When the wanted 2D shape was created, the user could press the "See in 3D" button and the 3D shape was created as it looks in Figure 30.

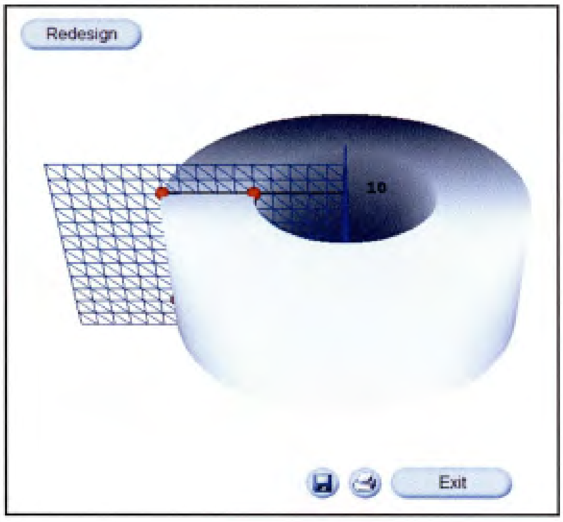


Figure 30: Sample of the **Math Wheel application showing the creation** of a 3D shape

Subsequently, the user could drag and rotate the 3D shape giving top, side and front views of the object. The user could save the whole object or print a picture of it. If the shape was not the one expected to be, the user could modify his/her 2D shape by clicking on the "Redesign" button.

Didactical situation: There were two tasks associated with Math Wheel application. Task C was an introductory task that aimed to familiarise students with the application, while task D was the main task, which was taken into account for the research's purposes.

C. Go to "Start", "All programs", "Elica 5.6", "Dalest" and press on the application "Math Wheel". On the window that opens, there is a square paper on which you can draw a shape. The shape can be a triangle (having 3 angles), a quadrangle (having 4 angles) or a circle. Press the triangle button to create a shape with 3 angles.

1) Draw a right-angle triangle.

2) Now press the “See in 3D” button.

3) What shape is created?

4) What does it look like?

5) What happened to the first shape?

6) Press the “Redesign” button and create a second shape.

7) With the “See in 3D” button, see the shape that is created.

8) What does the application do to each shape?

D. Now we are going to do the opposite process. You are given some real objects (Figure 31):

Lamp Toilet paper Pencil

Figure 31: Pictures of the real objects given

1. Examine these objects carefully and then imagine the shape that you should draw on the screen to create the above shapes. Draw your ideas on the paper provided.

2. Now draw the shape on the screen using the Math Wheel application. Then press the “See in 3D” button, and see if the first shape is created.

3. If the shape is not the same as the first one, how can we fix it? What went wrong? Click on the “Redesign” button to modify your shape.

Task C was an introductory task and therefore the shape created was a simple triangle that when rotated produced a cone. The cone could then be considered as a birthday hat or any other real 3D object that had that shape. Task D however, included more complicated shapes for the user to create. Task D3 gave students the opportunity to modify their views and therefore gave feedback to the user. It also acted as an excuse for the students to express their thinking in words in order for the researcher to explore the way they experience dimension.

Aims of the tasks: The purpose of these tasks was to look at students' articulations of experience when they tried to reproduce a 3D shape on the computer screen by visualising a 2D representation of the object. The specific real objects used were chosen in order to have a variety of different shapes. The pencil is a solid cylinder, the lamp is part of a cone, and the toilet paper is a particular type of cylinder having a hole in the middle. In Task D, there was a transition of dimensions from 3D (real objects) to 2D (2D representations on 2D virtual world) and then to 3D again. This task also followed the task design principles mentioned at the beginning of this chapter. First, it had the potential to act as a window on students' dimensional experiences when moving from one dimension to another. The task acted as an excuse for the students to externalise in words their thinking while working. Second, the materials presented to students were real objects that they had to draw on the computer screen (a lamp, a toilet paper roll and a pencil), so there was no evidence of an illusion or misleading. Third, the task was designed as to be well planned, including some basic predetermined questions but also offering the opportunity to the students to express their thinking as the interview unfolded. Fourth, the students had a target to be accomplished, and that was to construct the 3D shapes on the 2D screen.

Finally, the Math Wheel is an easy application for primary school students to use, without the need of any further instruction.

5.3 Data analysis

The data were analysed according to the procedure of data analysis described in the previous chapter (see Procedure of Data analysis, Chapter 4, p. 111). However, the analysis of this particular situation was not progressed through all the stages of the procedure. After transcribing the interviews, reading and re-reading the transcripts and deciding which excerpts were included in the pool of meanings, it was decided that the pool of meanings was not yet sufficiently rich to enable the effective imposition of a structure. The 'richness' of the pool of meanings was judged according to (a) the coverage of the aspects of the orientation of dimensional experience and, (b) the variety in the representation of the same aspect in the pool. Even a cursory glance of the pool of meanings by my supervisor and myself, led us to believe that some obvious meanings were not yet represented in the pool and that it would make sense to gather data from the second and third situations before starting to organise the data into categories of description.

The classification of whether an excerpt was appropriate to be included in the pool of meanings was based on its relevance to the definition of dimension and the orientation of dimensional experience as described in Chapter 2 of the literature review. However, I reproduce here the table of the orientation of dimensional experience in order to offer a more visible idea of the procedure of data analysis (Table 3):

Table 3: Reproduction of the Orientation of dimensional experience

Orientation of this study towards dimension
•**the identification, distinction and creation of relationships between 2D and 3D (and other dimensions) space/objects**
•**the articulation of dimension as a property of space/object (in any level of formality)**
•**the development of geometrical intuition and spatial awareness**
•**the development of an informed background of many aspects of the world relating to dimension that might be used to stimulate and challenge the students**
•**the identification of what stays invariant and what changes in a set of transformations**
•**the development of the ability of reasoning and proof in geometrical contexts**
•**the development of the ability to visualise, draw and construct figures**
•**the representation of dimension-related concepts whose origin is not visual or physical**
•**the use of mathematical language for describing objects/spaces**
•**the creation of relationships between reality and abstraction regarding objects/spaces**

Adding to the above, the students' accounts were interpreted according to the structure proposed through the exploration of psychological theories (see 113-114). The next section presents a brief summary of the selection procedure of the accounts for built-in the pool of meanings by giving examples of excerpts. The children's sentences are coded with their name initials. In some cases I place my own intervention on the excerpts in order to give extra clarifications for the reader. My own insertions are coded *[in square brackets and in italics]*.

5.4 Getting started in building the Pool of meanings

The pool of meanings includes all the excerpts that show evidence of dimensional experience. Figure 32 is a representation of the pool of meanings as it was extracted from the transcripts. The pool of meanings was drawn up in that way as to show its lack of structure. At this stage of analysis, the pool of meanings does not prioritise any particular excerpt, thus the meanings are entered into the 'pool' arbitrarily. Only when meanings are compared and organised in the next phase of the analysis, a hierarchy is proposed.

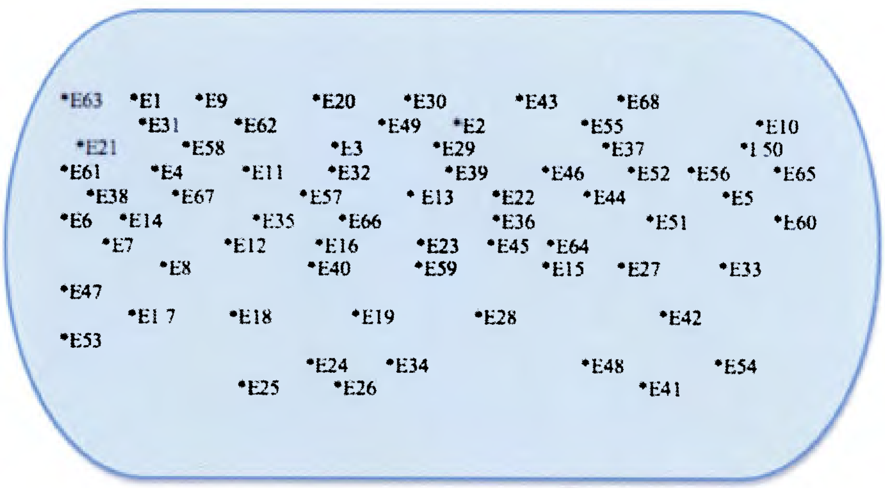
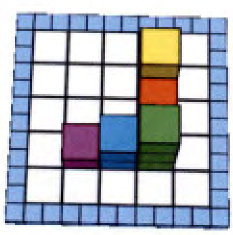


Figure 32: Situation I: The Pool of meanings represented by the arbitrary positioning of excerpts from the transcript. The * represents an excerpt and the code points to the specific excerpt. The lack of structure in the diagram emphasises that at this stage all meanings are regarded with equal weight.

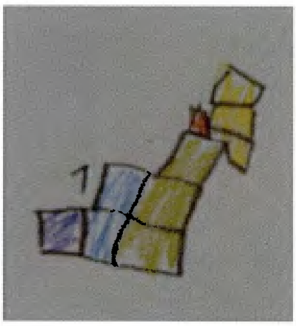
The above figure illustrates all the excerpts chosen to enter the pool of meanings. Each code (i.e. *E1, *E2, *E3…*E69) represents the corresponding excerpt from this situation (Elica applications thus the E used for coding) as presented in the appendices (Appendix 3, p. 400). In the following paragraphs, I present some examples from the procedure of selecting excerpts, pointing to how an excerpt qualified to enter the 'pool of meanings' and why some excerpts might have been rejected.

An aspect of the orientation of dimensional experience (Table 3) was the ability to visualise, draw and construct geometrical figures. Take for example the excerpt *E28. During the Cubix Editor application task, students had to visualise how a 3D shape was by reading its 2D presentations (front, side and top view). When they were asked to talk about the differences between looking at the shape from the front, from the side and from the top, it was argued that "*from the side view you can see only some squares, and on the top view you can see every single square*". So, while trying to visualise how the 3D shape looked, according to this excerpt, the top view is the most reliable. This excerpt was chosen to enter the pool of meanings because it concerns how students visualise 3D shapes from their 2D representations, which relates to the notion of dimension and our orientation in the literature.

During the same task (Cubix Editor application) students had to draw the 3D shape on paper by looking at its top, side and front view. If we take excerpt *E32 as an example, it is noted that the first drawing created included a mixture of the side and the top view.



Object on screen



Representation of the object on paper

Mi: This is the side view [Pointing to the whole shape on paper] and this is the back view of the side view [Pointing to the yellow and red cubes at the top].

Figure 33: Representation on paper

This again relates to the ability of visualisation and drawing mentioned above. Not only *E32 showed how the 3D shape is visualised but also how it can be represented on a 2D surface. This is relevant to dimension because it shows first the way we move from higher dimensions (3D) to lower ones (2D) and second how we represent a space/object of a higher dimension to a lower one.

Another example of an excerpt included in the pool of meanings is *E18. Again during the Cubix Editor application, students had to construct the 3D shape on the computer screen as they visualised it from its three views on paper. In *E18, the 3D shape was constructed ignoring the dimension of depth. All the blocks were drawn on the same line-level even though the green ones should have been in front.

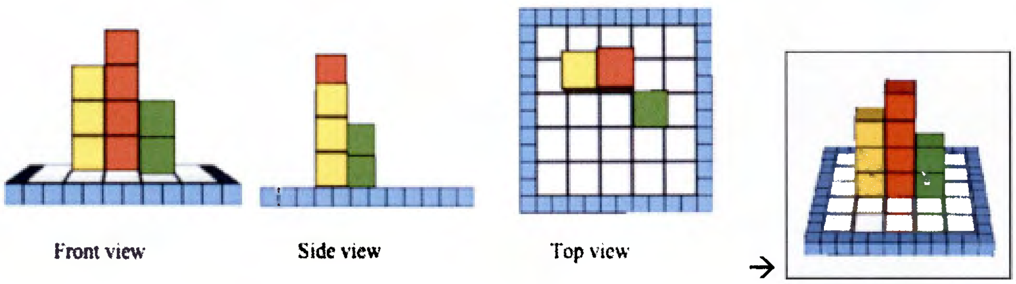
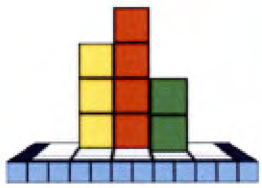


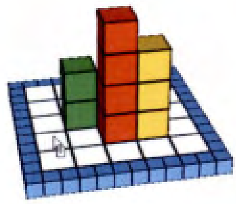
Figure 34: Cubix Editor Task B - Constructing object 1

Visualising and constructing figures is directly relevant to 'the ability of visualising, drawing and constructing geometrical figures' of the orientation of dimensional experience in Chapter 2 and thus it was chosen to enter the 'pool'.

The excerpt *E49 is another example referring to the visualisation ability. After constructing the 3D shape, the students rotated the Cubix Editor platform so that the front view of the shape would be exactly the same as the front view of the paper:



Front view as shown on paper



Object on screen

Researcher: The side view is the same. Let's see the front view. Is it the same?

S: No. If we look here [showing the front view on paper], it [the green block] looks kind of forward. We should put the greens on the other side and the yellows on the that side [showing the opposite side].

[Although the answer is right they can't imagine that this is another front view of the shape]

Figure 35: Cubix Editor Task B - Constructing Object 1

Not realising that the shape may have more than one front view relates to the ability of visualisation, and even though it is not directly related to dimension but to geometry in general, it was considered as a worth-including excerpt.

Relating to visualisation and construction but from another perspective was the excerpt *E24 for instance, where during the Math Wheel application task students had to construct a 3D real object given on screen, by using a rotation of a 2D shape. In *E24, a circle was chosen to be rotated in order to create the toilet paper:

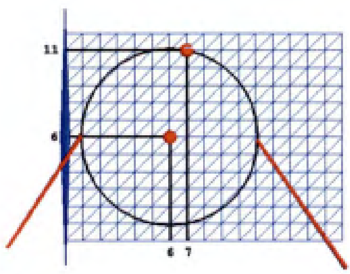
Researcher: What do we need to use to create the toilet paper? [on math wheel]

K: A circle.

A: Yes, maybe a circle.

Again this relates to the ability of visualising how a 2D shape might look after the transformation of a rotation where it creates a 3D shape, something directly relevant to the movement between dimensions.

The excerpt *E58 is also an example related to our orientation's 'identification of the restrictions of movement in a spatial environment' and that is why it was chosen to be included. For creating the lamp, the students created a circle having in mind that they could draw vertical lines on it – on a 2D surface- and eventually create the lamp. In a 2D surface, the degrees of movement are restricted and the students chose to ignore that.



S: If we can bring a line here, and here [showing a movement as the red lines I created at the next figure].

Figure 36: Math Wheel Task D: Creating the lamp

Relating to the part of the orientation about the 'creation of relationships within a spatial environment', in excerpt *E10 students formed a type of abstraction situated to the context it took place in order to reach their target. The statement "*the more to the left, the smaller the hole in the shape*" was helpful for creating the right version of the shape and although it does not relate directly to dimension, it relates to geometry and the way students make sense of the spatial environment they work in.

Another aspect of our orientation was the 'recognition of what stays invariant and what changes in a set of transformations'. In our interview, the transformation used was the rotation of a 2D shape to create a 3D shape. In excerpt *E34, it is shown that what the students considered as a change between 2D and 3D was that the 3D was "*longer*", "*bigger*" and it "*spread out more*" than the 2D shape:

Researcher: What does the application [Math Wheel] do to the shape?

Mi: It makes a different shape.

Ma: It makes it longer.

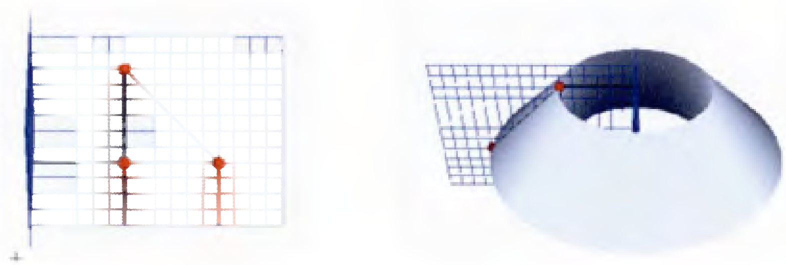
Researcher: What do you mean by 'longer'?

Mi: Like bigger.

Ma: Yeah, like, it spreads out more.

These characteristics were given to the shape because the dimension was changed and thus this excerpt was considered as relevant to be entered in the pool of meanings.

Other meanings relating to aspects of our orientation were included in the pool of meanings. For instance, in the excerpt about the identification of shapes, the students characterised the rotated triangle as a circle in an immediate and automatic response [*E6]:



Researcher: What shape is created?

C & M: A circle!

Figure 37: Math Wheel Task C - Rotation of triangle

However, these were excerpts which were not considered as relevant enough to enter the pool. An example is when students were working with Math Wheel to create the 3D shapes, most of the time students used a trial and error procedure, changing shapes without any explanation hoping to get to the right answer. This is a general mathematical procedure followed by students while solving a task and does not relate exclusively to dimension. Of course when there was something relevant to dimension it was included in the pool, but on the whole trying and changing procedure was avoided.

Another example of an excerpt which was 'rejected', was during Math Wheel where students drew the two holes of the lamp equal although the upper hole should be smaller. This might be important for accurate drawings in a product design business but it was considered not to be relevant enough to enter the pool. Last but not least, a third example of a 'rejected' excerpt can be seen during the Math Wheel application task where students mentioned that the fence where the 2D shape turns around is a "*symmetrical line*". This excerpt might be relevant to geometry in general but does not relate to dimension in such a way as to include it in the pool. In this section, I illustrated a representation of the pool of meanings together with some examples of excerpts that qualified to enter the pool, and excerpts that were avoided to be included in the pool. These examples offer an idea of how the selection procedure into the pool took place. The summary in the next section discusses the limitations of this situation leading to the design of the second situation.

5.5 Summary

Situation I was very useful for gathering excerpts for the Pool of Meanings. The tasks used enabled me to reflect on students' experiences about dimension. The Math Wheel task offered an experience of dimension as a quality of space occupied by a shape, while the Cubix editor task focused on 3D shapes and the reading of their representations. In general, students' actions followed a trial and error technique especially for the second task, and their meanings were often implicit in their actions rather than explicitly articulated through the spoken language. Sometimes, it felt as though there might be a wealth of meanings that remained obscure. Although the tasks were very interesting to students, they were not as rich as I had hoped in exposing their experiences in depth, failing in that sense to act as effectively as I had hoped as windows on experience of dimension. In judging the 'richness' of the pool, I noticed lack of representation of aspects of the orientation of dimensional experience and not having the same aspects representing multiply into the pool.

Thus, I decided that it would be too soon at this stage to begin a deeper analysis of the pool of meanings. Consequently, the next situation was designed with the intention that the tasks would provide me with greater access to experience of dimension, allowing the pool of meanings to be enriched before any consideration of how to structure those meanings, according to the phenomenographic procedure.

Chapter 6: Situation II
Interviews

6.1 Overview

After conducting the first situation it was decided that the 'pool of meanings' was not rich enough to start a deeper analysis. Therefore, the aim of the second situation was to enrich the pool of meanings with experiences of dimension. Learning from the previous situation, I had to design the next one in such a way as to act as a wider 'window' on students' dimensional experience. The interview was considered to be the most effective -at this stage- method to be followed in order to explore students' experience on dimension. 26 meanings were generated from four 10-year old students from the St Cyprian's primary school in London who according to their teacher can be characterised as being middle-ability. The students worked in pairs, having two pairs in total in this particular situation. Due to practical issues it was not possible to include students from Cyprus as well. The interview was the main method in this study and each interview lasted around 30 minutes. In the following sections, the method of data collection and the procedure of data analysis are discussed in depth.

6.2 Method

The purpose of the questions used in the interview was the exploration of individuals' experiences of dimension, including accounts of how they thought about dimension, inside and outside the school environment. The interview plan created was based on the 'Interview as a method' section in Chapter 4, and was also influenced by the kinds of questions of a clinical interview suggested by Hunting (1997). He proposed that the questions designed should: (i) be open ended so that students are allowed some freedom to choose their own preferred ways of responding, (ii) maximise opportunity for discussion or dialogue so that though processes can be revealed, and (iii) allow both student and interviewer to reflect on their respective thought processes (p. 153).

Similar to Situation I, the interview was semi-structured, having some predetermined questions but also allowing the researcher to add or modify the questions as the interview unfolds. The questions were posed in an open-ended way permitting a range of possible answers from the responders. The interview started by introducing to the students a concrete situation – a problem they had to solve, as proposed by Bowden (1996) and Marton (1994). As soon as the students got into the situation, more questions regarding dimension followed.

For the introductory task, students were given various 2D and 3D shapes which they had to divide into two categories. Physical materials were used as they were (a) concrete real objects that students could use without the danger of creating any illusions; and (b), materials that students were already familiar with from everyday life and school. Learning from the previous situation, I have noticed that students were more familiar in using real objects than any other tools (see Cubix Editor task in Chapter 5). The purpose of the task was to introduce students to the notion of dimension, thus it had to be as a simple task as possible.

The interview plan followed Marton's (1994) second suggestion of starting with a general question of the type 'What do you mean by…?', in our case 'What does the notion of dimension mean to you?'. Although some of these questions might have been examined through lessons in class (e.g. What is the difference between 2D and 3D? How many dimensions does a line have?), most of them were questions that the individuals might have never thought about (e.g. Can you think of examples in the real world, which are not 3-dimensional? How many dimensions does a reflection in the mirror have? How many dimensions does a shadow have?). This variety of the questions aimed to explore experiences both everyday experiences as well as experiences from schooling. What is more, the questions students never thought about could trigger their thinking leading to a search for connections with previous experiences, which would be exposed.

6.3 Interview plan

It was considered wise to start the students' interview with a task in order to stimulate their interest and understand their level of experience with dimension. Therefore the task below was an introduction to the notion of dimension by asking the students to classify various shapes into two categories and justify their answers:

Task: The students were given various shapes (2D and 3D ones) and they were asked to divide them into two categories and explain the reason for dividing them like they did. The purpose of the task was for the children to mention any word related to dimension and through the above task the following questions were discussed:

1. What does the notion of 'dimension' mean to you?
    a) What does the notion of 'spatial dimension' mean to you?
2. Do you think dimension is an important idea?
    a) Why/Why not?
3. What do you think is the difference between 2D and 3D?
    a) Can something have 0 dimensions?

b) Can something have 1 dimension?
c) Can something have 2 dimensions?
d) How many dimensions does a point have?
e) How many dimensions does a line have?

4. Can you think of examples in the real world, which are not three-dimensional?
    a) How many dimensions does a reflection in a mirror have?
    b) How many dimensions does a shadow have?
    c) How many dimensions does the page from the student's notebook have?

The variety of the questions led the students to use their prior knowledge and familiarity with dimension to think about them. And that was when their dimensional experiences were exposed for exploration.

6.4 Enriching the pool of meanings

As aforementioned in the first situation, the pool of meanings includes the students' experiences of dimension. A full version of the pool of meanings can be found in the appendices attached (Appendix 3, p. 400). Due to its extended length it could not be presented in the main body of this report and thus, I prepared a representation of it (see Figure 38). The characteristics of the pool of meanings are as described in the first situation. The data were analysed according to the procedure of data analysis described in Chapter 4 and like Situation I, this situation was not progressed through all the stages of the analysis. After transcribing the interviews and reading the transcripts and deciding which meanings were included in the pool of meanings according to our orientation of dimensional experience, the analysis was intermitted until the third situation's research was conducted. However, the new version of the

Pool of meanings was enriched by more excerpts, which were gathered from this situation.

Figure 38 illustrates the new version of the pool of meanings. The excerpts which entered during the first situation are represented by *E (E from Elica applications) while the excerpts entered during this situation are represented by ◆I (I from Interview). Consequently, each code *E1, *E2, *E3…*E68 and ◆I1, ◆I2, ◆I3 …◆I26 represents the corresponding excerpt from the first and the second situation accordingly. In the following paragraphs, some examples of the excerpts selection procedure are presented pointing to what made an excerpt to qualify or not to enter the pool.

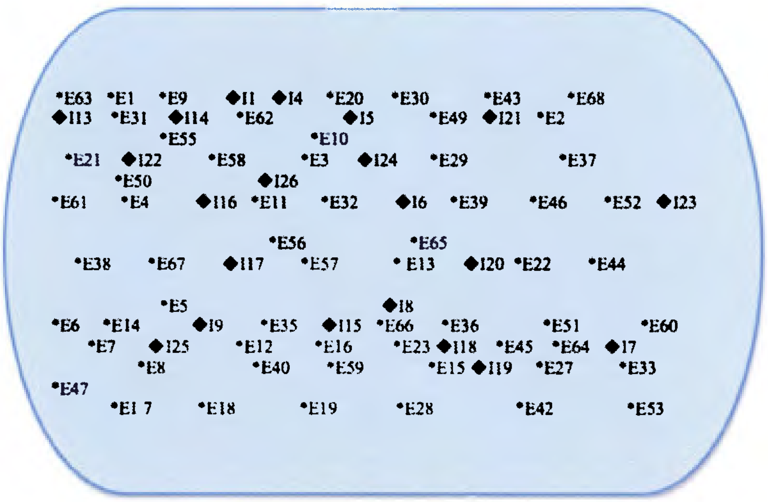


Figure 38: Situations' I & II Pool of meanings

To begin with, an excerpt, which related to the identification of shapes and was included in the 'pool', was the ◆I1 where it was argued there are two types of shapes: "*We have flat shapes and 3D shapes*". The second example relates to the part of our orientation regarding the properties of shapes and it is presented through the excerpt ◆I22:

Researcher: Can you think of examples in the real world which are not three dimensional?

N: The earth...it is probably 0 dimensional because it doesn't have any corners.

Researcher: Chelsea do you agree with that?

C: Yes because it doesn't have any corners or sides, straight lines around it so it is probably going to be 0 dimensions because it is only like a sphere, it doesn't have any corners or edges.

N: If it is 2D or 3D, it could be flat or standing up but like a sphere won't stand up or be flat it is just going to be a sphere and it is going to be round and in 2D and 3D there's corners and edges so only in earth and other planets there isn't so it is 0 dimensions.

In the excerpt above, the students classified the objects according to their properties i.e. 2D shapes are flat, 3D shapes are standing up, and 0D shapes are the ones which do not have any corners or edges. This also relates to the relationship between reality and abstraction as discussed in the orientation in the literature. A stronger example on this was the extract ◆I6 where the students define 2D shapes as bending and flexible, and as having materialistic properties:

Researcher: What do you think is the difference between 2D and 3D?

P: Well 2D is flat and it is bending sometimes, it is flexible. But if it solid and it is 3D you can't bend it and it stands up and it

K: It has thick so you can't bend it like you can bend a paper and make it into a different figure, you could feel it but you can't really change it into another shape unless of course if you break it, you could always change this (2D) into a different shape. If it is a 2D shape, you can't really change it into another shape but you can add stuff into it and use it for another shape.

Another example of an excerpt included in the pool, related to the creation of relationships between objects. In ◆I20, the students argued that if a point has 1 dimension, then a square, which has four corners-points must be 4-dimensional:

Researcher: How many dimensions does a point have?

N: One.

C: Probably only one because it doesn't have sides and it only has a corner on the top of it.

N: Or not because if you just have like a square that has 4 sides like 4 corners and it stands for one dimension, it will be 4...because it has 4 sides, so does it mean it is 4D...so I think it is 0.

The excerpt ◆I3 was also included because it relates to the relationship between reality and abstraction. For instance, the students tried to relate dimension to everyday materials giving the example of a microphone as a 3D shape:

Researcher: Before you said that a shape is 3 dimensional. What is a dimension?

Ka: I think it's...I think we are 3 dimensional as well actually, we are not just flat like a piece of paper but we pop up.

P: It is like it stands up. It's not something flat like that [showing the paper] but it stands up like the microphone.

Although the excerpts in the examples above entered the pool of meanings because they related clearly in some way to our orientation of dimension as presented in the literature, in some others the relation was not so clear. For instance, in excerpt ◆I10, even though the students 'misunderstood' the question, their answer regarding how a shape could be 2-dimensional and 3-dimensional at the same time was very interesting and thus it was decided to be included in the pool:

Researcher: Can something have 2 dimensions?

P: Maybe if it is a cube and a sphere and you cut them in half and you have one part from them and you put them together then maybe that can have 2 dimensions or maybe you could get like a flat shape and stick it onto a cube or something these would be like two dimensions because one part it is 2D and the other part it is 3D.

Similarly other excerpts qualified just because they expressed worthy of note experiences of dimension. For example, in ◆I8, the students talked about the difference between the real world compared to the digital world of the computer:

K: If you were trapped inside the computer some of the things in there would probably be...or have lots of dimensions and I think it might also have 0 dimensions. If you were trapped there you might have seen it.

Researcher: How?

K: It is kind of hard to explain but...

P: It is like a different world.

K: Yes, it is kind of a different world because we got experience from the world and that is how me make the computer and lots of different other things so in the computer it is like a digital world...and if you have to create a poster inside the computer sometimes you will have these kinds of different writing that goes across the page, and sometimes it actually looks like a 3D to you but it is really 2D...

P: Maybe to you it would look 2D but if you enter the computer you will find out what dimension it is because the screen it shows you a different picture than being inside the computer.

In contrast, some quotes did not qualify to the pool because they referred to mathematics or geometry in general or even considered not to be relevant to our study. For example, students talked of the 'D' in front of 1,2,3 as showing dimension and that without it the naming of the shapes would have no meaning i.e. 2/3 shapes instead of 2D shapes/3D shapes. Even though the way we name the symbol of dimension is an important notion, and it is mentioned in some excerpts, it was considered irrelevant to include all the excerpts referring on it.

Moreover, due to the open-ended questions of the questionnaire, students had the opportunity to express themselves in long conversations where sometimes they were just re-visiting what they had already said. These long excerpts were sometimes not of any significance to enter the pool. For instance, a couple of students estimated the dimension of a shape according to the number of edges or corners it had. This specific meaning was clear in many answers about dimension, and thus sometimes it was considered not necessary to be included again.

6.5 Summary

Situation II was very useful for gathering excerpts for enriching the Pool of Meanings. The aim of this situation was to expose in depth the experiences on dimension of the individuals. This situation resulted in meanings dominated by spoken language and not so much on actions as the previous one. This gave the opportunity to the respondents to talk about non-visible and abstract terms and thus, the pool of meanings was further enriched. However, they were times it felt as if there was a rich meaning but the individuals had a difficulty in expressing themselves only in words, and thus the meaning remained vague. Although the questions were interesting to them, they did not act as a 'large window' as I expected them to be.

In judging the 'richness' of pool, once again there was insufficient coverage of the aspects of the orientation of dimensional experience and also the pool lacked in having the same aspect of orientation representing multiply. Consequently, I decided to enrich the pool of meanings more before I start organising it into categories of description. Thus, a third situation was designed aiming to act as a 'wider window', exposing the experiences of dimension deeper.

Chapter 7: Situation III
Flatland

7.1 Overview

After conducting the two situations the pool of meanings was enriched from experiences on dimension. However, it was decided that a further situation was needed in order to have a satisfactory amount of meanings which cover a substantial part of the orientation towards dimensional experience (see Table 3, p. 133). The main criticism of Situation I was the lack of spoken language, which was dominant during Situation II to an excessive degree. The aim of the third situation was to involve interviews but also to give the individuals an experiential situation about which they could talk.

73 meanings were generated from four 10-year old students, the same students that were used for the Situation II. The students worked in pairs, having two pairs in total in this particular situation. The interview was the main method in this study. Each interview lasted around 30 minutes. In the following sections, the method of data collection and the procedure of analysis are discussed.

7.2 Method

Situation III used media for creating active thinking activities. A film acted as a resource for exploring how students experience cross-dimensional situations. A film was chosen because it offered an experience that could go beyond what individuals already experience in their everyday reality. As Champoux (1999) argued "films offer many opportunities to create powerful metaphorical images of abstract theories and concepts" (p. 7). Consequently, a film could lead to a rich discussion of the subject to which the individual is exposed. This situation used the film *Flatland the movie* (Travis, 2007) which is a computer-animated adaptation of Abbott's mathematical adventure novel *Flatland: A romance of many dimensions* (Abbott, 1884).

Description: 'Flatland: A romance of many dimensions' is a book written by Edwin A. Abbott in 1984. The main character of the book is a square living in Flatland, the two-dimensional world. The book is divided into two parts. The first consists of an exploration of Flatland and its inhabitants, while the second part describes how the square meets other worlds and other people such as Monarch in Pointland, King in Lineland and, a Sphere in Spaceland. A description of these four worlds will follow, showing also some pictures of people as they appear in Flatland the movie (Travis, 2007), a film based on the book by A. Abbott. The picture below is taken from the book of Flatland and includes all four worlds (Figure 39):

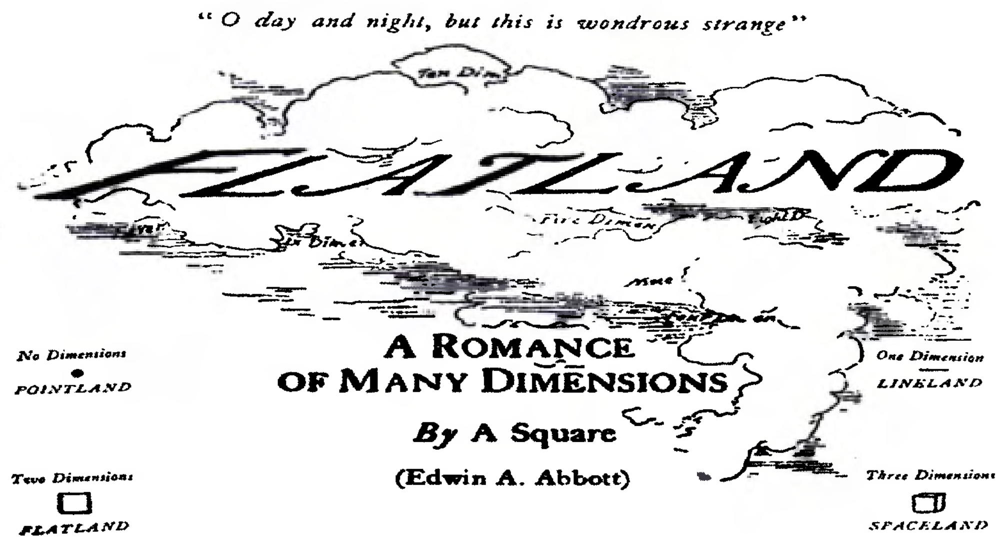
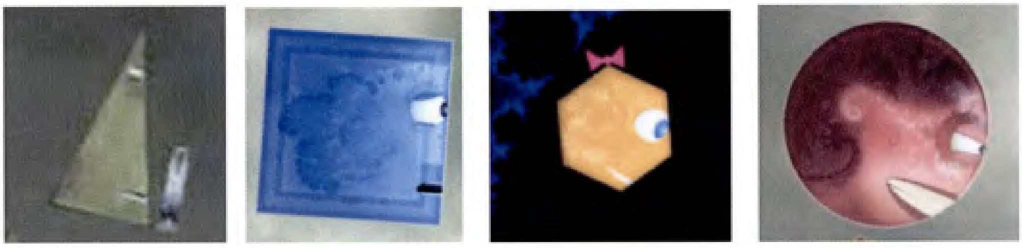


Figure 39: The four worlds in the book of Flatland

In order to understand what exactly Flatland is, I use the quote of A. Abbott as he described it in his book:

> 'Imagine a vast sheet of paper on which straight Lines, Triangles, Squares, Pentagons, Hexagons, and other figures, instead of remaining fixed in their places, move freely about, on or in the surface, but without the power of rising above or sinking below it, very much like shadows-only hard with luminous edges- and you will then have a pretty correct notion of my country and countrymen' (1884, p. 15)

In Flatland people are straight lines and polygons. Women are straight lines, the weak sex in Flatland. However, the film follows a more equal view among sexes having both men and women as polygons. The more sides you have, the cleverer you are and the higher your social status is. The soldiers and the lowest classes of workmen are triangles with two equal sides and a base. Middle class consists of equilateral or equal-sided triangles, and the professional men and gentlemen are squares or pentagons. Above these, it's the Nobility where there are many degrees starting from the hexagons and from rising in the number of sides until they receive the honourable title of Polygonal. The highest class of all it's the Circular or Priestly order, which consists of polygons with so numerous small sides that cannot be distinguished from a circle. The law of Nature in Flatland requires that every male child should have one more side than his father and in a step forward in the nobility scale, for example a square has a pentagon son. According to Flatland the movie, the characters look like in Figure 40.



Triangle	Square	Hexagon	Circle
(Soldier)	(Merchant)		(President)

Figure 40: Pictures of the characters in Flatland the movie

There are limited movements that people can make in Flatland's space: backwards/forward, right/left. The word 'upwards' is forbidden by Law. Houses in Flatland are pentagons. Squares and triangles are too pointy to become houses because they are dangerous. This is how a house in Flatland looks:

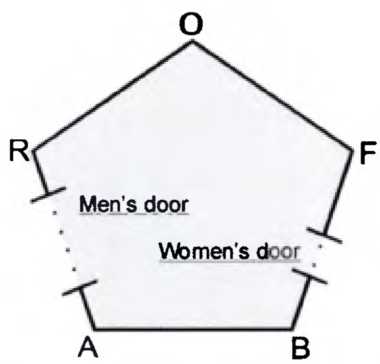


Figure 41: A house in Flatland

As illustrated in Figure 41, the two northern sides RO and OF represent the roof and on the east there is a small door for women and on the west a bigger door for men. It is raining from the north; therefore houses are protected that way. What people see of each other in Flatland is just a straight line (see Figure 42):

Figure 42: Vision in Flatland

They identify each other either by 'feeling' the edges of each other or by the light of the fog. The brightness of the visible side shows the shape type of the people.

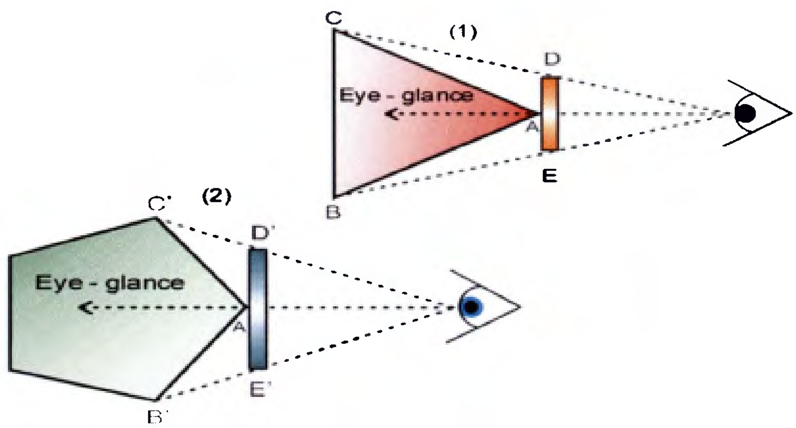


Figure 43: How shapes identify each other in Flatland

Apart from Flatland, the other worlds mentioned in the book and in the film are: Lineland, Pointland and Spaceland. According to the story of both the film and the book, the Square visits all the above places. Lineland consists of many points, which live moving forwards and backwards on a straight line. The book describes that small lines are men and the points are the women. On the contrary, the film presents all the inhabitants of Lineland regardless of their sex as line segments. All are restricted in motion and eye-sight to that single Straight line. The law of Nature in Lineland requires that every man weds two wives but this is not the case according to the film. This is how Lineland is illustrated in the book:

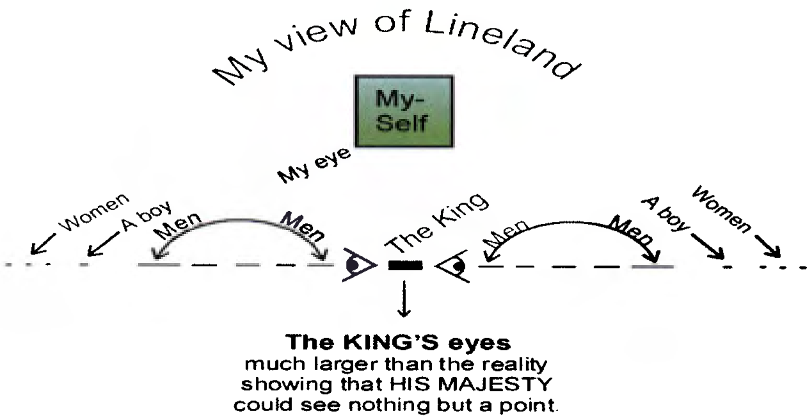


Figure 44: Illustration of Lineland

Because people in Lineland see each other as points, they calculate the length of each line by their voice. And this is how the King of Lineland is presented in Flatland the movie:

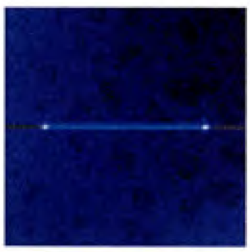


Figure 45: King of Lineland

Another world of the book is Pointland. There is only one inhabitant in Pointland, therefore it is considered as the monarch. This is how it is presented in Flatland the movie:

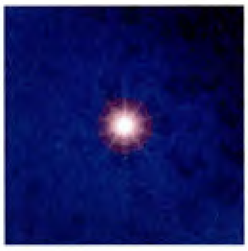


Figure 46: The monarch of Pointland

The last world that the square visits is Spaceland. Before travelling to Spaceland, the square has an interesting conversation with his granddaughter Hex (name taken from Hexagon). The granddaughter is insisting that there are more dimensions than two while the square is sure that space consists of just two dimensions:

> Well, then, if a Point by moving three inches, makes a Line of three inches represented by three; and if a straight Line of three inches, moving parallel to itself, makes a Square of three inches every way, represented by three-to-the-second; it must be that a Square of three inches every way, moving somehow parallel to itself (but I don't see how) must make Something else (but I don't see what) of three inches every way — and this must be represented by three-to-the-third (Hex in Johnson, 2007).

The Sphere comes to Flatland, and takes the Square in Space in order to show him the world of three dimensions. The sphere can see the inside of all the things and people in Flatland because it is watching them from the top. The square, at first, sees the Sphere as a circle, so the Sphere moves upwards and downwards Flatland so that the diameter of the circle would get smaller and bigger. Through the Square's eyes the Sphere is a magician. After the square's exploration of Spaceland, the square keeps in mind only a phrase: "Upward, not northward". But when the square tries to explain the three-dimensional world to its people, he gets imprisoned for the rest of his life. The picture that follows shows how the Sphere is presented in Flatland the movie:

Figure 47: A Sphere

Didactical situation: In the beginning of the task students were informed about what the film's plot was about and the researcher tried to explain how Flatland looks like in order for the students to have a basic idea of the plot. The film was broken into six parts and at the end of each part there were questions accordingly. The film was broken in order for the students to remain active during the interview process and being able to respond to the details of the questions they were asked. At the end of the film, the students were asked to talk about their general views of the film, whether they liked it or not and to talk about any points they found interesting. Each of the six parts is described in the following paragraphs illustrating the questions, which accompany it, followed by a brief description of its aims.

7.3 Interview plan

Task: You are going to watch a part from Flatland. A film based on the novel by A. Abbott 'Flatland: A romance of many dimensions'. Then we are going to discuss what you saw.

Imagine a vast plane, a world of only two dimensions, where lines, triangles, squares, pentagons, hexagons and other figures live and move freely about. Women in Flatland are lines and the pentagons are men. This is a story about a shape called A. Square that lives with his family in Flatland. Shapes in Flatland know nothing about the 3-dimensional world until one day a Sphere appears to A. Square and tries to convince him that the third dimension exists.

Chapter 1: The Dream (00:00-02:07)

Chapter 2: Laws of Flatland (02:08-04:29)

1. What do you think the film is about?
2. How would you describe the world of Flatland to someone?
3. What did you like and what did you dislike from this world?
4. How does this world differ from our world?
5. What are the laws of Flatland?
    a) What shape would a child of a square be?
    b) How is the social class of people formed?
    c) How are people categorised?
    d) Who are the leaders of Flatland? Why?

Aim: The first part, which consisted of two chapters of the film, was an exploration of the world of Flatland. The questions used reflected to students' experiences of this two-dimensional world and the differences it had from our world. It acts as an introduction to the notion of dimension, which was discussed after the second part.

Chapter 3: The ministry (04:30-07:52)

Chapter 4: Dimensions (07:53- 11:45)

6. What did the Circle of the Circles claim? (33A)
7. What questions does Hex have?
8. What do you think is a 'dimension'?

9. How is a 'line' created?
10. How is a 'square' created?
11. What shape, do you think, a square moving can create?
12. Why did the Square get upset with his granddaughter?

Aim: The second part drew the attention to the questions Hex (the granddaughter of the square) had. She tried to explain what a dimension was and the Square gave her the example of how a line and a square were created. Students' experiences were explored when they talked about the same object in various dimensions (point, line, square, cube etc.).

Chapter 5: Pointland and Lineland (11:46-14:50)

13. What did the Square dream of?
14. What are the names of the two worlds he travelled to?
15. How is the world of Pointland?
    a) Why did the king of Pointland not give any attention to the Square?
    b) Why the king of Pointland keep repeating ME and why did he not like the word 'TWO'?
16. How is the world of Pointland?
    a) Why was the King of Lineland not able to see the Square?
    b) What did the Kings of Lineland think of Space?
    c) Why did the King of Lineland know only the directions 'left' and 'right', and why did he not understand 'above' and 'below'?
    d) Why did the King of Lineland become aggressive with the Square?
    e) How did the Square become visible to the King of Lineland?

Aim: The third part of the film was an introduction to worlds of other dimensions than two. Students' experiences were exposed while discussing the worlds of Pointland and Lineland in which inhabitants lived differently than in Flatland and our world.

Chapter 6: A visitor (14:51-17:11)

17. What happened when the Sphere appeared?
    a) What did A Square see when he looked at the Sphere?
    b) Why do you think A Square saw the Sphere getting smaller and bigger and changing its size?
    c) How, do you think, the Sphere was able to 'see' through everything in Flatland?
    d) Why did the Square see the Sphere as a line/circle?
    e) Why, do you think, it was so difficult for the Square to understand the word 'upwards' and why did he keep saying the word 'northwards'? Is 'upwards' and 'northwards' the same thing?

Aim: The fourth part explored how students' might experience the differences between three-dimensional and two-dimensional objects pointing to their degrees of freedom in movement and sight.

Chapter 7: Spaceland (17:12- 22:29)

18. Why was the Square surprised when he left his world?
19. Draw Area 33H. Can you explain its shapes?
20. Where did the Sphere take him?
21. Describe Spaceland.
    a) What types of shapes existed in Spaceland?
    b) What do you mean by '3-dimensional shapes'?

Aim: The fifth part was an exploration of the world of Spaceland, which was similar to our world. Students were asked to make a comparison between Flatland and Spaceland and to explain the difference in the shapes existing in these two worlds. Moreover, Area 33H consisted of shapes from various dimensions such as points, lines, squares and cubes and students had to describe this area to the researcher.

Chapter 8: The secret (22:29- 25:23)

Chapter 9: The rescue (25:23-28:00)

Chapter 10: Arthur's trial (28:00- 32:43)

Chapter 11: The 4th dimension (32:43- 34:44)

22. Why although the Circles knew that 3 dimensions exist, they did not want people to discover it?
23. Where did Hex go? What happened?
24. What happened in Arthur's trial?
25. How did the Square vanish from court?

Aim: The last part of Flatland was mostly focused on the disappearance of the Square from the court in Flatland. Students were asked to explain how the disappearing happened and to justify how the Square managed to hide himself in the third dimension. This helped in exposing students' experiences regarding the degrees of freedom in various dimensional domains.

General questions after watching the whole film

What is the film about?

What did you like most about this film?

Did you learn anything by watching this film? What?

In this interview I have focused on dimension. Do you want to add anything about your views on the idea of dimension? Perhaps your ideas have changed…

Aim: The purpose of the general questions was to examine whether the ideas students had on dimension had changed after watching the film. And if they did, in which sense these changes were made and what part of the film helped in modifying their views.

7.4 Data analysis

The data were analysed according to the procedure of data analysis described in the previous chapter (see Procedure of Data analysis, Chapter 4). After transcribing the interviews, reading and re-reading the transcripts and selecting excerpts for the pool of meanings, the pool was rich enough to move to the next step of the analysis which was the organisation of the pool leading to the categories of describing dimensional experience. However, in this section I will discuss the selection of the excerpts of this specific situation.

Similarly to the other two situations, the classification of whether an excerpt is appropriate to be included in the pool of meanings was based according to its relevance to the definition of dimension and the orientation of dimensional experience as described in Table 3 (p. 133). Likewise, the students' accounts were interpreted according to the structure of experience as discussed through the psychological theories in the same chapter pointing to the intuitions, visualisation abilities and prototypes. The next section presents a brief summary of how the excerpts from this situation were selected to enter the pool of meanings.

7.5 Enriching the Pool of meanings further

The actual pool of meanings including the excerpts can be found in the appendices due to its large size (Appendix 3, p. 400). Here, I present a representation of the pool like I did with the other two situations, which is now extended due to the new excerpts taken from the third situation (Figure 48).

As aforementioned, the excerpts which were taken from the first situation are represented by *E (E from Elica applications) while the excerpts taken during the second situation are represented by ◆I (I from Interview). The third situation's excerpts are represented by ❖F (F from Flatland). Thus, each code *E1, *E2, *E3…*E68 represents the corresponding excerpt from the first situation, each code ◆I1, ◆I2, ◆I3….◆I26 from the second situation and each code ❖F1,❖F2, ❖F3…❖F73 represents the corresponding excerpt from the third situation. In the following paragraphs, some examples of the excerpts selection procedure are presented pointing to what made an excerpt to qualify or not to enter the pool.

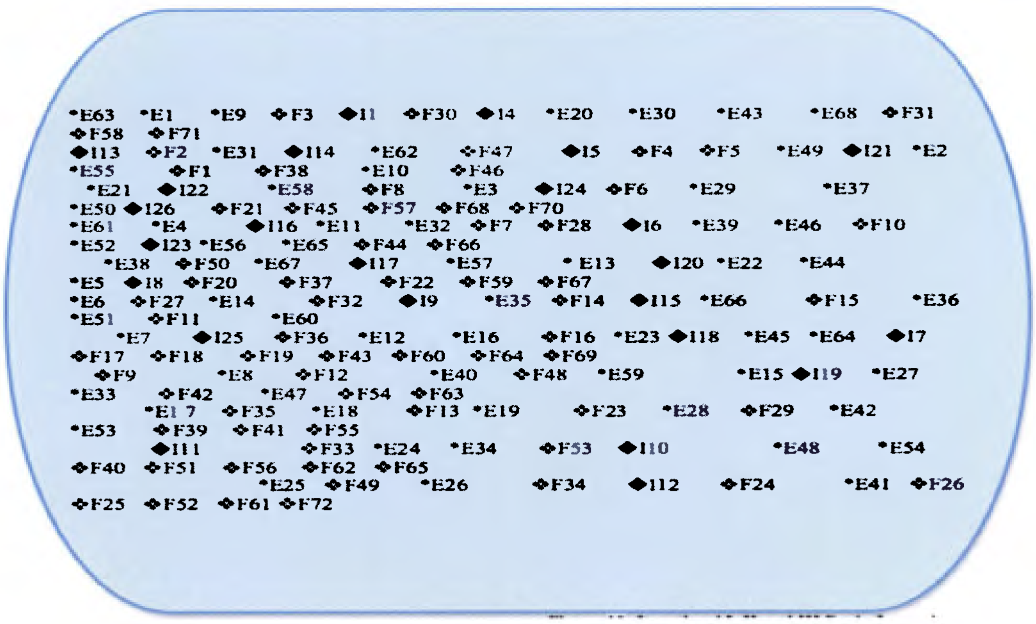


Figure 48: Situations' I, II & III Pool of meanings

The excerpt ❖F38 is the first example I am presenting as qualified for the pool:

> *Researcher: Do you want to add anything about your views on the idea of dimension?*
>
> *P: A place that there are different categories of shapes like a 3D dimension or a 4D dimension*
>
> *K: Different styles*

Researcher: What is Style? You mean different type? Of what?

K: Dimension? Point, Line, Square, Cube...how do you call a 4D shape?

The students here identified the point, the line, the square, and the cube as shapes with a different dimension, but they also talked about them as a hierarchical, from the lower to the higher dimension, leading to the question of how we call a 4D shape. This obviously can relate to the 'identification of geometrical shapes' discussed in the orientation of dimensional experience in the literature, and thus copes with our orientation.

The second example I present also relates to an aspect of our orientation and that is excerpt ❖F8. Students expressed a classification of 2D and 3D shapes even though their argument was not strong probably because their teacher had told them as a rule:

K: I think it was Mr Teacher that told us this earlier on in the year that if it is 3D it has edges I think and if it is 2D it has sides or points.

P: Yeah, if it is 2D it has sides and if it is 3D it has edges. Or was it the other way around? I can't remember.

Talking on the properties of classifying shapes, excerpt ❖F32 was included in the pool because it expressed some characteristics for categorising shapes: *"3D shape stands up, it is a shape that is not flat, it stands up right and you can hold it, you can break it into parts and it has got edges and sides some ...it's got sides and it's got faces and vertices and corners"* At the same time, it was relevant to the relationship between reality and abstraction (see Orientation) as the characterisations expressed a materialistic attribute. For instance, 3-dimensional shapes can be held and can be broken.

A fourth example is the excerpt ❖F68 where the student described the restriction of vision of the Flatlanders:

> *Researcher: What did you like most about this film?*
>
> *N: The way how it was flat, because how could you like have a flatland, because you are flat, you are like that (bug's eye view) 2D it is like you are cross-eyed, they don't have any bends or corners you are just a square like that so you see like cross-eyed, because that's how big you see like that (showing cross-eyed with his hand) but when you move your eyes you see this his left and this his right (turning his head)*

The excerpt above relates first to the 'identification of restrictions of freedom in a spatial environment' because it refers to the limitation of vision, and second, to the 'representation of dimension-related concepts whose origin is not visual or physical'. How an object behaves in a plane is not something you can easily have a visual or a physical experience on, but the film offered this to students who took the opportunity to discuss it. Adding to this, excerpt ❖F58 showed a visualisation aspect when students visualised how a Flatlander could see a sphere:

> *Researcher: Imagine you are the square in Flatland and a Sphere comes. What do you see?*
>
> *C: I only see him as a circle. And because you are flat when you are trying to look at it, you only see like a circle, he is actually round and his height is from up and down, and unlike other circles he can go rolling round, and with the priests they can't do that, they can't roll around because they are flat.*

Last but not least, excerpt ❖F1 is an example showing the impact that the 2D-perspective film had on the already familiar experiences of students on films:

> *P: normally in movies and films nowadays that we've got they've always got 3D people and objects, and that's weird because I haven't seen a film that is 2D and it's like the house that they are in and we get view from the top, and it is not like we are standing up, the ground is here (showing surface of the table) and it is like house lays from the ground. Like they are on the ground, they don't have stairs or anything,*

The above relates to our orientation regarding the 'development of an informed background of many aspects of the world relating to dimension that might be used to stimulate and challenge students'. This excerpt showed that this 'different' film stimulated the students, and thus it was included in the pool.

The examples I presented so far qualified to enter to the pool because of their relation to our orientation of dimension as discussed in the literature. Nevertheless, some excerpts were included because they related to our epistemological definition of dimension (see Chapter 2). For instance, excerpt ❖F73 showed that the student made the distinction between a 3D shape and a 3D world. That relates to our discussion about dimension as a quality of an object/space:

> *Researcher: Why was the Square surprised when he left his world?*
>
> *P: Basically the same as Kai, he is quite shocking because he hadn't seen a 3D shape before or a 3-dimensional world besides from his.*

In contrast to the above, I also present some excerpts, which did not enter the pool and I explain the reasons for that. The interview in this third situation is contextualised to the film of Flatland. Therefore, many excerpts from the transcripts were focused exclusively to the plot of the film without expressing any ideas related to dimension and our orientation. Consequently, these types of quotes were avoided to be included in the pool. An example of an excerpt rejected for that reason is:

> *Researcher: Where did Hex go? [during the trial]*
>
> *C: She went to go and see the Area of 33H, and all the guards came to get her because she was an intruder and she never knew about that her grandfather Arthur came to save her and everything and she was pushed into a big cube so she wouldn't be found as an intruder and Arthur put himself forward so she couldn't die.*

Furthermore, sometimes the questions posed were not leading students to talk about dimension but they were included in the interview plan in order for the questions to have a logical flow as the film unfolded. Therefore, sometimes, even whole questions and answers were omitted from the pool, as they were considered as irrelevant. For example, the following excerpt did not enter the pool because of the above reason:

> *Researcher: Why did the Square get upset with his granddaughter?*
>
> *N: Because it would be really bad, because he knows what his granddaughter thinks about because 3 units of 3 squares it is actually 3-dimensional and that is why he is scared because it is the same thing that hex's mother has created, his granddaughter created the same thing as her mother and that's why she died because she showed it to a priest and we don't know what happened to her.*

It is worth mentioning here, that there were excerpts omitted due to similar reasons mentioned in the previous situations. For example, excerpts relating only to mathematics or to geometry in general were avoided. And also, excerpts whose meaning was repeated throughout the interview were further filtered avoiding in that way the repetition of the same sentences each time.

7.6 Summary

The purpose of this situation was to gather more meanings in order to enrich the pool before the next part of the analysis begun. Situation III succeeded in exposing experiences of dimension and as a result the pool was extended. A significant advantage of this situation was that it involved a stimulus for the students to talk about and thus it shifted from the 'pure' interview, which was dominant during Situation II. However, similar to Situation II, the students were passive during the interview. The film was interesting but there was nothing the students had to do rather than discussing it. After conducting these three different situations, I learned that if you give the students a problem to solve, for instance something to build or design, they would be more active and probably they would expose their experiences more.

In this part, the revised pool of meanings was presented including the excerpts from all three situations and the selection procedure of these qualified excerpts was illustrated by some examples. The next chapter presents how the pool was organised into categories of description pointing to the similarities and the differences of the meanings, which were articulated.

Chapter 8: Creating the categories of description

8.1 Overview

The final version of the pool of meanings included 168 excerpts in total: 69 from the first situation, 26 from the second, and 73 from the third. These excerpts were placed arbitrarily into the pool without prioritising any particular ones. As it was mentioned before, for a meaning to qualify into the pool it had to relate to the orientation of dimension in Table 3 (p. 133). The next stage of the analysis involved organising the pool of meanings. Section 8.2 presents how the organisation of the meanings was conducted, based on their similarities and differences after comparing them. Subsequently, section 8.3 illustrates the categories of description were formed through the organisation and the grouping of the meanings.

8.2 Similarities and differences among meanings

Comparing the 168 excerpts between them was a long and time-consuming process. First, it was because of the large number of excerpts and second, because of their nature. To begin with, it should not be assumed that each excerpt represented a different meaning. On the contrary, it was noted that some excerpts could have supported the same meaning or even a single excerpt could have supported two or more different meanings. Such duplication is addressed subsequently when categories of description are extracted (see later this chapter). Due to the limited space of the thesis, it was not possible to include all the comparisons of the excerpts therefore in this section I present and discuss a selection of excerpts as examples. The discussion points to excerpts with a similar meaning and excerpts having a different meaning, in order to illustrate the procedure of how the pool was organised and the categories were formed.

To begin with, take for example the excerpt ◆I9. The student argued that by adding 0D and 2D together could make 1D: "*both like 0-dimensional and 2D mixed together they make one-dimensional*". This statement showed an experience of dimension involving a type of measurement where dimensions could be added. Now, look at excerpt ◆I19. The students there stated that the dimension a shape had, depended on the number of the corners and edges of the shape. Students supported that a circle was 0-dimensional because it did not have any corners. In the specific excerpt, dimension was expressed as counting, where the more the corners a shape had, the bigger its dimension. Comparing these last two excerpts, the attribute in the first example related to the adding of dimensions while the second related to the adding of parts of the shape. Thus, one could say that different meanings attributed to these two excerpts; therefore, these excerpts were initially placed in different categories.

After revising the meanings and the large number of categories created, some meanings seemed to have more similarities than differences with one another. Take for instance, the excerpt ◆I10. A student argued that a shape could have 2 different dimensions, for instance being 2D and 3D at the same time:

> *P: Maybe if it is a cube and a sphere and you cut them in half and you have one part from them and you put them together then maybe that can have 2 dimensions or maybe you could get like a flat shape and stick it onto a cube or something these would be like two dimensions because one part it is 2D and the other part it is 3D.*

This showed an outcome of a measure compared to the other two excerpts above, which referred to the actual act of measuring and counting. Thus, in revising our categories, the first two excerpts ◆I9 and ◆I19 were placed in the same category as they expressed dimension as an act while the third was placed in a different category as it expressed dimension as an outcome of an act. So instead of having three different categories, the above meanings were merged in two.

Even though the categories were revised, their number was still large. Thus, I followed this process again and again and combined more categories. Take for example the excerpt ❖F66: "*2D shapes are always flat and 3D shapes are always standing up*". This excerpt illustrated a more static expression of dimension as an attribute for separating shapes to '*flat*' and '*standing up*' ones, showing that no action or measurement were needed for defining the object. The excerpt ❖F66 looked motionless compared to the previous excerpts ◆I9, ◆I10 and ◆I19, which although they had their differences – some referring to an act while others to an outcome of an act- they all expressed dimension in a more dynamic sense in the form of an action. Consequently, the excerpts ◆I9, ◆I10 and ◆I19, which related to a type of action, were placed in the category 'Dimension as Action', while the excerpt ❖F66 was placed in a different category. As a result, instead of having four different categories of meanings in our examples, we had two.

There were many dilemmas like the above while organising the pool and developing the categories of meanings. However, after continuous revision and comparison, the categories became smaller in number but richer in meanings. Before introducing the categories however, I present some more examples of dilemmas that arose during this procedure of analysis. In last example presented before, excerpt ❖F66 was thought to express a static view of dimension, in the form of an automatic response from the participants. Take now the excerpt *E20. As soon as the student saw the round shape created from the rotation of a triangle, she shouted "*It became a circle*". The first example reflected an attribute of dimension by arguing that 2D shapes were 'flat' while the 3D shapes were 'standing up'. However, the second excerpt referred to the identification of a shape. In the first one, the student was able to identify the shapes even though his criteria were materialistic, while in the second one, the student named a 3D shape as a circle almost automatically showing no need of further explanation. Thus, those two excerpts represented two different meanings, and consequently belonged to separate categories.

Similar to the excerpt *E20, were also the excerpts ❖F7 and ❖F31. In ❖F7 students argued that "*a pyramid [is] the main shape of Flatland*" (instead of a triangle). In ❖F31, students were asked to explain what 3D shapes were, and they replied: "*Cube!*". There was a dilemma of whether these specific prototypical responses should have been included in the categories. After revising, it was noticed that most of these prototypical responses related to other categories of description and thus, they did not constitute an independent one. However, there were excerpts that did not belong to any other category, like for example the last three excerpts above. It was decided that not to include these in the formation of the categories because they did not show any considerate argument, compared to the other excerpts of the pool. An example of an excerpt showing a considerate argument was the excerpt ◆I17:

> *Researcher: Do you think dimension is an important idea?*
>
> *N: Yes, because the only real part of shapes is like 2D and 3D. There are always two separate categories every time you do shapes; sometimes it is 2D and sometimes 3D.*

Researcher: So, for what is it important?

N: It is important when you have to separate shapes.

Researcher: What do you think Chelsea [asking the second child]?

C: If you didn't have the D bit at the end, like the dimension, and you had 2 shapes and 3 shapes, you wouldn't know what it is going on about.

In excerpt ◆I17, the student posed a thoughtful argument, explaining the classification of shapes into 2D and 3D by using language used in school. This excerpt showed a more considerate reaction by the students while searching of an approach to classify shapes, compared to the previous excerpts that reflected an automatic, and probably prototypical expression of geometrical shapes.

Continuing from excerpt ◆I17, dimension there was expressed as an attribute for classifying objects. However, in excerpt ◆I4, the student argued: "*if a sphere is kind of like a circle, in a different dimension it could be quite different. Well, if it is a cube, it could also be a flat square*". This excerpt differed from the previous one because it referred to a classification of domains and not shapes. Similar was the excerpt ❖F21 where the students talked of a hierarchy of relationships between Arthur-Lineland and Sphere-Flatland:

Researcher: What happened when the Sphere appeared?

K: He didn't know there's anything above...like the line before when Arthur came; kind of the same thing is happening for Arthur, because the Sphere was round and it had much of itself, Arthur couldn't see her because he was like 3-dimensional so the Sphere had to be like that to speak to him [showing the Sphere on the table] so it was half of it, half of it was above and half of it was below so she could speak to Arthur.

P: It is like Arthur, [the kind of Lineland] could see him [Arthur] if it was left or right but couldn't see it if it was above or below. What I mean by below is that below [takes the paper up the table and shows the space under the paper] and not next to him.

Concerning the above, it was considered reasonable to have two categories, one showing the hierarchy of shapes according to their properties, and a different one showing the hierarchy of shapes according to the domains they belonged. However, after revising the meanings, the excerpt *E54 challenged the above categorisation. In *E54, the students characterised the circle drawn on Math Wheel as a '*thin circle*' which after rotation became a '*thick circle*'. This showed a classification of the shapes according to their thickness without referring to any clear logical hierarchy of geometrical shapes according to dimension. In contrast, the previous excerpts even though they showed a different classification, they referred to dimension as hierarchy. The three excerpts ◆I4, ◆I17 and ❖F21 showed a hierarchy of dimension through their answers, thus their categories were merged in one. The category was named 'Dimension as Hierarchy' and it included expressions of both talking about dimension as a property for classifying shapes and as a property for classifying domains. On the contrary, excerpt *E54, did not show any clear logical hierarchy between the geometrical shapes and thus, it was not included in the same category as the previous three.

Taking again the last excerpt as an example, in *E54 students talked of shapes as if they were real objects. They referred to thickness, which is a materialistic attribute that cannot be used for classifying objects that are not physical. Similarly, in ◆I12 it was argued that "*a thin line could have zero dimensions and then a quite thick line could have two dimensions*". The excerpt ❖F32, however, did not express dimension in regards to thickness, but it involved talking about shapes as if they had materialistic attributes. It included, for instance, expressions such as "*you can hold it*", "*you can break it into parts*", that again referred to physical characteristics of objects. At first, the above excerpts were placed in a different category because of the different material characteristics they referred to. However, after revising, their similarity, which was to express dimension as if it was a material's attribute, was stronger than their differences. Thus, they were placed in the same category even though the materialistic characteristic that each one referred to was different, such as thickness and touch. From this exploration of meanings, the category 'Dimension as Material' was created which included expressions of dimension as any type of a materialistic attribute.

After revisiting the excerpts and their corresponding meanings, the excerpt ◆I19, which belonged to the 'Dimension as Action' category - it referred to the counting of corners for identifying the dimension of a shape – was reconsidered because the counting of the corners showed a type of materialistic attribute as well. Similar to this example, other excerpts included in other categories were reconsidered, and some of them were also integrated to this one.

Another dilemma concerning 'Dimension as Material' was about some excerpts similar to ❖F2. In ❖F2, students argued that people in Flatland *"it's like a fish. They are just moving in and out of...."*. In this example, students not only referred to materialistic attributes but they also gave examples of objects in the real world. At this point, the excerpts that talked about real-world objects and the ones that related dimension to a materialistic attribute were considered as different.

In ❖F36, students talked about 0-dimension, 1-dimension and a 4-dimension and argued that "*there's a possibility of thousands of dimensions*". In this particular excerpt, students' expressions of dimension showed a type of generalisation, which was not in terms of material characteristics. There were other similar excerpts expressing this as well, thus, I had to revisit the grouping of the categories again in order to find a way to incorporate these meanings. After looking back to the initial categorisation, the distinction between expressions of dimension relating to real objects and the ones related to materialistic attributes did not look as strong as before, therefore, these two categories were merged into one, 'Dimension as Material'. Subsequently, a new category was added, 'Dimension as Abstraction' as the distinction between reality and abstraction was considered more essential.

After revising the categories, another dilemma arose: many excerpts, such as ◆I21, which referred to the counting of corners for identifying the dimension, were included in 'Dimension as Action' category because they showed a type of counting. Nevertheless, they were also included in the 'Dimension as Material' category, because they expressed dimension as a materialistic attribute. One might also support that they showed a type of abstraction even though it was situated in a materialistic perspective. However, the type of abstraction they expressed was different than the one that 'Dimension as Abstraction' possessed before. Therefore, a different category was created, 'Dimension as a situated abstraction', which referred to the expression of abstractions in situated terms.

Then another type of meanings created a dilemma. While working on the Elica application in *E10, students formed some abstractions which helped them to create their shapes. For instance expressions of the type '*The more we do the 2D shape on the left, the smaller the 3D shape would be on the top*' showed again an abstraction situated in the specific software application. This type of situated abstractions was different from the previous one, as this one referred to abstractions situated in the specific software while the other referred to abstractions based on dimension as a material attribute. However, this difference was not significant for creating a different category of situated abstraction, thus excerpts which were similar to *E10, were placed in the already existing 'Dimension as a situated abstraction'. It is also worth-mentioning here, that after revisiting and revising the categories further, the categories 'Dimension as Abstraction' and 'Dimension as a situated abstraction' were merged into one. Both of them referred to abstraction and this characteristic was considered more significant for integrating rather than separating them.

Nevertheless, in excerpt ◆I5 the students argued: *"if we were 1D we would be unseeable (not visible) and if we were 2D we would be really flat and we couldn't really survive in there"*. Even though, the expressions of 1D and 2D were considered as expressing a generalisation, they also showed an articulation of dimension as a world (a domain). Thus, this excerpt was thought to be different from the ones in 'Dimension as Abstraction' and therefore, was placed into another category.

In ◆I5, students expressed the hypothesis of how we (as humans) would be if we were not living on the 3-dimensional earth. The articulation of dimension as a domain was also evident in excerpt ❖F38 where the student argued that dimension is "*a place with different shapes 3D, 2D and 4D*". Although ◆I5 referred to dimension as an attribute of space while ❖F38 referred to dimension as the space itself, these two categories merged after revision, because they both articulated 'Dimension as a State'.

A dilemma arose regarding the expressions of meanings similar to *E58. In the specific excerpt, students worked on Math Wheel application, which itself can be considered as a 2D domain. While creating the lamp, a student argued:

> *S: If we can bring a line here, and here [showing a movement as the red lines I created in Figure 49].*

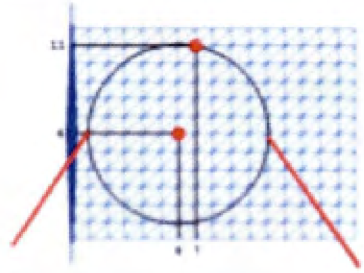


Figure 49: Creating a lamp in Math Wheel

The students did not identify the restrictions of movement in the 2D domain above therefore they thought that by adding more lines to the 2D shape it could become 3D. There were other excerpts as well, which referred to the restrictions of the domain that the students were working in. At the beginning of the categories' classification, excerpts of this type were placed into a separate category. However, they referred to a domain – although students worked in that domain – similarly to the excerpts in the 'Dimension as State' category; therefore, these two categories were merged into one 'Dimension as State'.

On the other hand, in excerpt ❖F58, the student referred to the domain of Flatland but her focus was on the description of shapes' behaviour in the specific domain and not explicitly on the domain itself:

> *Researcher: Imagine you are a square in Flatland and a Sphere comes. What do you see?*
>
> *C: I only see him as a circle. And because you are flat when you are trying to look at it, you only see it like a circle, he is actually round and his height is from up and down, and unlike other circles he can go rolling round, and with the priests they can't do that, they can't roll around because they are flat.*

None of the previous categories – even the 'Dimension as State' one- seemed to embrace this excerpt's articulation adequately. Looking at ❖F58 from another perspective, it explained a cross-dimensional situation. Due to vision restrictions, if a sphere entered Flatland, the Flatlanders would see a circle instead. In this specific excerpt, students talked of how a sphere (3D object) could be represented in a 2D world (as a 2D object). This cross-dimensional representation was also expressed in excerpt *E1 when students read the 2D representations of a 3D object:

> *Researcher: What's the difference in the shape when you look at it from the side or in front?*
>
> *C: When we look at it from the side, we look at 1,2,3 (counting blocks) while when we see it from the front, we see it as a whole.*

In the excerpt above, the students argued that the side view of the shape showed a part of the actual shape compared to the front view that showed the whole shape. Even though it was not possible to see the whole 3D shape from the front view, the students identified the restriction of vision that the representations have. Although the excerpts ❖F58 and *E1 were taken from two different situations, they both reflected meanings related to cross-dimensional situations and more specifically, to the restriction of vision. Therefore, they were both entered into the 'Dimension as Cross-dimensional' category.

The aim of the previous paragraphs was to give an idea to the reader of how the organisation of the pool took place. Reasonably, the first attempt of organising the pool included a large number of categories, which were further compared and merged into a smaller number. Many dilemmas arose during the procedure of categorising, and the examples above showed how some of them were resolved. The final version of the organised pool of meanings is presented in Table 4. The organised pool of meanings as presented in Table 4 below illustrates the categories in which the excerpts were distributed according to their similarities and differences. As aforementioned, it was possible that one excerpt expressed more than one meaning and thus included in more than one category. The categorisation was based on the clearest and most transparent meaning that the excerpts reflected. In the next section the characteristics of each category are going to be defined, pointing to what makes a category different than the other ones.

Table 4: Organised excerpts to categories of description

Dimension as Action	Dimension as State	Dimension as Material	Dimension as Abstraction	Dimension as Cross-dimensional	Dimension as Hierarchy
*E11 ◆I9 ◆I10◆I19 ◆I20◆I21 ◆I22◆I23 ◆I24◆I25 ❖F8❖F41 ❖F42❖F45 ❖F48	*E36 *E37 *E58 *E59 *E63◆I4 ◆I5◆I8 ❖F1❖F4 ❖F12❖F13 ❖F14❖F15 ❖F16❖F17 ❖F18❖F19 ❖F23❖F24 ❖F25❖F26 ❖F33❖F38 ❖F39❖F42 ❖F51❖F52 ❖F53❖F54 ❖F55❖F56 ❖F57❖F58 ❖F59❖F60 ❖F61❖F61 ❖F63❖F64 ❖F65❖F68 ❖F69❖F73	*E54 *E6 ◆I1◆I2 ◆I3◆I6 ◆I7◆I11 ◆I12◆I13 ◆I14◆I16 ◆I18◆I19 ◆I22◆I26 ❖F2❖F8 ❖F16❖F29 ❖F30❖F32 ❖F42❖F66	*E9 *E10 *E12 *E13 *E15 *E23 *E38 *E41 *E45*E60 *E61*E65 *E68 ◆I9 ❖F20❖F34 ❖F36❖F47 ❖F65❖F70 ❖F35❖F44 ❖F72	*E1 *E2 *E3 *E4 *E5 *E7 *E8 *E11 *E14*E16 *E17 *E18 *E19 *E21 *E22 *E24 *E25 *E26 *E27 *E28 *E29 *E30 *E31 *E32 *E33 *E34 *E35 *E36 *E37*E39 *E40*E42 *E43 *E44 *E46 *E47 *E48 *E49 *E50 *E51 *E52 *E53 *E55 *E56 *E58*E59 *E62 *E63 *E64 *E67 *E69 *E66 ◆I4◆I8 ❖F3 ❖F9 ❖F10❖F11 ❖F21❖F22 ❖F23❖F24 ❖F37❖F40 ❖F58❖F59 ❖F67	◆I4◆I17 ❖F5 ❖F6 ❖F21❖F27 ❖F28❖F41 ❖F43❖F46 ❖F49❖F50 ❖F71

8.3 Characterising the categories

After the excerpts were grouped according to their similarities and differences, the focus shifted to the characteristics of each group followed by the distinction of features between the groups. In order to do that, the core meaning of all the excerpts in a certain category was extracted, defining the criteria attributes of each group in contrast to the other groups. This was how the 'categories of description' were created showing the variation in which dimension can be experienced. The labeling of the categories was based on Lybeck's (1988) suggestion of 'something (x) is seen as something (y)'. In this study the 'something (x)' was defined as 'dimension', thus the categories described the various forms that dimension was seen and expressed.

Starting with a relatively large range of categories of description, after continuous refining, I ended up with six categories, which could not be filtered further. The next sections discuss each category of description by drawing on examples from the excerpts as presented in the pool of meanings.

8.3.1 Dimension as Action

Seeing dimension as an action included both the expression of dimension as an act and as an outcome of an action. Dimension as an act referred to the measurement of distances while dimension as an outcome referred to the result of the act.

Students tended to see dimension as an arithmetic term in which actions could be carried out. For example, a couple of students were counting the angles the real 3D shape had in order to choose a 2D shape in Math Wheel with the same amount of angles [*E11]. Likewise, a way of finding the dimension of a shape depended on the corners, the edges or the sides the shape had [◆I19, ◆I20, ◆I21, ◆I23, ◆I24, ◆I25, ❖F8, ❖F41, ❖F42, ❖F45, ❖F48]. The act of adding and subtracting dimensions was noted first, when it was argued that if a point has one dimension then the square which has 4 points should be a 4-dimensional object [◆I20] and second, when it was supported that 0D and 2D mixed together could make a 1D shape [◆I9].

Dimension as an outcome of an act was noted when it was stated that an object can have more than one dimension, i.e. being 2D and 3D at the same time, if the shape consisted of both 2D and 3D shapes [◆I10].

The significant outcome of the analysis was that the individuals did not express dimension only as a measurement, but on the contrary, their answers varied having multiple ways of seeing dimension. The same child could express different articulations of dimension depending on the nature of questions posed. For instance, although in some questions the children tended to focus on the measurement aspect of dimension, in others, they referred to dimension as a place or something abstract. The following sections describe these other articulations of dimension.

8.3.2 Dimension as State (place)

The students sometimes talked about dimension as a way of a measurement's outcome or as actions that could be carried out, while in other times, this view was challenged by the same students who focused on the domains (worlds) of dimension. Talking about dimension as if it was state was visible in students' answers [◆I8, ❖F1, ❖F4, ❖F12, ❖F13, ❖F14, ❖F33, ❖F38, ❖F39]. For example, students argued that dimension was a place with different shapes [❖F38].

Another example of seeing dimension as a state was when a pair of students talked about how we (as humans) would live if we were in a 1D world in which we would be 'unseeable', or in a 2D world in which we would be really flat [◆I5]. At some points, the students' answers showed the distinction between the object itself and the domain it belongs to [❖F42, ❖F73]. This tendency of shifting between talking about the object and the domain of the object was present during the interviews. However, during the Flatland situation, when there was a discussion about the actual domains of Pointland and Lineland - which in the film were presented as worlds- students sometimes talked about them as if they were objects [❖F51, ❖F52, ❖F54].

The place where the object was located was considered to be a very important element for defining the number of dimensions the object had. For instance, it was argued that the number of dimensions an object had depended on where the object was [◆I4, ❖F65]. Locating an object on a dimensional domain and identifying the restrictions of movement in each domain was also something students mentioned [❖F18, ❖F19, ❖F23, ❖F24, ❖F55, ❖F60, ❖F61]. For example, during Flatland, when students were asked to explain how the Square was able to see the Sphere getting smaller and bigger, they referred to the moving of the Sphere as 'up and down' or 'forward and backwards' [❖F23]. However, when students had to work on a 2D screen environment during the first situation, they did not consider the restrictions the application had. In this sense, they failed in drawing the right 2D shapes [*E36, *E37, *E58, *E59, *E63].

Another way of talking about dimension as a state was in relation to position [❖F26, ❖F53, ❖F62, ❖F63]. For example, students during the film were able to distinguish that a Flatlander would interpret the word 'below' as 'in front of', and to explain how 'under' is represented in space. This also occurred with the word 'upwards' which for Flatlanders meant 'northwards' [❖F26]. Students also referred to the limitation of vision for certain domains [❖F15, ❖F16, ❖F17, ❖F25, ❖F56, ❖F57, ❖F58, ❖F59, ❖F61, ❖F64, ❖F68, ❖F69]. For example, students stated that when the Square was in the 3D world, he was able to look up and down, something he was not able to do when he was in Flatland [❖F64].

8.3.3 Dimension as Material

Seeing dimension as having material attributes was very common among the children [*E54, ❖F29, ❖F30, ❖F32, ❖F66, ◆I1, ◆I2, ◆I3, ◆I6, ◆I13, ◆I14, ◆I18]. To begin with, students related dimension with the reaction to the touch of a shape. For instance, smaller dimensions such as 2D were thought to be more flexible and bending, and as the dimension increased to 3D, shapes became thicker and therefore one could "*feel them more*" [◆I2]. Adding to this, it was stated that the smaller the dimension of a shape, the easier it was to be transformed and changed into another shape by bending [◆I6, ◆I13, ◆I14]. On the contrary, higher dimensions were more 'hard' to be changed unless they were 'broken' into parts [◆I6, ❖F32]. This showed the materialistic way of seeing dimension as part of everyday life where 'thin' or 'flat' 2D objects exist, such as a piece of paper, and 'thick' 3D ones, such as a glass vase [*E54, ◆I1, ◆I2, ◆I3, ◆I18, ❖F66]. An articulation of thickness was also expressed when it was argued that a thin line had 0-dimensions while a thick one had either 1 or 2 dimensions [◆I12].

Another element of defining dimension, strongly supported by the students, was the ability of 'seeing' a shape. The smaller the dimension, the greater difficulty of seeing the shape. In other words, smaller shapes had smaller dimension. When a pair of students was asked to explain whether there was something that had 0-dimensions, they replied that it was something so flat one could barely see it [◆I7]. Students' explanations using vision also implied the idea of thickness as it was described in the previous paragraph. The students' examples on 0-dimensional objects that follow showed this relationship clearly: "*A really thin yo-yo string*", "*It is something that it cannot stand up*", "*A crack in the wall that you can't see it that much*", "*A thin line with a pencil*", "*A string of hair, you can't really see it but it is there*" [◆I7, ◆I11]. Students thought that extra dimensions gave extra qualifications to the shapes. They supported the idea that the more faces, vertices or edges a shape had, the bigger its dimension would be [◆I19, ◆I26, ❖F8, ❖F42]. For instance, a shape that did not have any vertices or edges, such as a circle and a sphere, was a 0-dimensional shape.

Students also related dimension to the real world [❖F2, ❖F16] by giving examples of real objects and phenomena, and by explaining dimension as if it were a tangible object. In contrast to the materialistic experiences of dimension, students also expressed some abstractions related to dimension, which are explored in the following section.

8.3.4 Dimension as Abstraction

While exploring dimension theoretically, many students not only had conflicting views but also some of their beliefs had been mathematically incorrect. For example, it was argued that a straight line had 0 dimensions while the curved one was 1-dimensional [◆I7]. In general, students were impressed when they were introduced to more dimensions than their familiar ones [❖F34, ❖F36, ❖F47], and also when they saw shapes moving between dimensions, for instance a 2D shape going to 0D, 2D and 3D [❖F70].

However, most of the students' generalisations were implanted into the specific situation or task they encountered. To begin with, views of the type "*the smaller the dimension, the more flexible and bending the object is*", "*the smaller dimension a shape has, the easier it is to be transformed*", "*the more dimension, the thicker the object*", "*the thicker a line, the greater its dimension*", "*the smaller dimension requires a greater difficulty of seeing the shape*", "*the more vertices or edges an object has, the greater its dimension*" were abstractions noted in the interviews which were based on students' materialistic experiences. Adding to this, students created abstractions influenced by the film Flatland such as "*In Flatland there are only 2D shapes but in Spaceland there are only 3D shapes*" [❖F65]. Or, they talked about the various dimensions they met in the film by arguing that everyone knew their dimension better than any other dimension [❖F35]. Other examples referred to "*the more corners you have, the more angles you are and the more clever, the more smart you are*" [❖F44], and also that the point was flat because that was how it was shown in the film [◆F8].

What is more, students also created some abstractions in order to make sense of the environment they were working on. Many situated abstractions were noted during the work with the Elica applications in the first situation [*E10, *E12, *E13, *E15, *E23, *E38, *E41, *E60, *E61, *E65]. Some examples were "*the more to the left, the smaller the hole in the shape*", "*the smaller the number on the fence, the smaller the shape*", "*the further from the fence, the bigger the shape*". These abstractions were very helpful for the students to create the right version of the requested shapes, however they were situated to the software tools. For instance, when at the end of the interview, students were asked to reflect on what they had done and explain which shapes they had used, they gave different answers with no reference to the 2D shapes they had constructed [*E9, *E45, *E68].

8.3.5 Dimension as Cross-dimensional

The students talked of dimension as a component in cross-dimensional situations. In general, students faced a difficulty in visualising what the 3D shape was like by reading its 2D representations (Cubix Editor application task-Situation I). The difficulty was first noticed when at the beginning of the first situation students were asked to talk about the differences of looking at the various views of a shape. The various views were considered by the students as independent components of the shape, some of them having more privileges than others. For example, it was argued that the side view showed a "*different shape*" having a limited view compared to the others [*E27], while the top view was considered to show a "*bigger shape*", "*every single square*" or the "*whole shape*" [*E16, *E28, *E46, *E48]. One of the students even described the shapes by looking at their top view [*E4, *E5] while some others supported that the front view showed something more than the other views [*E1]. A confusion was noticed regarding which view was the right one to draw in order to represent their 3D shape: switching between the top view and the front view [*E29], or between the top view and the side view [*E32]. It is also worth mentioning a pair of students, who copied all three views on paper in order to represent the shape [*E47].

Even though the top view was considered as the most privileged, the front view was the one mostly used in drawing [*E2, *E17, *E29, *E32, *E33, *E50, *E52]. The front view was also used for constructing the 3D shape on screen as well. Students ignored the dimension of 'depth' while reading representations, and thus presented the 3D shape as it looked by its front view on paper [*E18, *E19, *E30, *E48]. At times when depth was not ignored, it was not estimated correctly, having the blocks misplaced on screen [*E51]. In general, students faced a difficulty in reading representations. Students rotated the Cubix Editor platform so that the view of the shape was exactly the same as in paper [*E3, *E31]. The lack of visualising that a top view rotated remained still a top view, or not realising that a shape might have more than one front or side views, was apparent in students' answers [*E49].

Moreover, when looking at a 3D real object, students tended to look at its top view, which was 2-dimensional. This was noted during the Math Wheel application task where students had to construct a 3D real object given on screen, by rotating a 2D shape. The circle was used as a starting point for creating a lamp [*E22, *E36, *E37, *E58], the toilet paper [*E14, *E24, *E40], and the pencil [*E25, *E64]. An assumption could be that students looked at the top view of these objects, which was a circle. What is more, students did not consider the restrictions of movement that the specific software had, asking for extra lines and shapes to complete their drawings [*E26, *E59, *E63]. A pair even put a real pencil on screen in order to see if it fitted to their 2D drawing [*E43]. Even though some students created some logical relationships between the 3D real object and the 2D shape on screen, such as counting the angles of the object [*E11] or arguing that the 2D shape was the half of the 3D object, or even mentioning the line of symmetry [*E44, *E56, *E63], others did not see any relation at all [*E69].

Another aspect included in this category was students' expressions regarding what stays invariant and what changes in a set of transformations. In Math Wheel, the transformation used was the rotation of a 2D shape to create a 3D shape. Students showed signs of familiarity with the Math Wheel application when they rotated 2D shapes [*E7, *E21, *E39, *E67] and transformed shapes into a 3D [*E55, *E67]. However, when they were asked to explain what happened to the 2D shape, their answers varied as if it was not clear to them.

During the same task, students tended to focus on only one part of the 3D shape and made modifications to the 2D rotated shape, only to satisfy the specific part they had in mind. For example, while constructing the lamp, students tended to look at the size of the hole rather than the whole shape, and similarly, while constructing the pencil, students focused on the size of the peak rather the whole shape [*E42, *E62, *E66]. At times, students' expressions could not be fully explained, and thus, avoided for further discussion at this stage of the research. An example of these expressions was when students argued that the lamp looked like a triangle being split up [*E35].

Other visualisation difficulties were noticed during the third situation with the use of Flatland. For instance, students could not visualise that the 'mega-square' was a 3D cube [❖F10]. A second example was when students supported that the Square in Flatland saw the Sphere as a circle, influenced by their top view as observers of the film [❖F22, ❖F58, ❖F59]. However, at times their visualisation skills were correct. For example, they realised that a 3D sphere had more freedom than the 2D square because the sphere could see from above and below, and they also realised why the sphere could appear smaller or bigger to the square [❖F21, ❖F23, ❖F24, ❖F58, ❖F67]. In general, students were impressed by the way Flatland was represented in the film [❖F3, ❖F40], and they were excited at having the opportunity to visualise other than their familiar dimensions [❖F9, ❖F37].

The notion of cross-dimensionality was strong among students also when creating relationships. Students referred to the relation between 2D and 3D. For example, they talked about 2D faces of 3D squares and they gave the example of cuboids, which had a square as each face. Also, they talked about having the 'same shape' in various dimensions. For instance, a square could be in 2D and a cube could be a square seen in 3D [◆I4]. Students felt the need to relate several objects to each other or to itself, and also to recognise that some properties of an object remained the same even if the object changed orientation, or dimension. The students also mentioned that a 2D shape folded could create a 3D shape [❖F11]. Their need to explain how an object's dimension can be converted into another dimension led them to a discussion about the evolution of computational geometry and 3D graphics. While talking about dimension, one pair gave the computer as an example [◆I8]. It was argued that although writing on screen was 2-dimensional, there were specific applications that could make it look like 3D. Talking of transformations, some students considered the 2D shape to be an incomplete version of the "whole" 3D [*E8, *E21, *E34]. In contrast, other answers expressed the 3D to be a completely different shape than the 2D one, without having any relation [*E34, *E53].

8.3.6 Dimension as Hierarchy

Dimension was also expressed as hierarchy. In general, dimension was seen as a property for distinguishing shapes [◆I4, ◆I17]. It was also defined to include different categories of shapes [❖F46, ❖F71]. Students were able to make the connections across cross-dimensional situations such as expressing a hierarchy of point, line, square, cube, 4D shape [❖F28] or arguing that the situation of the Sphere in Flatland was similar to the one of the Square in Lineland [❖F21]. However, at other times, they did not make connections. For instance, when they were asked to say what a moving square would create, the students mentioned a pentagon, a hexagon or a 3D shape without being sure which was the right answer or being able to say what type of a 3D shape it would be [❖F49, ❖F50]. Similarly, it was difficult for some students to add the cube at the end of the sequence "*point, line, square…*" and they mentioned the "*mega-square*" instead [❖F27]. Similarly, students argued that the pentagon could be added after the "*triangle, square…*" sequence [❖F43]. Difficulties were also noticed when students classified shapes according to their sides. For instance, they supported the idea that a circle had 0 sides [❖F6, ❖F41] and that the decagon had the most sides of all the shapes considered [❖F5].

8.4 Summary

The aim of this chapter was to illustrate the six basic categories of describing dimension as they were extracted from the pool of meanings. Each category's main features were defined by referring to examples of excerpts. The basic components of each category as described before are presented in the Table 5:

Dimension as Action

dimension as an act

•the number of angles of 2D shapes remain the same even after they transform into 3D

•the more corners, edges and sides of a shape, the greater its dimension

•dimensions can be added

•a shape can have more than 1 dimension at the same time

dimension as an outcome

•the difference between 2D and 3D is depth

•in volume we have 3 dimensions

Dimension as State

•dimension is a place with different shapes

•the plane is a 2D object while space is a 3D

•2D are represented on paper, 3D are presented in the real world

•the number of dimensions an object has depends on where the object is

•the restrictions of a domain (motion, vision, orientation) depend on the dimension it belongs in

Dimension as Material

using everyday examples

•2D people move like fishes and see like a caterpillar

materialistic attributes

•the thicker the object, the greater its dimension

•the greater the dimension, the more you can feel the shape

•the less the dimension, the more flexible the shape is

•the smaller dimension requires a greater difficulty of seeing the shape

•extra dimensions give extra qualifications to the shapes

•nothing can be 0D because it would have no matter

Dimension as Abstraction

mathematical definitions

•everything should have at least 2 dimensions

•a line can be 1D or 2D

•a point can be 0D or 1D

situated abstractions (see Dimension as Material as well)

•*In Flatland there are only 2D shapes but in Spaceland there are only 3D shapes*

•everyone knows their dimension better than any other dimension

•the more to the left the 2D shape is, the smaller the hole in the 3D shape

•the smaller the number on the fence, the smaller the 3D shape

•the further the 2D shape is from the fence, the bigger the shape

Dimension as Cross-dimensional

•relation 2D-3D

representations

•the side view shows a "different shape"

•the side view has a limited view compared to others

•the top view shows a bigger shape

•you can see the whole shape from the top view

•the front view shows something more than the other views

•the front view as the one used in drawing

transformations

•a circle rotated creates a lamp or a toilet paper or a pencil

•the 2D shape rotated is half of the 3D created

•a 2D shape folded can create a 3D one

moving between dimensions

•the 3D sphere had more freedom than the 2D square

•using the word mega-square instead of cube

•a square is in 2D and a cube is a square seen in 3D

•3D to be a completely different shape than the 2D one without having any relation

•the 2D shape to be an incomplete version of the "whole" 3D

Dimension as hierarchy

dimension as a property for classifying shapes

•point, line, square, cube, 4D shape

•cube described as mega-square

Table 5: Categories of description

I would like to add here that other possible ways of experiencing dimension are not excluded though a phenomenographic approach presupposes that the range of experiences is limited. It is a limitation of this study that further meanings of dimension might exist and may not have been revealed due to the restriction of the tasks used or even due to the nature of the participants in this study.

Furthermore, the aspects of dimensional experiences noted were not of the same frequency. Some of them did not appear very often, while others, were articulated only by specific children. Additionally, the experiences of dimension were presented in various forms. Most of them were in the form of thoughts apparently dependent on the situation in which they emerged.

The next chapter first compares the categories of description to the elements of the definition of dimension and the orientation of dimensional experience as extracted from the literature review (Table 3, p. 133) trying to point out any omissions in the experiences noted. Second, this study looks within each situation, identifying the role of setting in the formation of dimensional experience. The purpose of the next chapter is to form a discussion of Phase 1 that could inform the Phase 2 of this phenomenographic study.

Chapter 9: The emergence of situation as an organising idea

9.1 Overview

This chapter presents a discussion of the categories of description by looking back first, to the orientation of dimensional experience in Section 9.2 and second, to the definition of dimension in Section 9.3 as formed through the literature review. The purpose of this discussion is to identify which elements of the orientation of dimensional experience and the definition of dimension are missing or which are not clearly inferred by the categories of description. Subsequently, Section 9.4 takes a micro-perspective on the process of looking within situations as required for Phase 1, pointing to the role of the setting in the formation of dimensional experience. The aim of this chapter is to illustrate the transition from looking *within situations* to look *between situations*, which was the focus of Phase 2 of this study.

9.2 Looking back to the orientation of dimensional experience

The purpose of Phase 1 was to explore the variation in the learners' experience of the situation and to examine whether the participants were oriented towards the phenomena that were present there (Marton and Booth, 1997). In order to do that, three situations were designed in order to extract children's experiences of dimension and six categories of description were formed as the result of the phenomenographic analysis. This section consists of a comparison between the meanings included in the categories of description, and the orientation of dimensional experience as formed in the literature review in Chapter 2 (see Table 3, p.133).

Comparing the meanings generated to the orientation of dimension, some meanings were included but others were not as expected. In the following paragraphs, each component of the orientation, as presented in Table 3 (p.133), is discussed by drawing on the relevant meanings generated as presented in the categories of description.

9.2.1 The identification, distinction and creation of relationships between 2D and 3D (and other dimensions) space/objects

To begin with, all three situations generated meanings referring to the identification, the distinction and the creation of relationships between 2D/3D (and other dimensions) spaces/objects. Students distinguished between 2D and 3D shapes, even though at times they faced a difficulty in naming the shape (i.e. cube as 'mega-square'). Most of their reasoning on identification was in the form of prototypes such as 'shapes that pop up', 'standing up shapes' and 'flat shapes'. In addition, some situated abstractions were also noticed relating to the identification and classification of shapes, for instance, "*the more corners the shape has, the greater the dimension*' or '*the more the dimension, the thicker the shape*" (see Categories of description – Dimension as Abstraction/Dimension as Material).

Students' experiences of dimension were challenged when they found themselves exposed to a cross-dimensional situation, and this acted as a stimulus to create a variety of relationships on dimension. Relationships were created both within a dimension and across dimensions, even though some of them were not mathematically correct (see Chapter 8 p. 170 Categories of description – Dimension as Hierarchy/Dimension as Material/Dimension as Abstraction). Multidimensionality was another way of expressing a relationship, which was evident when students talked about 0D and 1D, and the possibilities of more dimensions.

Dimension was argued to be a property for defining and classifying shapes. It was considered as a physical characteristic of real objects such as thickness and touch (see Chapter 8 p. 170 Categories of description – Dimension as Material). It also acted as a property for making the connection in a progressive sequence of objects or spaces, i.e. point, line, square, cube (see Chapter 8 p. 170 Category of description – Dimension as Hierarchy).

9.2.2 The articulation of dimension as a property of space/object (in any level of formality)

According to our orientation, dimension was a property of object/space. However, students either considered dimension to be an attribute of shape or an attribute of space, never connecting both. Thus, there were meanings focusing on expressing dimension as a property of a shape or as a property of location and space. Therefore, the expression of dimension as a property of both object and space was not considered to be an evident element in the categories, and it constituted an essential aspect for further examination during Phase 2.

9.2.3 The development of geometrical intuition and spatial awareness

Intuition and spatial awareness were notions that could not be easily distinguished within the meanings generated. However, the three situations designed offered environments in which these two abilities could be further developed. Focusing on the notion of spatial awareness, Situations I and III offered two different ways of visualising space: Situation I showed, through the Math Wheel application, how a 3D space could be formed by rotating a 2D shape, and Situation III showed evidence of expressing dimension as a quality of space (see Chapter 8 p. 170 Categories of description – Dimension as State). Even though looking at dimension as location was mostly promoted by the worlds of Flatland during Situation III, similar meanings were also generated from the tasks in Situations I, and II.

During the tasks, students worked on various dimensional domains and this experience helped them to infer meanings regarding the degrees of freedom of moving in space. Even though the students responded quite well on Flatland's restriction of movement questions, when they had to work themselves on a 2D software domain, like the Math Wheel application, they faced a difficulty in identifying the restrictions of movement of the domain. Similarly, while working on a 3D software domain like Cubix Editor they faced difficulties in depicting depth (see Chapter 8 p. 170 Category of description – Dimension as State/Dimension as Cross-dimensional). Thus, an element of the orientation missing, which was considered in Phase 2, was to have the students themselves identifying the restrictions of movement in the domain they were actively working on.

9.2.4 The development of an informed background of many aspects of the world relating to dimension that might be used to stimulate and challenge students

I believe that the three situations designed did not efficiently develop an informed background of aspects of the world relating to dimension. The two tasks of Situation I related to real life phenomena because they showed the concept of rotation as well the reading of front/side/top representations. However, it was not supported that students had made the connection with the outside world. Added to this, students also did not seem very challenged during these tasks, and most of their articulations showed that the purpose of the tasks was not clear to them.

On the contrary, Situation III was challenging for the students because it included a film about a 'world', which was not similar to their own. However, the film was too abstract for them to link it to their world. Therefore, this was one more element of the orientation, which was not presented clearly through the meanings. Thus, the aspect of relating dimension to everyday phenomena that could stimulate students' interest was considered for Phase 2.

9.2.5 The identification of what stays invariant and what changes in a set of transformations

Meanings referring to the ability of identification of what stays invariant and what changes in a set of transformations were generated mostly during the Math Wheel application, which involved rotating a 2D shape to create a 3D one. For instance, students argued that the properties of a 2D shape remain the same after rotation (see Chapter 8 p. 170 Categories of description – Dimension as Cross-dimensional).

During the rest of the situations, students talked about shapes in various dimensions but not about the concept of transformations itself. Therefore, this was one more element of the orientation that was not present adequately and was taken into consideration for the next phase of the study.

9.2.6 The development of the ability of reasoning and proof in geometrical concepts

All three situations generated meanings showing the children's ability to reason and prove. Children's generalisations and abstractions were situated in the situation where they found themselves. For example, while working with Elica in the first situation they created abstractions situated in the specific application, and similarly, while working with Flatland they created abstractions situated in the film.

Students formed some materialistic generalisations on dimension, like "*2D shapes are flat shapes*" or "*3D shapes are shapes that pop up*" (see Chapter 8 p. 170 Category of description – Dimension as Material), or even some more general ones, like "*dimension includes different categories of shapes*' or '*a shape in various dimensions*" (see Chapter 8 p. 170 Category of description – Dimension as Hierarchy). Moreover, students sometimes tended to give automatic responses to specific questions, and this gave an insight into the prototypical thinking they had. For instance, automatic expressions of the type 'A circle!' while looking at a round 3D shape were very common. To sum up, experiences of dimension relating to generalisation and abstraction were in the form of: (a) situated abstractions and (b) automatic prototypical responses.

9.2.7 The development of the ability to visualise, draw and construct figures

The three situations revealed something of the children's abilities to visualise, draw and construct figures. Students found it difficult to represent depth during the Elica applications. The difficulty was to represent depth while drawing the 3D shape on Cubix editor by reading its 2D representations (see Chapter 8 p. 170 Category of description – Dimension as Cross-dimensional).

The above leads us to the ability of visualisation, which was another component of the orientation of dimensional experience. The analysis showed that students were able to visualise how certain shapes looked in various dimensions (see Chapter 8 p. 170 Category of description – Dimension as Hierarchy/Dimension as State). However, they only referred to certain typical shapes, such as a square looking like a cube in a 3rd dimension.

Students considered the restriction of vision between dimensions. During Situation III, they identified the difference in vision between themselves, the King of Pointland, the King of Lineland, and the Flatlanders, and they gave possible reasons for why this was happening. When they were asked to say what they liked about the film of Flatland, they mentioned the words 'flat' and 'point of view' (see Chapter 8 p. 170 Category of description – Dimension as cross-dimensional). Students referred to the size of the object, its position and its orientation as elements that could restrict vision. Although students identified the difference in vision between themselves and the 'people' in the film, while describing the Square's point of view they were influenced by their own visual perspective.

As for the construction of figures, only Situation I gave this opportunity to the students, although it was still in restricted terms (by rotation in Math Wheel and by placing ready-made cubes in Cubix Editor). So an element not clearly explored during these three situations was having the children actively draw various dimensional figures in various dimensional domains. In that sense, an element considered for Phase 2 was to have the students being actively involved during the task.

9.2.8 The representation of dimension-related concepts whose origin is not visual or physical

The representation of dimension-related concepts whose origin is not visual or physical was portrayed during the film Flatland, where students had to make sense of representations of objects/spaces in various dimensions such as the King of Poinland, and the King of Lineland (see Chapter 8 p. 170 Category of description-Dimension as Cross-dimensional). Moreover, while representing dimension as a domain, which was one of the components of the film 'Flatland', the students were surprised by the way the 2nd dimension was represented in the film, and they mentioned that they had not seen a film in 2D before. Thus, it was concluded that this element of the orientation was embraced through the meanings.

9.2.9 The use of mathematical language for describing objects/spaces

All three situations offered space for mathematical language to be used in order to describe spaces/objects. The study showed that although the children did not use the formal language of plane and space, they were in a position to talk about the domain of dimension by wondering what it would be like to live in a 1D or 2D world (see Chapter 8 p. 170 Category of description – Dimension as State). In general, the language used in all three situations was in the form of abstractions situated in the setting where they first took place (see Chapter 8 p. 170 Category of description – Dimension as Abstraction).

9.2.10 *The creation of relationships between reality and abstraction regarding objects/spaces*

The creation of relationships between reality and abstraction was an evident component to identify in the meanings. Students often related objects to known objects. For instance, they argued that the round 3D shape created by Math Wheel was a 'thick circle'. Students compared objects and their behaviour, such as movement, to similar objects of their world such as "*It's like a fish*" or "*It's like a caterpillar*" (see Chapter 8 p. 170 Category of description – Dimension as Material). However, the students did not clarify the distinction between the representation of an object and the object itself, therefore being unsuccessful in distinguishing the tangible objects from the non-tangible ones (see Chapter 8 p. 170 Category of description – Dimension as Abstraction/Dimension as Material).

9.3 **Looking back to the definition of dimension**

During Chapter 2 of the literature, dimension was defined as a quality of space/object. This distinction was made in order to justify that in one context dimension might be regarded as object, while in another it might be regarded as space. Similarly, children sometimes talked of dimension as if it was a property of objects, while at other times they referred to it as a property of space. As aforementioned, the duality object/space was not shown in the meanings generated because dimension was expressed as a quality of space or as a quality of object independently. Thus, this was an element missing that was considered for the next phase.

In Chapter 2, dimension was also defined as portraying a quality of freedom and capacity. Dimension was expressed as quality of freedom during Situation I and III. Situation I offered the user the ability to work in a 2D environment during the Math Wheel task, and in a 3D environment during the Cubix Editor task. However, the children did not seem to generate meanings relating to the restrictions and affordances of working in various spaces themselves. Meanings were generated from Situation III only through expressions about the differences in the degrees of freedom of movement in the various worlds of Flatland. In spite of this, in this situation the students were passive viewers of the film, which did not offer opportunities for active involvement in motion. Therefore, this was an element of the definition not observed fully through the meanings and was taken into account for Phase 2.

The idea of capacity was clearly represented through the Math Wheel task. However, students' experiences were mostly articulated by trial and error actions, which did not show any clear reference to the specific idea. Flatland the film, on the other hand, presented various dimensional worlds and referred to many shapes in various dimensions. Nevertheless, similar to the idea of freedom mentioned before, the students were passive receivers of information and there were few if any opportunities for active participation. Consequently, students created some mathematically wrong generalisations of the type "*A sphere could go into the 0D and 1D world*". Therefore, this was one more element of the definition, which students did not clearly refer to, and together with the aspect of freedom, were considered for Phase 2.

9.4 Looking within situations

The comparison made between the meanings expressed in the categories of description and the orientation of dimensional experience, showed that some elements of the orientation were absent from the categories of description, or not clearly articulated. During the comparison, it was noticed that each situation had the potential to promote certain meanings. Thus, there was a reason to believe that the setting had a more significant impact on the formation of individuals' experiences and thus, a more comprehensive examination of its influence was conducted. This section (a) presents an analysis of this study's settings and (b) discusses how each setting differentiated the generation of meanings.

In Lave's (1988) terms, this analysis sets out a dialectical relationship between *activity* and *setting*. The purpose was to examine how individuals abstracted knowledge having in mind the influence of the setting in which the experience took place. This study considered that each situation involved a different setting having the potential to trigger specific experiences, which may have differed from those in other settings. More specifically, the analysis of each situation took into account (a) the representation of dimension in the tasks, (b) the types of vocabulary resources available to students and (c) the level of students' involvement. Since a set of categories of description was proposed, one might have conjectured that, if differences between the settings did exist, there would be discrepancies between how often they appeared in the various situations. Such differences in frequencies of occurrence pointed to the relationship of the setting embedded in each situation and its influence on the generation of dimensional experiences.

9.4.1 Situation I

The method used was the interview accompanied by tasks based on the computer software Elica. The two basic tasks of the interview made use of the Math Wheel and the Cubix Editor applications of the software.

The setting

The Math Wheel task offered an experience of dimension as a quality of space occupied by a shape (rotating a 2D shape to create a 3D one), while the Cubix editor task focused on 3D shapes and the reading of their representations (front, side and top views). The students explored the production of 3D shapes in a 2D real environment (paper), in a 2D virtual environment (computer screen) and in a 3D real environment (plastic blocks). There was a transition of dimensions from 2D (2D representations on the 2D real world) to 3D (real objects) and from 3D (real objects) to 2D (2D representations on 2D virtual world) again.

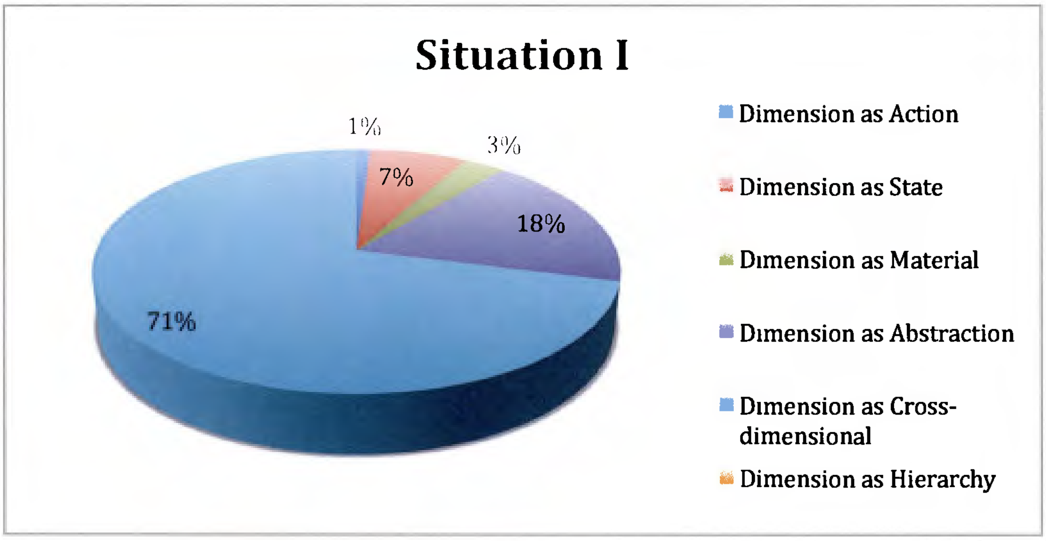
In general, students became acquainted with the Cubix Editor application quite easily despite some difficulties they had in depicting the depth of the 3D shape by reading the 2D representations. On the contrary, during the Math Wheel tasks, students seemed unfamiliar with the rotation of 2D shapes, and thus faced a greater difficulty in producing their 3D equivalent. Consequently, students' actions followed a trial and error technique especially for the Math Wheel task.

Students' meanings were often implicit in their actions rather than explicitly articulated through the spoken language. Sometimes, it felt as though there was a wealth of meanings that remained obscure.

Categories of description promoted

Table 6 shows how students' dimensional experiences (meanings of dimension) were distributed into categories of description during Situation I. As the pie chart illustrates, 71% of the meanings generated were categorised as *Dimension as Cross-dimensional.* This confirmed the aims of the specific tasks, which were based on the transition between 2D to 3D and vice versa. The cross-dimensional tasks helped the students to talk about the switch between dimensions. The Cubix Editor task gave students the opportunity to talk about how the same object could be presented in 2D and in 3D, while the Math Wheel task promoted the idea of creating 3D shapes by rotating 2D ones.

Table 6: Situation I distribution of categories



On the contrary, the three categories promoted the least were the *Dimension as Action, Dimension as Material,* and *Dimension as Hierarchy* that had no meanings. It was not surprising that although students' involvement in the task was dominated by actions on screen such as drawing and measuring blocks, their language did not show any meanings of dimension as action or measurement. This validated the assumption that the language remained embedded in the action and was not expressed in words. There was a significant percentage (18%) of expressing dimension as *Abstraction*. Students created situated abstractions during the Math Wheel task in order to make sense of the software and the task. They connected the 2D shape to the rotated 3D one by forming some 'rules', such as "*the more to the left the smaller the hole created*", which helped them to solve the tasks.

9.4.2 Situation II

The method used was the semi-structured interview influenced by Hunting's (1997) idea of clinical interviews. A task preceded the interview in order to introduce the students to the notion of dimension.

The setting

The introductory task consisted of various 2D and 3D objects on the table, which students had to divide into two categories and extract in that way the idea of dimension.

The interview plan included questions that students might have been familiar with at school, but most of them were questions that students had probably never thought about, such as "How many dimensions does a reflection in the mirror have?" and

"How many dimensions does a shadow have?" The aim of the first type of questions was for the students to reproduce what they had learnt at school and to become comfortable in discussing the topic further. The purpose of the latter type of questions was for the students to try to use what they already knew about dimension in order to explain situations in new contexts that they had not experienced before.

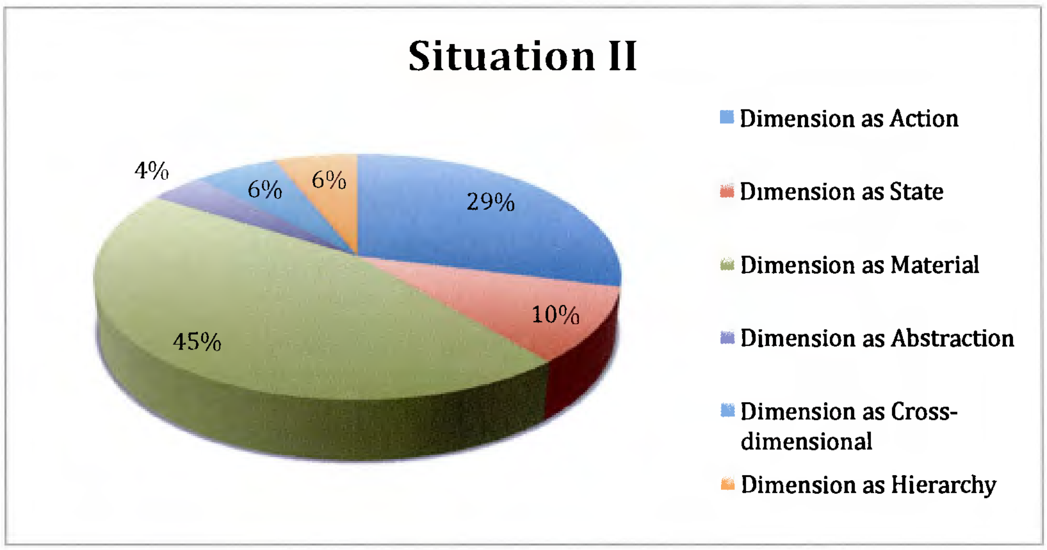
All the students divided the shapes into 2D and 3D during the introductory task and thus, it was easy for them to refer to the notion of dimension. The meanings extracted were dominated by spoken language and not so much on actions as in the previous situation. This gave an opportunity to the respondents to talk about non-visible and abstract terms. However, the students did not have sufficient language to talk deeply about these terms, and thus their meanings remained vague.

Categories of description promoted

Table 7 shows the distribution of the categories of description during Situation II. As the table illustrates, the two dominant categories were the *Dimension as Action* and the *Dimension* as Material.

There was a tendency of talking about dimension as if it was a real object having materialistic attributes (45% *Dimension as Material*). There were two reasons for why this might have happened. The first was that the introductory task prior to the interview might have influenced the students. In the specific task, real objects were presented to students and there was a possibility that students continued using these 'materialistic' terms during the interview, thus influencing their expressions of dimension. The second reason was that students talked of dimension as material because this was how dimension was presented to them in the real world. In order for the students to talk about this abstract term, they had to relate it to examples of their everyday lives.

Table 7: Situation II distribution of categories



Considering the last point above, students' expression of *Dimension as Action* was not surprising either. Students related dimension to their everyday lives, and this included both their experiences at home and at school. Seeing dimension as a measuring unit was promoted at schools, and this explained the high percentage of the *Dimension as Action* category.

Students' meanings about various dimensions were expressed both in hierarchical terms, starting from lower and moving to higher dimensions (6% *Dimension as Hierarchy*), and also moving between these dimensions (6% *Dimension as cross-dimensional*).

9.4.3 Situation III

The method used was the interview accompanied by the film Flatland. The film was broken into parts and each part included questions relating to the plot of the film, which indirectly implied reference to dimension.

The setting

Flatland the film acted as a resource for exploring how students experienced a cross-dimensional situation, which was ignored during the previous situation. The film helped the students to go beyond what they had already experienced in their everyday lives, and to perturb their thinking leading to a rich discussion about dimension. More specifically, this setting was designed to explore students' experiences about:

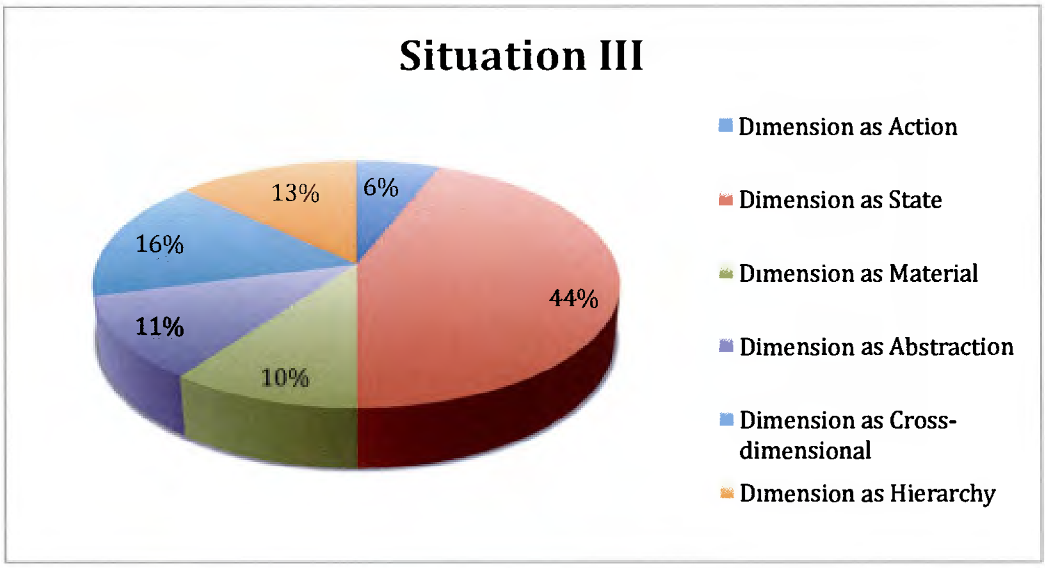
- Various dimensions of the same object (point, line, square, cube etc.).
- Worlds of dimensions other than two.
- Differences between three-dimensional objects and two-dimensional ones pointing to their degrees of freedom in movement and sight.

The film acted as a stimulus for the students to talk, and thus it shifted from the 'pure' interview that was dominant during the second situation. The disadvantage of this situation was, similar to Situation II, that the students remained passive during the interview because the setting did not include anything that the students had to do or construct. Thus, their actions were restricted to just discussing the plot.

Categories of description promoted

Table 8 shows the distribution of the categories of description during Situation III. There was a significant rise from the 6% that the category of *Dimension as Cross-dimensional* was during the previous situation to 16% in this situation. This showed that the setting did influence the students' ability to talk about dimension in cross-dimensional terms.

Table 8: Situation III distribution of categories



As the table demonstrates, there was a significant rise of expressing dimension as a State (44% compared to the 10% and 75% of the previous situations). This showed that the specific setting influenced the students towards talking of dimension as a place or a domain. Indeed, dimension was mostly presented as a state during Flatland the film. The worlds of Pointland, Lineland, Flatland, and Spaceland represented dimension as if it was a world, and this influenced students' expressions. Apart from the large percentage of the "*Dimension as State*" category, the distribution of the rest of the categories was balanced.

9.5 Summary

This chapter presented a discussion of the findings of Phase 1 by comparing the meanings that constituted the categories of description to the orientation of dimensional experience and the definition of dimension. From this comparison, the following elements were not visible in students' articulations of dimensional experience and thus, were taken into consideration for the Phase 2 of this study:

- The development of situations expressing an informed background of many aspects of the world relating to dimension that might be used to stimulate and challenge the students.
- The identification of what stays invariant and what changes in a set of transformations
- The construction of various dimensional figures in various dimensional domains.
- The expression of dimension as a quality of both object and space.
- The development of active situations for expressing dimension as quality of freedom of movement.
- The development of active situations for expressing dimension as quality of capacity.

Subsequently, Phase 1 explored the variation in the learners' experience of the situation, and discovered that the learners were largely oriented towards the phenomena that were present there. The exploration of the meanings' frequency within each situation, gave an insight into the significant role that the situation had to the formation of experience. Considering the settings in which the meanings of Phase 1 took place, it seemed that the frequency of generating meanings belonging to a specific category of description was dependent on the task on which the individuals were working. For instance, most of the meanings relating to *Dimension as State* were extracted from the Flatland task (Situation III), in which there was an obvious orientation towards the worlds of dimension. *Dimension as Material* meanings were mostly generated by the interviews (Situation II) in which the individuals had to talk about dimension as a notion in general, and thus they linked it to their everyday reality. *Dimension as Cross-dimensional* meanings were mostly generated through Elica (Situation I) where students had to move from 2D to 3D and vice versa. Phase 1 looked at the categories of describing dimensional experience *within each situation*, and illustrated that students were oriented toward the phenomena that were present there.

Subsequently, Phase 2 aimed to explore the variation in the children's experience of the phenomenon by looking *between the various situations*. By considering the role of the setting in the formation of dimensional experience as well as taking into account the omitted elements of the orientation of dimensional experience and the definition of dimension, this study moved a step forward by designing a new situation aiming to embrace the new outcomes. The intention of the new situation, Situation IV, was to act as a window on experiences of dimension not observed before. The role of the setting was thus revisited, by taking a deeper micro- perspective on the role of designing of the fourth situation. Subsequently, a revisiting of the existing structure of categories of descriptions took place leading to a discussion about the factors that influenced the formation of dimensional experience.

Phase 2

Chapter 10: An approach for moving between situations

10.1 Overview

Phase 1 looked within each situation and identified the significance of the setting in the formation of experience. The aim of Phase 2 was to explore the variation in children's experience of dimension by looking between the various situations. In the process of doing that, this study considered the importance of the setting to the formation of meanings as extracted during Phase 1, and as a first step designed a fourth situation aiming to generate meanings, which reflected the missing elements of the orientation as extracted from Phase 1.

This chapter presents the exploration for an approach developed first, to look between situations, and second, to form the theoretical basis on which the fourth situation could be built. Phenomenography together with the designing and modelling theories were considered to form the starting point of the elements that could constitute the last situation. By considering the importance of the situation to the formation of experience, the role of 'window' is revisited in section 10.2, pointing to its significance for the designing of Situation IV.

Subsequently, Section 10.3 goes back to the previous situations and presents how their affordances and pitfalls were considered in order to encourage or avoid their presence in the new situation's design. It also suggests possible ways for integrating the missing elements of the orientation into the research design. Finally, Section 10.4

takes a deeper micro-perspective into the principles that were taken into consideration while choosing the software and its tools for the next step of study.

10.2 Designing a window for abstraction

Phase 2 continued following a phenomenographic approach, aiming to situate students in communicative practices and attempting to understand how they made sense of dimension "across conceptual and linguistic boundaries" (Säljö, 1996, p. 24). However, the fourth situation had to be designed in a way that would have the potential to generate dimensional experiences relevant to the missing elements of the orientation (Table 9).

Table 9: Missing elements from the orientation of dimensional experience

MISSING ELEMENTS FROM THE ORIENTATION OF DIMENSIONAL EXPERIENCE
1. The development of situations expressing an informed background of many aspects of the world relating to dimension that might be used to stimulate and challenge the students.
2 The identification of what stays invariant and what changes in a set of transformations
3. The construction of various dimensional figures in various dimensional domains.
4. An expression of dimension as a quality of both object and space.
5. The development of active situations for expressing dimension as quality of freedom of movement.
6. The development of active situations for expressing dimension as quality of capacity.

In a sense, Phase 2 tested the conjecture that it was possible for children of this age to experience these other aspects of dimension, in effect to provide evidence that the lack of previous articulation of these elements was related to the structures within the settings rather than, for example, developmental aspects of the children. It followed the view that through experimentation students might be led to reconsider their initial understandings, and probably reconceptualise the given phenomenon (Marton, 1992). Thus, Situation IV took a deeper micro-perspective into the role of the setting by examining how the role of designing and modelling could influence children's dimensional experiences.

In relation to the missing meanings of dimension, the purpose of this new situation was to expand children's *contextual neighbourhood*, which, according to Pratt and Noss (2010), "captures the domain over which the idea has been encountered and found to be powerful by the child in explaining the on-screen behaviour" (p. 94). Thus, a domain needed to be designed intending to expand the contextual neighbourhood of children's experiences of dimension:

> [...] *designing for mathematical abstraction* becomes a challenge to create a domain for the articulation of situated abstractions, and the means by which the contextual neighbourhoods can be refined and expanded. (Pratt and Noss, 2010, p. 97)

Building on the idea of designing for mathematical abstraction, the new situation was designed intentionally to refine and expand the experiences of dimension; and, a rich way to do this, was to use the rationale proposed by Noss and Hoyles (1996) regarding the notion of 'windows on thinking-in-change' (p. 9).

Although the first three situations were designed to have the potential to act as 'windows' into students' dimensional experiences, the role of the 'window' had to be revisited in more depth in order to embrace their outcomes. One element considered was that Situation IV had to act as a 'wider' window into students' experiences of dimension than the previous three situations.

Building on the notion of 'windows', Noss and Hoyles (1996) argued that the researcher gets to examine the thinking processes by introducing new ideas and trying to understand how the thinker connects these notions with his/her previous knowledge. As a result, the window itself and the way it is designed influences the experiences to be extracted. Therefore, its designing was considered a vital element of the research process and had to be re-examined. According to Noss and Hoyles (1996):

> Windows are for looking through, not looking at. It is true that windows mediate what we see and how we see it. Equally, windows can, at times, be objects for design and study. But in the end, what counts is whether we can see clearly beyond the window itself onto the view beyond (p. 10).

Consequently, Phase 2 had to design Situation IV in such a way as to be as 'transparent' a window as possible for the researcher to look into children's experiences of dimension. At the same time, a second aspect of the window was to engage students in experiences of dimension. In Situation IV, the window had to embed the notion of dimension having the expectation that the child would concretise these ideas through its use (Wilensky, 1991).

The notion of 'embeddedness' was first introduced in this study through phenomenography, when Marton et al. (1997) talked of the "situation in which the phenomenon is embedded" (p. 83). Looking from a pedagogical perspective, the notion of 'embeddedness' together with 'situated abstraction' could point to the situated *roots* of the mathematical notion and offer *routes* for developing a more developed and a sophisticated way of thinking about dimensional geometry (Mason et al., 1985).

However, in order to embed a phenomenon in a situation, one had to consider the ways in which the phenomenon could be modelled within the specific situation. This led to an investigation of the preceding research on how modelling could foster the utility of mathematical concepts (Ainley, Pratt and Hansen, 2006; Noss and Hoyles, 2006; Simpson, Hoyles and Noss, 2005). To begin with, modelling was construed as:

> The process of encountering an indeterminate situation, problematizing it, and bringing inquiry, reasoning, and mathematical structures to bear to transform the situation. The modelling produces an outcome--a model--which is a description or a representation of the situation, drawn from the mathematical disciplines, in relation to the person's experience, which itself has changed through the modelling process (Confrey and Maloney, 2007, p. 60).

Phase 2 followed the above idea of modelling, in order to design a situation that the children were more likely to problematise and reason about. Through this process new experiences were more likely to be generated; or even, students' existing experiences could have been further developed or expanded.

Exploring modelling further, the various ways through which dimension could be modelled were considered. First, it was decided that the models used had to represent the modelled world of the phenomenon as close to reality as possible in order for the students to feel confident in a familiar to them domain:

> …it will be most fruitful to introduce children to modelling practices through models that preserve resemblance because these models more rapidly sustain mappings between the model and the world. As children learn, over a number of cases, that resemblance is less fundamental than function, they will become increasingly prepared to work with kinds of models that do not preserve similarity between the model and the modeled world (Lehrer and Schauble, 2000, p. 108).

Second, the use of the computer was considered as the best medium to be used as a model of dimensional ideas, and through its use to extract students' articulations of experiences:

> I see the computer as helping in two ways. First, the computer allows, or obliges, the child to externalize intuitive expectations. When the intuition is translated into a program it becomes more obtrusive and more accessible for reflection. Second, computational ideas can be taken up as materials for the work of remodeling intuitive knowledge (Papert, 1980b, p. 145).

According to Papert, the computer had the potential to be used both for helping the students to externalise their experiences but also to develop them further. Likewise, the computer could act as a window both to students' experiences of dimension, but also to the researcher to actually observe the articulations of children's experiences on screen. As Noss and Hoyles (1996) argued, the computer "provides a screen on which learners can express their thinking, and simultaneously offers us the chance to glimpse the traces of their thought" (p. 6).

Indeed the proper use of ICT was noted to create new situations of thinking and creation of relationships (John and Sutherland, 2004) during Chapter 2 of this thesis. But, at this stage of the research, its use was re-examined in more depth by looking at the potential of computational environments and microworlds for modelling dimension.

Computational environments were believed to be able to model mathematical ideas which were previously inaccessible (Resnick, 1995). Relating this to Wilensky's notion of 'concreteness' (1991), computational environments could act as a domain which had the potential to bridge the 'concrete' with the 'abstract', by presenting abstract ideas as concrete and visible to the learner who could manipulate as if they were material entities (Alberti and Marini, 1995; Laborde and Laborde, 1995; Noss and Hoyles, 1996). In other words, computational environments could be used as domains, which could 'embed' abstract ideas. The idea of microworlds was then introduced, as the type of computational environments, which had the potential to 'embody' mathematical ideas:

> It is perhaps in this sense that we can speak of a microworld as "embodying" mathematics: not because of some reifying link between the representation and the mathematical entity, but because of the opportunity that such environments provide for learners to kinaesthetically and intellectually interact with the designers' construction of a system of mathematical or scientific entities. (Edwards and Benedickt, 1995, p. 150)

Likewise, Laborde and Laborde (1995) argued that geometrical concepts were 'built-into' a computational environment such as Cabri-géomètre, which controlled the software's behaviour. Adding to the above, the microworld also needed to have the potential to reflect a structure by connecting the various embedded ideas and identifying relationships between them (Edwards and Benedickt, 1995; Laborde and Laborde, 1995).

Considering the above, it was decided the software chosen for this study had to 'embed' the mathematical ideas underlying the missing elements of Phase 1, and also to show the connections between these mathematical ideas. The next step was to go back to the missing elements of the orientation and try to re-examine those through the perspective of designing for abstraction and modelling for embedding mathematics. The aim of this re-examination was to gather some useful components that could inform the choice of the software and the task design of Situation IV.

10.3 **Bridging the gap**

The experience gained by designing the previous situations was helpful for the designing of this one in order to avoid any pitfalls noticed, and to incorporate their affordances to the new design, having the aim to embrace the six elements missing from our orientation. It was noted that Situation I was based mostly in actions while Situation II was the exact opposite by having students only express themselves in words. Situation II did not include a stimulus for discussion, although *having something to talk about* and a story to tell was considered an important component in the designing (Confrey et al., 2010). Thus, the tasks designed for Situation IV had to offer children a stimulus for discussion. Adding to this, they needed to be purposeful, having a meaningful outcome for the child, but also the mathematical ideas embedded needed to have a useful meaning that the child could acquire (Ainley, Pratt and Hansen, 2006). This related to the first element missing from the orientation (Table 9, p. 218), which was the development of situations expressing an informed background of many aspects of the world relating to dimension that might be used to stimulate and challenge the students. Therefore, Situation IV needed to have a purposeful outcome, possibly relating to a real life situation, that could challenge the students to see the utility of the mathematical ideas embedded within it, and to exchange ideas while working on the target.

Situations II and III were criticised because the children were only passive responders. Likewise, the fifth and sixth element missing (Table 9, p. 218) requested a situation in which students would be actively involved. Therefore, one of the characteristics of Situation IV was to actively engage students in the situation in order to experience the restrictions of movement within a particular dimensional domain and the idea of capacity as extracted from the orientation.

The third absent element (Table 9, p. 218) related to having students constructing figures of various dimensions in various dimensional domains. This was also linked to the sixth element (Table 9, p. 218) that talks about capacity. Questions like "Which shapes can I construct on a 2D surface?" compared to "Which shapes can I construct on a 3D space?" were considered in the design process.

The second element missing (Table 9, p. 218) related to the identification of what stays invariant and what changes in a set of transformations. During the previous situations, it was noticed that the students' need of characterising cross-dimensional environments (both during Situations I and II) acted as an activation tool for their thinking, which until that moment was static. In trying to understand what changes and what stays the same in such situations their thinking was disturbed and challenged. For that reason, Situation IV considered the use of cross-dimensional tasks as a way of perturbing students' thinking.

The fourth point of the missing elements (Table 9, p. 218) related to the embracing of the idea of dimension as both a quality of object and space. Thus, Situation IV sought to consider how students articulated thoughts about object and space aspects of dimension when their experiences were shaped by a task designed to be a window on how they might bridged these two perspectives. In that sense, it took a more active stance than the earlier situations designed, actively seeking to perturb children's experiences, since the experience of object/space did not seem to occur in conventional schooling, or at least no such experiences had been reported in the first three situations.

Prodromou and Pratt (2006) demonstrated this idea of 'bridging' although their study referred to distribution and statistics. They argued that there were two perspectives of distribution, data-centric and modeling, and they suggested that "causality may be a significant agent in constructing a bridge" between these two perspectives (p. 86). Looking into this notion of bridging further, the framework proposed by Abrahamson et al. (2007), regarding learning axes and bridging tools was related. Abrahamson et al. (2007) suggested a framework where first the designer focuses on mathematical representations relating to the target concept which then divides into two main idea elements. With regards to this study, these two elements of dimension (i) as quality of space and (ii) as quality of object were already identified. The significant part of the Abrahamson's et al. (2007) work was the second step of the designing, where the designer creates bridging tools accompanied with learning situations where they lead into cognitive conflict:

> Next, the designer creates bridging tools, ambiguous artifacts bearing interaction properties of each of the idea elements, and develops activities with these learning tools that evoke cognitive conflict along the axis. Students reconcile the conflict by means of articulating strategies that embrace both idea elements, thus integrating them into the target concept. (p. 24)

The question raised at that point was, what kind were these effective bridging tools that could be designed in creating a situation where students' experiences could be perturbed? The next section describes the principles that the tools of the software needed to have in order to have the potential to act as bridging tools between the different ideas of dimension.

10.4 The bridging tools

Phase 2 needed to design the new situation in such a way that the child would have been able to generate the experiences of dimension not observed before and also to bridge the two disconnected ideas of dimension. The previous paragraphs expressed the importance of the tools – in this case, the software tools- that could help the students to achieve these targets. Orhun (1995) supported that these 'devices' could be the communication medium between the students and the researcher's intentions:

> The user engages in communicative actions not only with other users in dealing with situations but also with the devices in order to interpret the designer's intentions embodied in them. (Orhun, 1995, p. 309)

Bringing into the idea of 'bridging' (Abrahamson and Wilensky, 2007; Prodromou and Pratt, 2006), the 'bridging tools' had to be effective in creating a situation where students' thinking on dimension could be perturbed. Thus, a set of principles had to be noted in order to choose the best software and tools that could embrace the above objectives.

In order for the students to identify the researcher's intentions, the mathematical ideas embedded in the tools had to be visible to the child. The tools needed to have an *expressive power* (Abelson and DiSessa, 1986), and be transparent by making "visible its operations and how they are integrated with the embedded context" (Orhun, 1995). The mathematical ideas could become transparent to the child if the tools had restrictions. As Gargarian (1996) argued:

> Designing requires restricting design. Without restrictions, a designer would be unable to choose from the possible actions he could take; he would be paralyzed. Moreover, an action has little value if there is no way to evaluate the effect of the action (p. 132).

Indeed the restrictions of the tools could offer the opportunity for the user to search for a solution to the problem, and thus explore the environment and the ideas embedded further. Having the shape right, for instance, could act as a stimulus for a further exploration of the tools and the mathematical potentials they embed.

Talking of restrictions, the idea of 'messing up' identified by Healy et al. (1994), and the notion of 'figure' as a bridge between unrestrained drawing and the mental geometric ideal (Laborde, 1995b), was considered appropriate. As Healy and Hoyles (2002) pointed out:

> The distinction between drawings that can be 'messed up' and figures whose geometrical properties are retained under dragging can be exploited as a window through which students can come to appreciate the theoretical aspects of geometry (p. 236).

The restrictions of the tools could also lead to restrictions of activity, having the potential to create a window. For instance, trying to find the shape 'right' could act as a window to the mathematics embedded and this process could give the opportunity to the child to generate new experience or even modify previous conceptions:

> While working with computer software, pupils adapt their strategies to its constraints and functioning mode, and the new meanings are generated by this continuous process of adaptation, thus constructing new knowledge (Osta, 1998, p. 130).

Likewise Papert (1980b) and Edwards (1995) talked of the idea of debugging as the process through which the children search of what they made wrong and try to find a way to fix it. Although Papert talked about debugging in a LOGO environment, this process could also happen in any other expressive software:

> Typically in math class, a child's reaction to a wrong answer is to try to *forget* it as fast as possible. But in a LOGO environment, the child is not criticized for an error in drawing. The debugging is a normal part of the process of understanding a program. The programmer is encouraged to study the bug

> rather than forget the error. And in the Turtle context there is a good reason to study the bug. It will pay off (Papert, 1980b, p. 61).

By interpreting the feedback from the manipulations of the tools, the child was more likely to reconsider its understanding of the domain of the software.

Last, by re-considering the idea of 'bridging' in order for the tools to have the potential to bridge ideas, first the tools needed to have the capability to link more than one mathematical idea between them, and second, a combination of tools needed to reflect a more complex mathematical notion.

Gathering the principles explored in the previous paragraphs, the software chosen had to incorporate computational tools which embedded ideas of dimension not observed before, in a transparent and visible form for the child to notice; it needed to have restrictions which could act as obstacles to the children's construction and were more likely to lead them into a debugging process aiming to extract the mathematical ideas embedded within. Last, the embedded mathematical ideas had to be linked within the tools in a way that students could connect and create relationships leading to a logical reasoning.

10.5 Summary

This chapter set the theoretical foundations through which the fourth situation could be designed. By considering the role of the setting as extracted from Phase 1, the new situation examined more deeply the significance of designing for abstraction, and the embedding mathematical ideas through modelling. Looking from a designer's perspective, some important affordances, which the software and tasks needed to satisfy were considered in order to embrace the six elements missing from the orientation. First, it was noted that the tasks needed to have something that the children could talk about, pointing to the ideas of purpose and utility. Also, the software and tasks had to model real-world phenomena as close to reality as possible in order for the students to connect with their everyday experiences. Active situations needed to be designed, having the students actually constructing objects/spaces, and the exposure in cross-dimensional situations was considered suitable. Last, the software and tasks needed to have the potential to perturb students' experiences and offer them an environment through which they would have the potential to bridge ideas.

Subsequently this chapter took a deeper micro-perspective into the principles that the software tools needed to have. Tools needed to 'embed' abstract mathematical ideas, which they could present in a transparent and visible way for the students to connect. The importance of having restrictions and tool constraints that could lead into debugging processes was also noted. And finally, the tools had to also be 'bridging tools' aiming to show relationships between mathematical ideas. The next chapter discusses the designing of Situation IV, giving information on the software chosen, the tasks designed and the tools used.

Chapter 11: Situation IV Google SketchUp

11.1 Overview

So far, Phase 1 showed evidence of six different 'characterisations' of experiences of dimension. Comparing the categories of description to this study's orientation of dimensional experience and definition of dimension, there were six main elements that were 'missing' (Table 9, p. 218). This new situation aimed to offer students an active situation that could embrace and build bridges to these six elements relating to dimension. The theories on designing and modelling as discussed in the previous chapter extracted some useful guidance on the designing of the particular study, especially for choosing the particular software, designing the tasks and identifying the bridging tools.

This chapter describes the designing of Situation IV. In brief, experiences of dimension were generated from twelve 10-year old students, different than the ones used for the other situations, but from the same school in London. The students worked in pairs, having six pairs in total for this particular situation. The interview was the main method in this study following the idea of clinical interview as described by Hunting (1997) accompanied by the completion of tasks with the use of the software Google SketchUp. Each interview lasted around 120-180 minutes.

Section 11.2 illustrates why the particular software was chosen in order to incorporate the necessary affordances as described in the previous chapter and presents a description of the main features of the software. Subsequently, Section 11.3 presents the interview plan showing the development of the tasks designed for this situation, and Section 11.4 examines the mathematics embedded in this situation by looking at the software's *dimensional tools* through a micro-perspective.

11.2 The software

To begin with, software was used because the use of computational environments was considered to offer dynamic situations for exploring experiences of dimension (Alberti and Marini, 1995; John and Sutherland, 2004; Laborde, 1995b; Noss and Hoyles, 1996; Papert, 1980b; Resnick, 1995). This situation used the software Google SketchUp for designing active thinking environments. Google SketchUp is 3D modelling software first released in 2000 as a general-purpose 3D content creation tool, easier to use than other 3D CAD programs. The version used for this study was SketchUp 7, which was downloaded free from the Google SketchUp website (http://sketchup.google.com/).

Google SketchUp was chosen because it was a 3D modelling software, which depicted objects and space close to reality. Thus, it offered a domain through which students could easily make connections with their familiar real environments. Although Google SketchUp was initially created for use within the disciplines of architecture and art, its features were easy to use even for primary school students, but of course with some modifications.

The use of CAD software was considered an effective tool for the teaching/learning of geometry as first it prepares students for many professions in which CAD is used (such as architectural, civil, and mechanical engineers as well as filmmakers, game developers, and related professions), but also it was thought to be very effective for developing skills of visualisation, representation and transformation of 3D objects (Osta, 1998). Similarly, during a study named the Math and More project, a simpler version of CAD named KidCAD was used for research on scale drawings with primary school students and it proved to be successful for developing the skills of representation and visualization (Watt, 1998). Likewise, SketchUp was thought to be simple enough for primary school students to use. A description of the software and its tool bar are presented in the following paragraphs.

The software interface includes a basic window and a toolbar (see Figure 50). The basic window is coloured green and light blue in a way to represent the ground floor and sky. In the Sketch environment there are three axes by default: the red, the blue and the green axes. The axes are perpendicular and their extensions are marked with dotted lines (negative direction). There is also a person standing 'on the ground', the latter being defined by the red and the green axes.

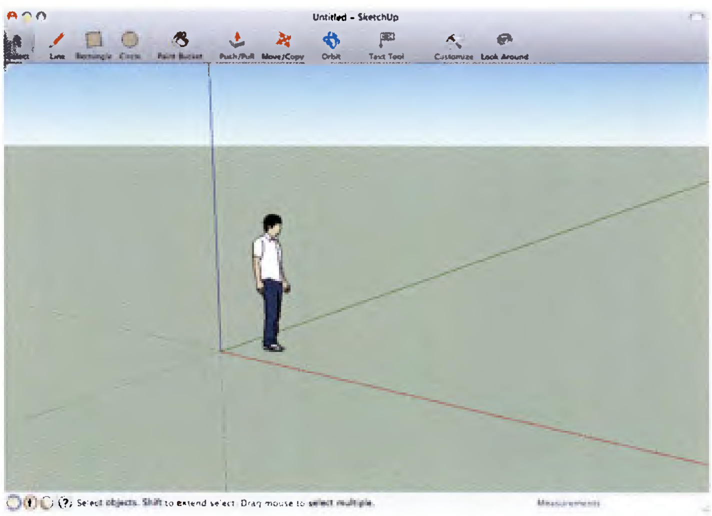


Figure 50: Software user interface

The toolbar is situated at the top of the basic window. It contains the tools and the menu items necessary for the user to become acquainted with SketchUp. The tool bar can be customised using the Customise tool. For the needs of this situation, only a selection of tools was used, which were: the Select tool, the Line tool, the Rectangle tool, the Circle tool, the Paint Bucket tool, the Push/Pull tool, the Move/Copy tool, the Orbit tool, the Text tool and the Look around tool (see Figure 51).

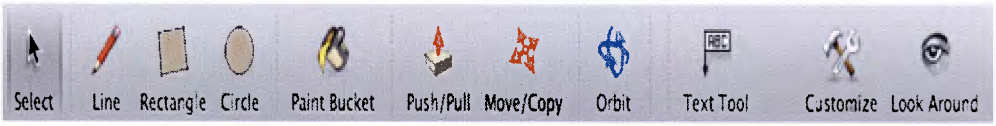


Figure 51: SketchUp's toolbar

Each tool was selected because it was useful for a particular task. Some of the above tools were more valuable than others as they served significant roles to the actual design of the tasks and the mathematics involved. More about these 'special' tools and how they related to the designing of the tasks is discussed in the next sections.

11.3 The tasks

Each task of the interview plan was designed according to the principles that Ainley et al. (2006) proposed for creating purposeful tasks (p. 35-36):

- It has an explicit end product that the pupils care about
- It involves making something for another audience to use
- It is well focused, but still contained opportunities for pupils to make meaningful decisions.

The main task designed was for the children to create a neighbourhood by using the particular software. This was the stimulus offered to students to link the virtual environment to their reality, and it also proposed something for the students to talk about, which was considered as an important element of the designing during the previous chapter. Furthermore, it was an end product that the students could find purposeful, and also students had to use the mathematics embedded in order to complete their task. (More about the 'special' tools and the mathematics they embedded are discussed in the next section.) What is more, creating buildings was considered to be an active situation through which students could construct shapes themselves and also could move around space themselves and experience its restrictions. It offered a cross-dimensional situation because building required creation of various dimensional shapes (1D (lines), 2D and 3D shapes) on both 3D space and 2D surfaces.

Google SketchUp 7 was the software used for the design of the tasks. It was also a requirement for students to learn its use in order to complete the tasks. As aforementioned, the main theme of the interview was for the students to build a neighbourhood and this was described through the first task. However, more tasks were created to enrich the interview plan by being more specific, leading students to the exploration of specific mathematical notions. Four different theme tasks were designed in total, each one of them consisting of smaller tasks and questions. The end product of the interview was described through the first task (Task A), where students had to use SketchUp to build their neighbourhood. Task B and C were brought into play according to the needs of the students, which could be either during this first task or after its completion. In other words, these tasks were complementary because they offered space for further exploration of specific tools or even notions. Thus, the sequence of the tasks was not as described below, and it actually differed among pairs of students. Task D was a discussion at the end about some general ideas of dimension, which appeared during the interview. In the following paragraphs, the interview plan used is described, including the tasks involved:

TASK A: Building a neighbourhood

Description: Here is an example of a neighbourhood a pair of students designed. They used special software for designing (The researcher shows the 3D example of a neighbourhood).

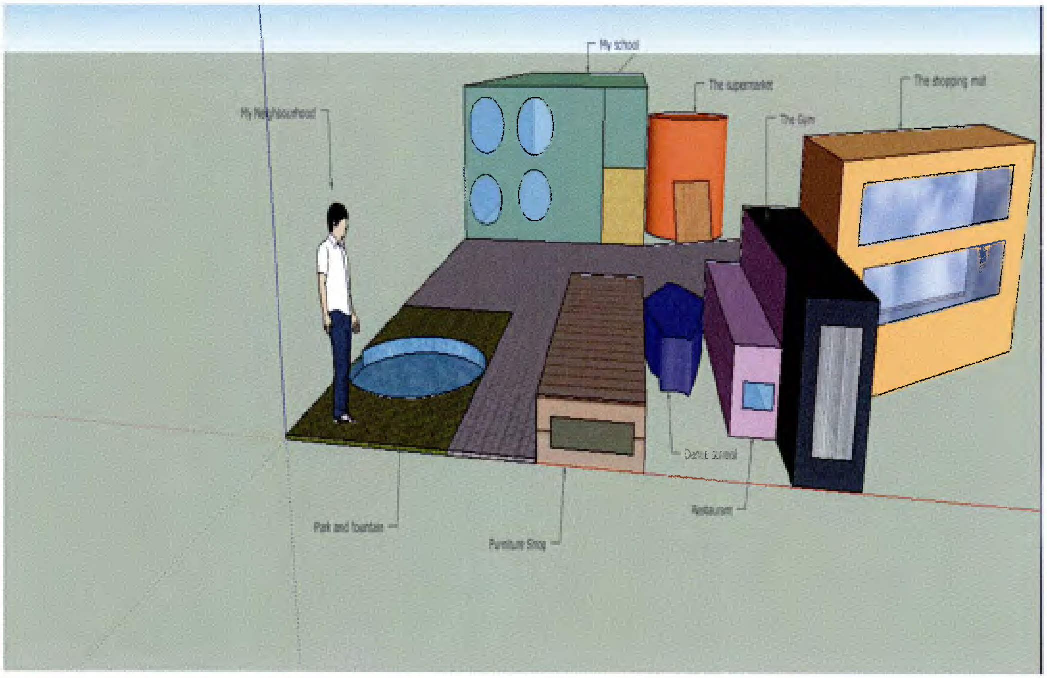


Figure 52: 3D example of neighbourhood shown to students

The task is to draw your own version of a neighbourhood like the example. For building the neighbourhood, students first designed it like this (the researcher shows the 2D example of the neighbourhood):

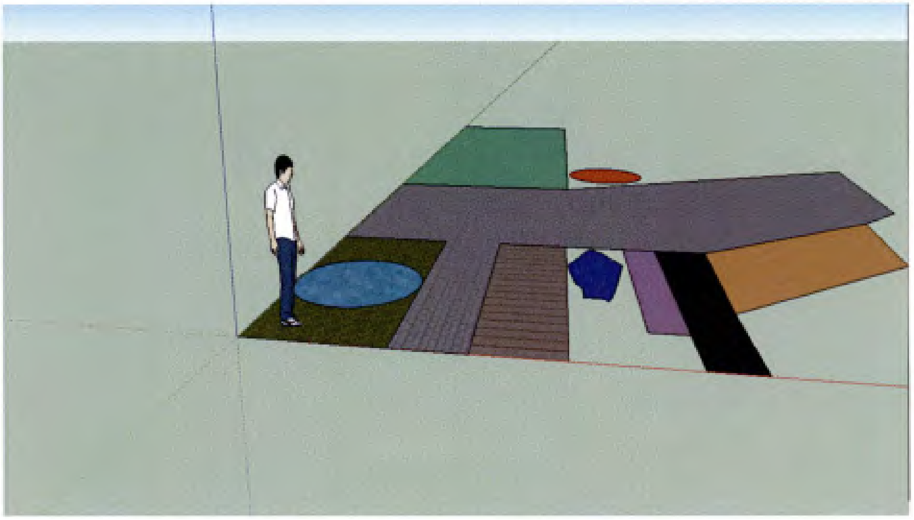


Figure 53: 2D example of neighbourhood shown to students

Use the tools on the top of the screen to do that. (I, the researcher, just show where the tools are, but I do not explain them to the students. I also reset the starting image in front of them and adjust it so that it would give them a larger area to draw.)

Use a special tool (push/pull button) to make it like that (like the 3D example in Figure 52*). Now, first describe your neighbourhood.*

(Designing starts)

Possible questions during the designing

- *What do you think is happening?*
- *This is a very big field. Choose a part of it to build your city. Why have you chosen this part? (Defining the plane they want to work with)*
- *What shape is your house/ supermarket/ school?*
- *Do all the buildings in your neighbourhood have the same shape?*
- *What is their similarity?*
- *What does this tool do to the shapes?*
- *Which part of your building do you think it would be the most difficult? Why?*
- *Which objects are you doing first? Why?*
- *Click the orbit button and look your world from different views. Is the whole world on the same level (plane)? What do we need to change?*
- *What do you think is the role of the red, green and blue lines? Why do you think they are there?*
- *Now look at this version of your world (2D).*
    - *How should we call this world?*

  - *Describe this world to one of your classmates who is not here (3D top view) / Imagine a man walking on the road of your neighbourhood. Describe what he sees.*
  - *What shapes belong to your 2D world?*
  - *Imagine you are a tiny bug walking on the road of your neighbourhood. Describe what you see (bug's eye view).*
- *Now press the pull/push button on your objects.*
  - *How should we call this world? (the 3D)*
  - *How is this world different than the previous one?*
  - *How is this world the same as the previous one?*
  - *Did the same shapes that exist in the 2D, existed in the 3D world? What changed?*
  - *What shapes belong to your 3D world?*
- *Explain to another student how you have created your world in order to do the same. What would you advise him/her to pay attention to? (any difficulties you had, how you sorted it out, limitations of the software etc)*
- *Which part of your building was the most difficult? Why? How did you sort it out?*

TASK B: Carefully structured problems

1. Create a rectangle. Is there any other way of creating a rectangle? (Rectangle button/ rectangle created by line segments)

2. Create a cube. Is there any other way of creating a cube?

(square and then push/pull tool or by using only the line tool or by using the rectangle tool)

3. Axes: Now look at these three shapes I have constructed.

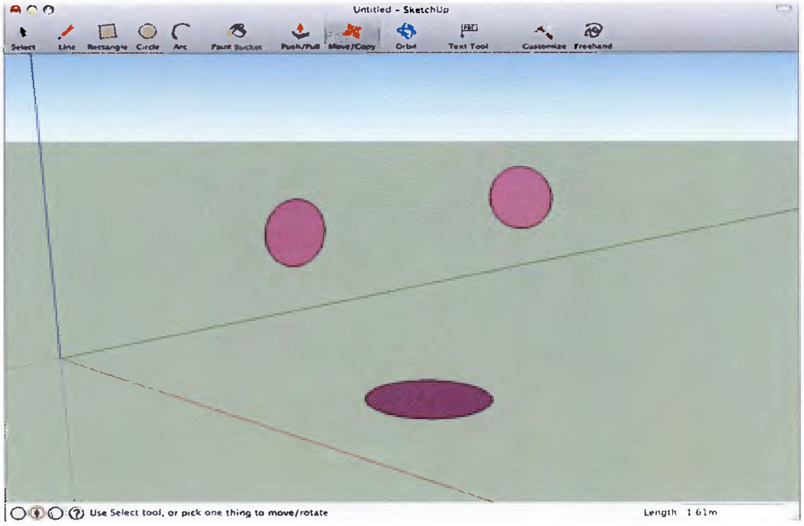


Figure 54: Task B3 the three circles before Push/Pull

-What shapes are they?

- What do you think it will happen if I press the push/pull button on them? In which direction are they going to be pulled out?

Now press the push/pull button.

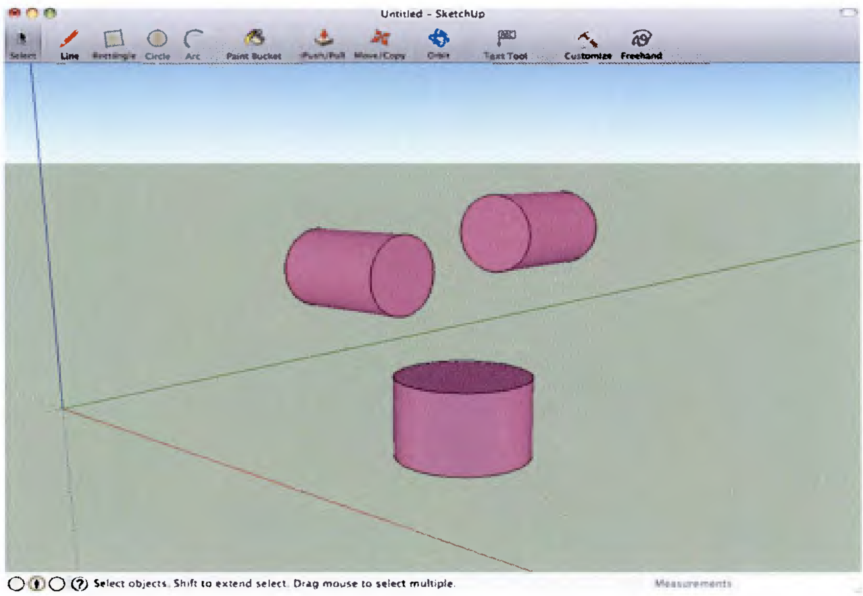


Figure 55: Task 3B the three circles after Push/Pull

-What happened?

- Is that what you expected?

- Why are they pulled out differently?

- Why do you think this happens?

4. Here are some incomplete shapes (Figure 56: Task B4 Incomplete frames). I would like you to complete them.

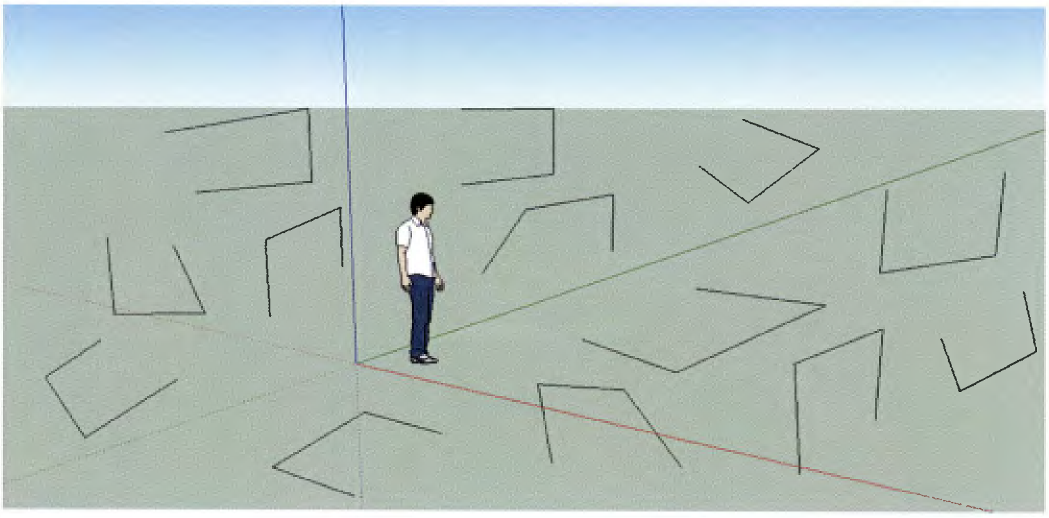


Figure 56: Task B4 Incomplete frames

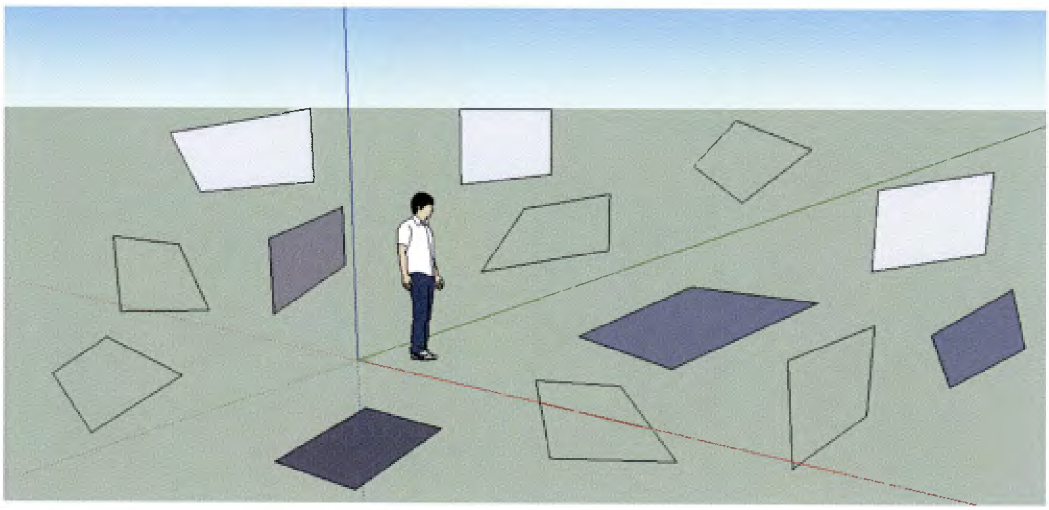


Figure 57: Task B4 Completed frames

- What do you observe? (Some got a colour while some others didn't)
- Why do you think this happened? (Getting some initial hypotheses)
- Now use the orbit button to turn around. What do you see?

- Why do you think that some shapes were coloured and some not?

- In a new window, draw a coloured shape.

- Now draw a shape that is not coloured.

5. Here are some shapes I drew (Figure 58: Task B5 Rectangles – Orbit)

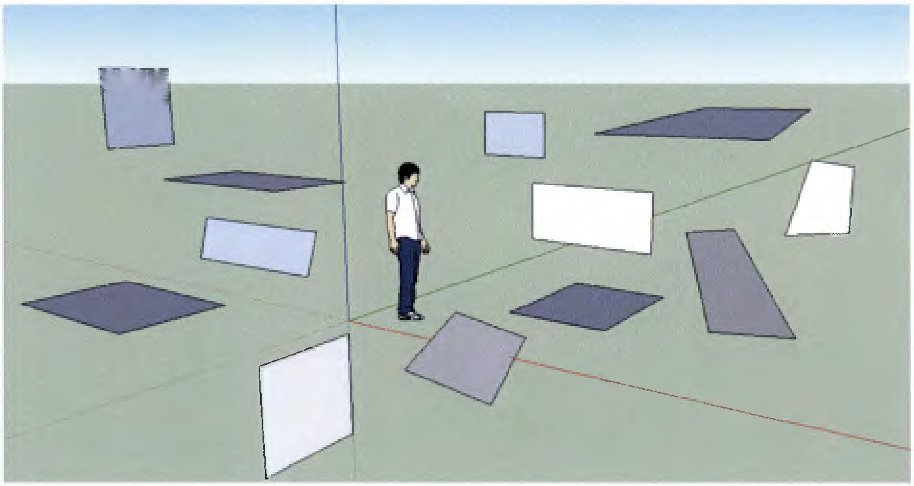


Figure 58: Task B5 Rectangles – Orbit

- What are their similarities?

- What are their differences?

- Use the orbit tool to look around. How are they different?

6. Here is the 3D neighbourhood I created. It is the same as the one I gave you on paper. Now I would like you to look at it by using:

(a) the Look Around tool, (b) the Orbit tool

- How do you look at it by using the Look Around tool?

- Is it the same as using the orbit tool?

- What are the differences in view?

7. Here are some circles I created (Figure 59: Task B7 Circles).

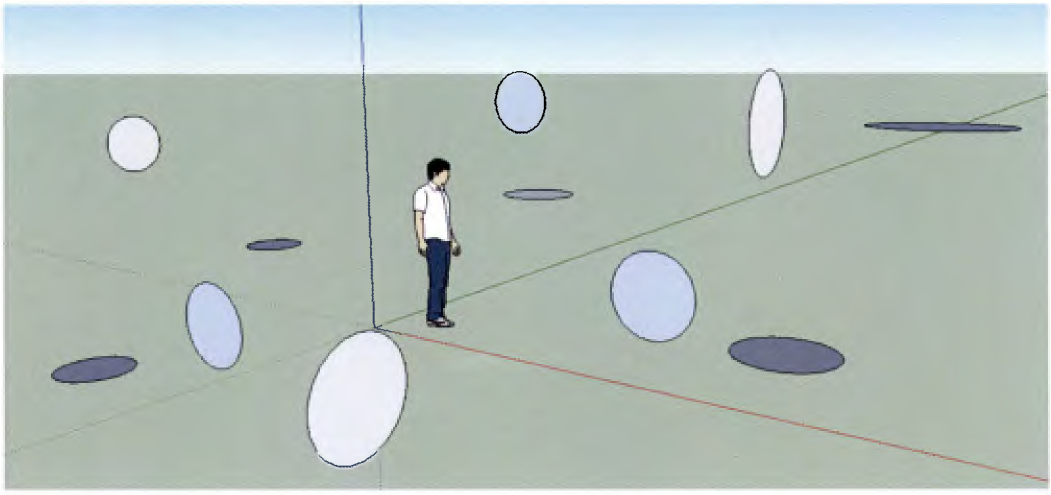


Figure 59: Task B7 Circles

- Are they the same?
- What are their differences?
- Now press the circle tool and hover the pointer on the circles. What do you observe? Have you noticed the colour of the pointer?
- Why do you think they have a different colour?
- Now, use the orbit tool to turn around. What do you see? How are the circles different now?
- In a new window, try to draw a circle that has (a) a blue colour, (b) a green colour (c) a red colour.

8. Here are three rectangles (Figure 60: Task B8 Turning rectangles).

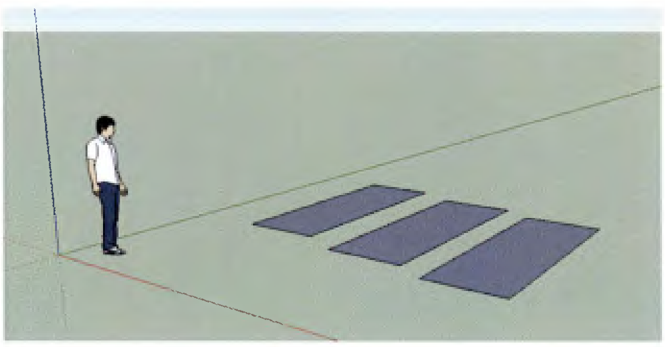


Figure 60: Task B8 Turning rectangles

- What colour are they?

- Now press the Move tool and press on the side of the first rectangle. Try to rotate it. What colour is it now?

- Now rotate the second one so that it gets a different colour from the other two. What colour is it?

- Why do you think their colour is different?

- What does the colour show?

- Use the push/pull button to them.

- What happened?

- What is the difference between the three shapes?

TASK C: Exposing views about 1D and 0D worlds/objects

1. How would a 1D world look like?

    (a) What shapes, do you think, might exist in this world? Why?

    (b) What would be the difference between this world and your 2D/3D world?

2. How would a 0D world look like?

    (a) What shapes, do you think, might exist in this world? Why?

    (b) What would be the difference between this world and your 2D/3D world?

TASK D: Revising

General questions in order to assess whether students' thinking about dimension has changed:

1. What do we mean by dimension?

2. What is the difference between the first world/picture you created (2D) and the second (3D)?

3. What is the difference between a square and a cube?

4. What shapes exist in the first world/picture (2D)?

5. What shapes exist in the second world/picture (3D)?

6. How can we create (a) a line, (b) a square and (c) a cube?

7. Give me an example of a 2D/3D shape. Give me an example of a 2D/3D world.

Through the above tasks, students were given the opportunity to work in a 3D environment and experience the capability this facility had to offer. Task A offered them the chance to explore both the 2D and the 3D design and make comparisons pointing to their potentials and constraints. It involved both differences in spaces (2D floor – 3D space) and in objects (lines, 2D shapes, 3D shapes). More specifically, students:

- experienced the restrictions of movement in a 2D compared to a 3D environment (working with 2 or 3 axes; working on a plane; working in space)
- experienced the restrictions of vision in a 2D compared to a 3D environment
- experienced how the objects were created in a 2D and a 3D environment (by aggregating other shapes; by dragging lower dimensional shapes)
- experienced which objects can exist in a 2D and/or in a 3D environment

Task B included some pre-constructed tasks that were used as an extra reinforcement into the ideas of dimension. Great emphasis was given to the notions of direction, position and orientation. Task C, on the other hand, moved a step forward by challenging students to make predictions of what/how 0D or 1D environments might be. In order to do that, students had to reflect on the differences between 2D and 3D objects/spaces.

So far, a macro view of the tasks and their potentials was presented. However, if a closer look is taken at each individual tool, each one was hiding a mathematical notion embedded within that students had to make sense of in order to complete the task. Thus, I come back to my previous discussion of tools' selection and I argue that some tools were selected on purpose in order to support the exposure of specific mathematical concepts. The tools selected for this cause, which formed the basis for the designing of the tasks and the interview plan, are explored in the next paragraphs.

11.4 Dimensional tools

The purpose of this final situation was to identify bridging tools (Abrahamson and Wilensky, 2007; Prodromou and Pratt, 2006) that could be used effectively in creating a situation where students' thinking on dimension could be perturbed. This study identified such tools through the software and more about these 'special' tools and how they responded to the principles set in the previous chapter are discussed in this section.

A *dimensional tool* is a tool whose use is likely to raise opportunities for the mathematical experience of dimension. This type of tool was part of Google SketchUp 7, either as individual tools on the toolbar or as characteristics of the software's environment. This situation made use of these tools because of the significance of the mathematics they embedded. However, each of these tools was identified as having the principles of the tools described in the previous chapter. More specifically, each tool had to present the mathematics embedded in a visible and transparent way that children could infer to, and it needed to have restrictions in order for the children to enter into a debugging process and explore the mathematics underlying the tools. What is more, these tools had to offer the potential for the students to link various dimensional ideas between them, leading to the creation of relationships. In the following paragraphs, each of these dimensional tools is explored by pointing out their *description*, *utility* and the *mathematics embedded*.

Through the *description*, I give an overview of how the tool is used, pointing out to its potentials and restraints. It is similar to a user manual for the software. And *utility* is used in the same sense as Ainley's et al. (2006) definition that "the learning of mathematics encompasses not just the ability to carry out procedures, but the construction of meaning for the ways in which those mathematical ideas are useful" (p. 30). In other words, the *utility* section describes how the tool was used throughout the interview plan and why its use was important for the completion of the tasks.

When I refer to *embedded* mathematics, I refer to the potential for the tool to raise opportunities for mathematical experience of dimension. This is of course my interpretation of that potential and whether that potential was realised or not – and exactly how – was a focus for the analysis. The majority of the mathematical concepts embedded in the tools were ideas relating to vector space as explained in the next section. I wish to emphasise that, in identifying embedded mathematics, I was not expecting young students to learn the mathematical notions identified. I was however hoping that my analysis would reveal various ways in which the students' experiences had the potential to be interpreted as situated accounts of mathematical experience as could be recognised by someone enculturated in those sophisticated mathematical notions. In other words, my interpretation of what students might have experienced is a light version of what is connected to abstract mathematics.

11.4.1 Object/Space and Vector Space

While working with Google SketchUp, I began to realise that certain tools seemed to have the special relevance to this study in the way that they embedded ideas about vector space. The 3D SketchUp domain could be seen as a vector space within which vectors as objects could reside. At the same time, vectors (the objects in Google SketchUp) could be seen as forming a basis that generates (spans) the vector space (that create other objects/spaces). In this respect, the mathematical idea of vector space articulated in a formal way what had been merging informally as a duality between object and space and as capacity. Dimension as an idea, formally the number of linearly independent vectors in the basis of the vector space, informally captures the scope for a space to contain equal or lower dimensional objects or be generated by combinations of those objects. By playing constructively in Google SketchUp, the students worked with informal tools, which in them embedded formal ideas of vector space. In the following paragraphs each dimensional tool is described pointing to its utility and the specific mathematics embedded within it.

11.4.2 Line tool (colour of line) 

Description: The line tool cursor is a pencil. While drawing, the colour of the line changes to mirror that of any parallel axis (red, blue or green). If a line is drawn that is not parallel to any of the three default axes then it has a black colour. Note that this colour attribute is present only from the beginning of the drawing process (1st click) and, as soon as the line is created (2nd click) then the line turns black (Figure 61). When the line is subsequently selected, it becomes blue.

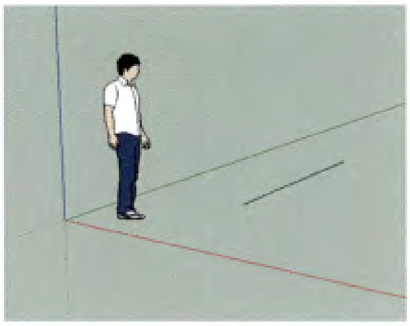


Figure 61: A line drawn using the Line tool

If one draws lines that form a closed shape, like a triangle or rectangle, SketchUp will automatically fill the enclosure to create a face. Three guidelines facilitate the simple creation of edges and surfaces:

1. Draw a closed loop of edges to create a surface. Your loop of edges must also be co-planar.
2. Watch the axis directions and use inferring to line up edges. Inferring is the ability to ask SketchUp to line it up for you. SketchUp snaps to the red, green and blue axis.
3. Always draw to and from existing edges, and do not draw new edges over existing edges.

Utility: The line tool could be used in a variety of tasks through the interview. First, during TASK A: Building a neighbourhood, students could use it for creating 2D shapes for their neighbourhood. Second, during TASK B2/3: Create a Rectangle/Cube, it could be used as one of the possible ways to create these shapes. Third, I have designed an extra task (see Task B4: Incomplete shapes) in which students could see the difference between co-planar and non-coplanar shapes. More specifically, a set of incomplete frames was drawn, which students had to complete. Some of them took a colour after completion and some others did not. Therefore, the student had to use the Orbit tool to check why, and thus how the co-planar edges notion was introduced.

Mathematics embedded: A user enculturated in mathematics might recognise two key mathematical notions 'embedded' within this specific tool. The two key mathematical notions are 'Direction vector' and 'Base vector', and these are explained, first formally and then more intuitively below:

(a) Direction Vector: is any vector $\overrightarrow{AB}$ that describes a line segment *D*, where A and B are two distinct points on the line *D*. If *v* is a direction vector for line *D* then all of the direction vectors of that line have the form of *kv* where k is any nonzero scalar.

In Euclidean terms, any line in 2D space can be described as the set of the solution of the equation $ax + by + c = 0$ where *a*, *b*, and *c* are real numbers. In that case, the one direction vector of (*D*) is (-b, a) and any multiple of this is also a direction vector. For instance, let $4x - 3y + 6 = 0$ be an equation of a line. Then (3,4), (6, 8) and (-3, -4) are all direction vectors of this line. Similarly, any vector in 3D space can be described as (x, y, z) where x, y, z are real numbers.

A direction vector in 3D can be defined through the equation of a straight line in 3D. A line that passes through the point with position vector ***a***, and is parallel to the direction vector ***b*** has vector equation $r = a + tb$.

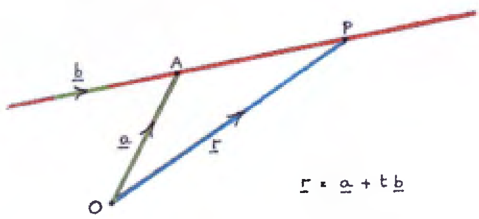


Figure 62: Equation of a line in 3D taken from http://www.netcomuk.co.uk/~jenolive/vect17.html

Thus a line passing through the point A, position vector $2i-3j+2k$ and which is parallel to the vector $i-j+4k$ has vector equation $r=2i-3j+2k+t(i-j+4k)$.

This may also be written in column vector from $r=\begin{pmatrix}2\\-3\\2\end{pmatrix}+t\begin{pmatrix}1\\-1\\4\end{pmatrix}$. In general, an n-dimensional vector is a vector $(x_1,x_2,\ldots,x_n)$ with n components.

Relating these to SketchUp, the Line tool offers the user the ability to create direction vectors in space. The colours of the three axes (red, blue, green) code default directions so that any line being created by the line tool can be identified even in the drawing process as a direction vector parallel or not to one of the default directions. For example, if a line segment being created is red, it is a direction vector parallel to the red axis. If a line segment being drawn is black, it is not a direction vector parallel to any of the three axes. The pictures that follow show examples of direction vectors of each of the axes correspondingly:

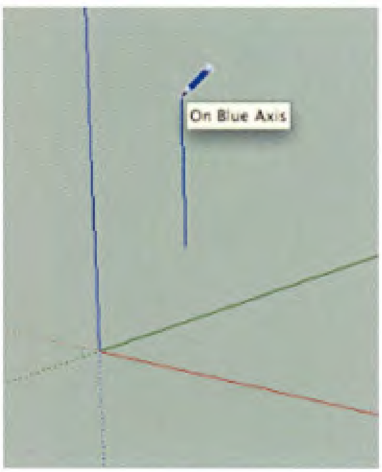


Figure 63: Direction vector parallel to the blue axis

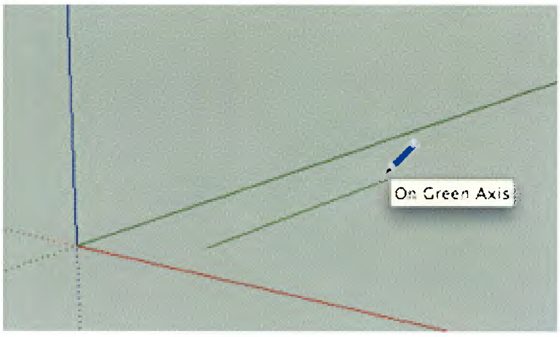


Figure 64: Direction vector parallel to the green axis

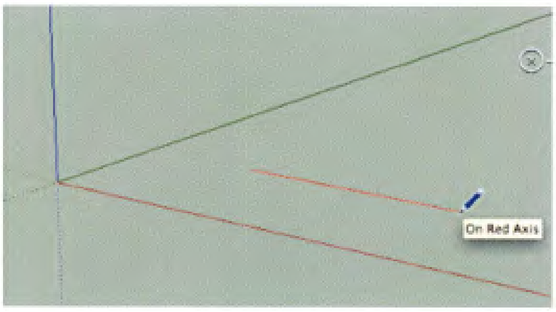


Figure 65: Direction vector parallel to the red axis

I would like to clarify here that the action of creating the line using the Line tool is the direction vector because while drawing the line, it takes the same colour as the default direction is following. However, after a line is drawn, its colour changes to black regardless of the direction vector used to create it. Concerning the above, the Line tool was thought of as supporting the construction of situated abstractions for direction vectors by allowing the student to create vectors and by drawing attention to their relationship with the axes.

(b) Base vector: Every vector space has a basis. A basis *B* of a vector space V over a field F is a linearly independent subset of V that spans (or generates) V.

The collection {i, j} is a basis for R^2, since it spans R^2 and the vectors i and j are linearly independent (because neither is a multiple of the other). This is called the standard basis for R^2. Similarly, the set {i, j, k} is called the standard basis for R^3 and in general,

$\{e_1 = (1,0,0,...,0), e_2 = (0,1,0,...,0), ..., e_n = (0,0,...,0,1)\}$ is the standard basis for R^n.

A base vector is a member of a basis for a vector space. A vector space V will have many different bases. However, there is always the same number of basis vectors in each of them. The number of basis vectors in V is called the dimension of V. For example, since $(-1, 2)$ is clearly not a multiple of (1,1) and since (1,1) is not the zero vector, these two vectors are linearly independent. Since the dimension of R^2 is 2, the two vectors already form a basis of R^2 without needing any extension.

Relating these to SketchUp, the Line tool 'embeds' the notion of base vectors by offering coloured direction vectors from 2 or 3 different axes respectively. For instance, to create a 2D shape, a student needed to use two different direction vectors whereas a 3D shape required three direction vectors, possibly one parallel to each of the three different axes. The following pictures show different bases for R^2 created by using different combinations of direction vectors parallel to the three axes:

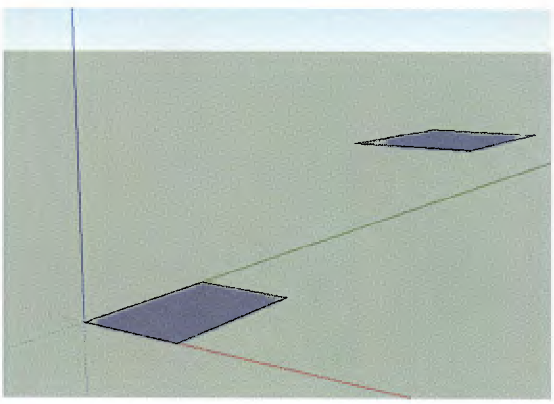


Figure 66: Rectangles created using red and green base vectors

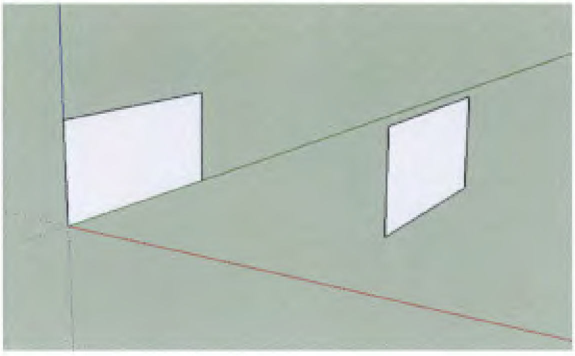


Figure 67: Rectangles created using green and blue base vectors

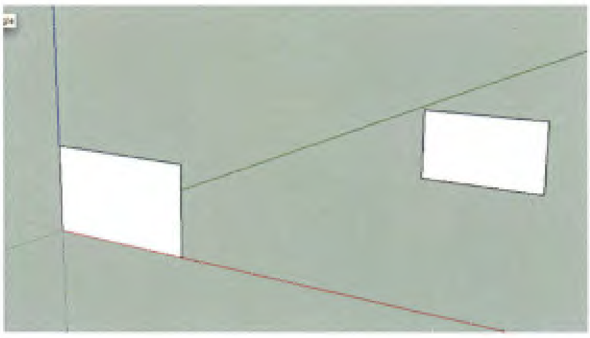
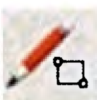


Figure 68: Rectangles created by using red and blue base vectors

11.4.3 Rectangle tool 

Description: The rectangle cursor is a pencil with a small symbol of a rectangle next to it:



To draw a rectangle in SketchUp, you click one corner, and then the opposite corner. You can draw rectangles on the ground plane or on any surface. Rectangles can be drawn along the axes directions. As you orbit around, you can draw rectangles in various orientations i.e. on the blue-red plane, red-green plane or blue-green plane. It is also possible to draw a rectangle on a different plane from the three above or on an existing surface. The following picture shows the creation of a rectangle in SketchUp (Figure 69):

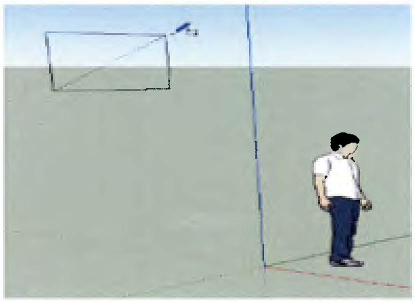


Figure 69: Creating a rectangle in SketchUp

Utility: The rectangle tool could be used for creating rectangles in TASK A: Building a neighbourhood. It could also be used for TASK B2: Create a cube, which was one of the ways of drawing a cube.

Mathematics Embedded: The notions of linear combinations of vectors and spanning are embedded within this tool. A linear combination is the sum of some set of vectors multiplied by a real number. For instance, let the field F be the set of R of real numbers and let the vector space V be the Euclidean space R^3. Consider the vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Then any vector in R^3 is a linear combination of e_1, e_2 and e_3. For example, if you take an arbitrary vector (a_1,a_2,a_3) in R^3 then:

$$\begin{aligned}(a_1,a_2,a_3) &= (a_1,0,0) + (0,a_2,0) + (0,0,a_3)\\ &= a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)\\ &= a_1e_1 + a_2e_2 + a_3e_3\end{aligned}$$

The linear span of vectors is the set of all linear combinations of the vectors $v_1,\ldots,v_n$ in an arbitrary vector space V. There are three theorems regarding linear spanning:

Theorem 1: The subspace spanned by a non-empty subset *S* of a vector space *V* is the set of all linear combinations of vectors in *S*.

Theorem 2: Every spanning set *S* of a vector space *V* must contain at least as many elements as any linearly independent set of vectors from *V*.

Theorem 3: Let *V* be a finite dimensional vector space. Any set of vectors that spans *V* can be reduced to a basis by discarding vectors if necessary. This also indicates that a basis is a minimal spanning set when *V* is finite dimensional.

Relating this to SketchUp, the rectangles created by the user can be considered as a subspace of a 2-dimensional vector space V and for creating this subspace the Rectangle tool uses a combination of two different direction vectors. It was thought that the Rectangle tool would support the construction of situated abstractions for spanning and I would like to add here that the word 'spanning' used in this particular tool is illustrating the process of generating surfaces and not explicitly the spanning set.

What is more, the notion of adding vectors also relates to this tool. Let $\vec{a}$ and $\vec{b}$ to be vectors that have different magnitudes and directions. The sum of $\vec{a}$ and $\vec{b}$ is $a+b=(a_1+b_1)e_1+(a_2+b_2)e_2+(a_3+b_3)e_3$. This can be represented graphically by placing the start of the arrow *b* at the tip of the arrow *a*, and then drawing an arrow from the start of *a* to the top of *b*. The new arrow drawn represents the vector $\vec{a}+\vec{b}$:

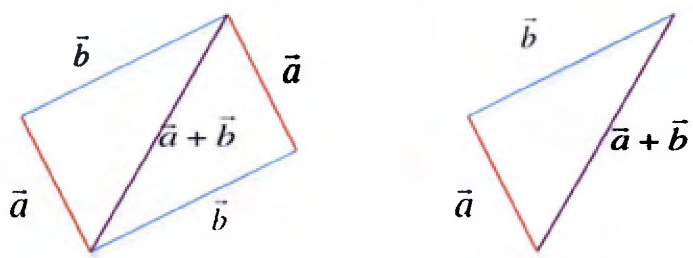


Figure 70: Adding vectors

For drawing a rectangle in SketchUp, the user first defines the common starting point of the two vectors and then the opposite corner. These two points create the $a + b$ vector. In other words, in order to create the rectangle, the user has to define the vector, which is the sum of the sides of the rectangle (the diagonal).

11.4.4 Shaded Surfaces tool

Description: Anything you can create in SketchUp is either an edge or a surface, and often an interesting combination of both. Edges can exist on their own; surfaces cannot. Each surface in SketchUp has a set of bounding edges. If any of these edges is erased, the surface is gone. The first guideline for creating edges, as aforementioned, is to draw a closed loop of edges in order to create a surface - note that this loop of edges must also be co-planar. If this rule is obeyed then the surface created becomes shaded. If it does not, that means that one of the edges belongs to another plane. This tool is directly related to the Line tool because the edges are drawn with the Line tool.

Utility: This property could be first noticed during Task A: Building a neighbourhood, while the users drew 2D shapes (surfaces). If a shape was created by edges which were not co-planar then it did not have a shade, and also:

(a) it could not be coloured afterwards with the use of the Paint Bucket

(b) it could not be pushed/pulled into 3D (see Push/Pull tool below)

This tool could be explored further through the Task B4: Incomplete frames as described above, where the user had to complete the frames in order for the surfaces to be created. If the surface got a shade then it is co-planar. This could be noticed by using the orbit tool to move around.

Mathematics embedded: Similar to the Rectangle tool, the notions of linear combinations of vectors and spanning are also embedded within this tool (see detailed description of the mathematics of these notions during the Rectangle tool). I would argue that the two tools are complementary as they express broadly speaking the same mathematical notions. In SketchUp, the 2D surfaces created by the user can be considered as a subspace of a 2-dimensional vector space V. Thus, for creating the subspace, the user has to use any combination of two vectors, including their multiples. In fact the Rectangle tool involves the user choosing a direction after which the second direction must be orthogonal. In comparison, the Shading on Surfaces tool offers the students open choice of vector directions and so more control over the process of generating surfaces. The closed and shaded surface can be considered as a representation of the spanning of the two base vectors. The potential for experiencing spanning in an intuitive situated way lies in the feel gained by observing how the edges of the rectangle combine, and result in a shaded surface whenever the edges are multiples of standard base vectors.

11.4.5 Circle tool (colour of pointer) 

Description: To draw a circle in SketchUp, you first click the centre point and then click an outside point to define the radius. The default circle tool actually creates a regular polygon with 24 sides. You can draw a circle on any surface over which you hover. Similarly to the Rectangle tool, when a circle is created, its surface becomes shaded (light grey, grey or white) according to the plane on which it lies.

However, the difference with the circles is that you can actually place the circle on a plane before it is created, by looking at the colour of the circle's circumference. For instance, if the cursor shows a blue circumference that means that the circle will be created on the green-red plane, if it shows a red circumference it will be drawn on the green-blue plane and, if green, it will be drawn on the blue-red plane. In other words, the colour of the circumference mirrors that of the axis perpendicular to the circle. The colour also shows the direction of the axis of any cylinder generated later by the Push/Pull tool (see Figure 71: Circle tool).

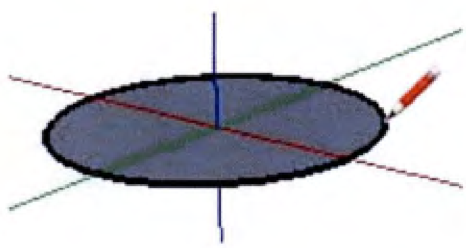


Figure 71: Circle tool

There is a possibility of creating a circle outside the three default planes. This can happen if you draw a circle on a 2D surface of a 3D shape. For example, if you try to draw a circle on the man who stands on the default screen, the circumference will take a black colour and that means that it is not on one of the three default planes.

Utility: The circle tool could be first used during the Task A: Building your neighbourhood, where the students had to create 2-dimensional shapes. Moreover, the change in colour of the pointer was introduced (in case it was not noticed earlier by the students) in Task B7: Circles, where students had to hover around the circles presented on screen.

Mathematics embedded: The mathematical notion embedded within this tool is the Normal Vector. A normal vector is a vector that is perpendicular to a surface:

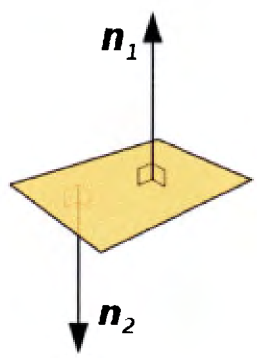


Figure 72: Normal vector

A normal vector to a plane specified by $f(x, y, z) = ax + by + cz + d = 0$ is given by:

$$\mathbf{N} = \nabla f = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

where ∇f denotes the direction of the normal. The equation of a plane with normal vector $\mathbf{n} = (a, b, c)$ passing through the point (x_0, y_0, z_0) is given by

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0.$$

(Weisstein).

In SketchUp, the cursor of the circle tool takes the colour of the normal vector of the plane on which the circle will be drawn. This is of course an efficient way of defining the direction of the circle. Normal vectors in the direction of the axes will appear red, blue or green, according to the parallel axis. For instance, a cursor with a red colour shows that the circle will be drawn on the blue-green plane having a normal vector in the direction of the red axis. Thus, the Circle tool was thought to support the construction of situated abstractions for normal vectors by linking the colour of the cursor of the circle tool while hovering to the colour of the three default axes.

The following pictures show the coloured normal vectors in the software:

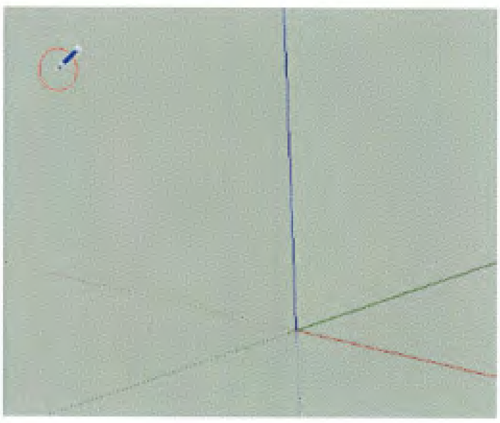


Figure 73: Normal vector in the direction of the red axis

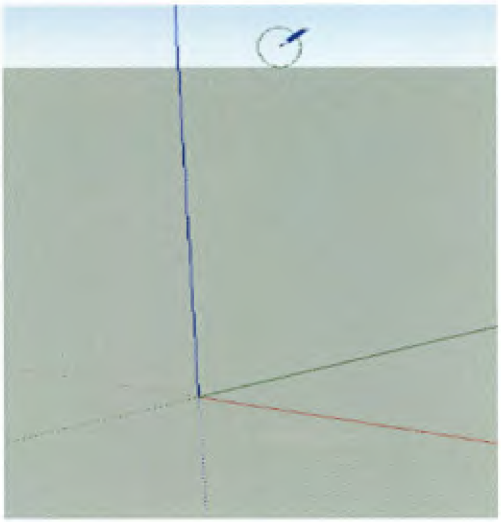


Figure 74: Normal vector in the direction of the green axis

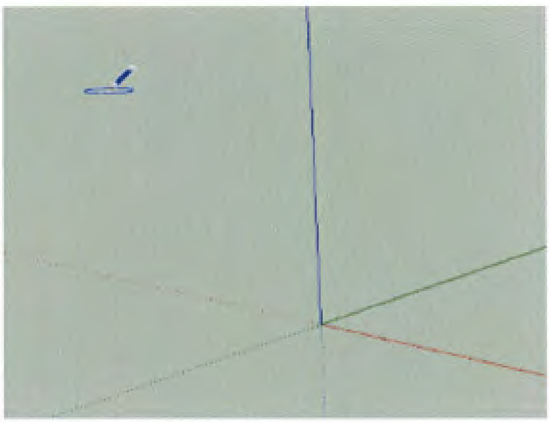


Figure 75: Normal vector in the direction of the blue axis

It is possible for a cursor to have a black colour when it is drawn on an object, and that shows that the normal vector is not parallel to any of the three default directions.

11.4.6 Push/Pull tool 

Description: This tool always pushes/pulls to a direction that is perpendicular to the surface. It can have a positive or a negative direction. The following picture shows how a rectangle on the green-red plane is pulled out to create a rectangular box or cuboids:

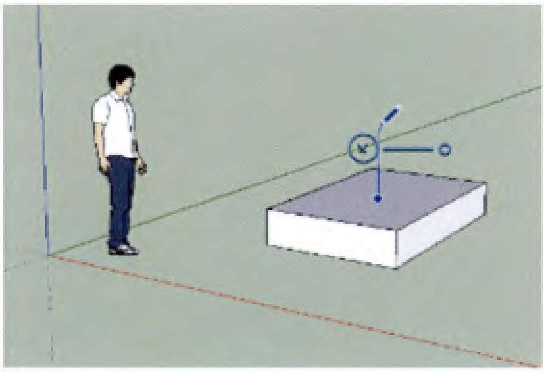


Figure 76: Push/Pull tool

As with the Circle tool, the cursor for the Push/Pull tool shows the colour of the normal vector to the plane of the surface.

Utility: The students pushed/pulled their 2D shapes in order to create 3D buildings in Task A: Building your neighbourhood. Furthermore, during the Task B3: Axes students were encouraged to notice the colour of the cursor as well as to predict where the circles would go after extrusion. Predictions were also made during the Task B7: Circles 2, where there were various rectangles in different orientations.

Mathematics embedded: The mathematical ideas embedded within this tool is the normal vector as described during the Circle tool before, as well as the extrusion as a transformation from 2D to 3D (continuing the theme in earlier situations of moving between dimensions).

In SketchUp, the surfaces are extruded along their normal vector, and there are three ways to do this (ignoring the negative directions). Surfaces can be extruded along normal vectors parallel to the red, the green or the blue axes.

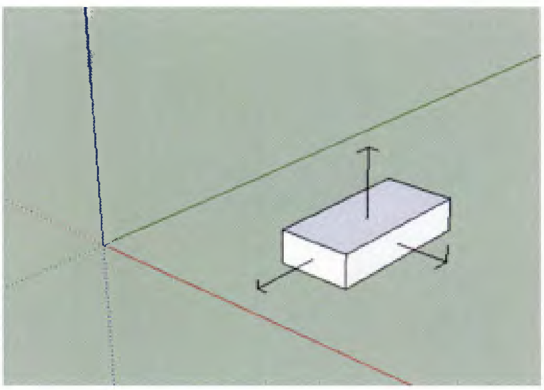


Figure 77: Extrusion directions

Concerning the above, the Push/Pull tool was thought of as supporting the construction of situated abstractions for linking the notions of normal vector to the extruding. For example, after using the Circle tool by hovering around and creating a 'blue' circle (Blue in the sense that the cursor of the Circle tool was blue meaning that the normal vector of the circle was the blue axis), then the Push/Pull tool will extrude the circle to the direction of the blue axis.

11.4.7 Orbit tool 

Description: In Physics, an orbit is the gravitationally curved path of one object around a point or another body, for example the gravitational orbit of a planet through a star.

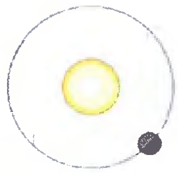


Figure 78: Orbit motion of earth around the sun

In a 3D drawing software, constructing 3D shapes only is not enough. There is the need of being able to look around these shapes as if they were real. In other words, you have to be able to 'move' in 3D space. The orbit tool offers this attribute to the user and thus the 2D screen becomes a 3D environment to work within.

Orbiting is like holding an object and turning it around. When you use this tool, you click and hold the mouse button and move the mouse around. Where your cursor is on the screen will be the centre of rotation.

Utility: Students were encouraged to use the orbit tool to check the view while constructing their neighbourhood in Task A: Designing of neighbourhood. The students were in fact likely to use the orbit tool during all the tasks of the interview plan.

Mathematics embedded: The orbit tool enables experience of co-planarity or lack of it. The mathematical notions embedded within it are the plane of rotation and the idea of perspective.

Plane of rotation: A plane of rotation for a particular rotation is a plane that is mapped to itself by the rotation. In 2 dimensions there is only one plane of rotation. In 3-dimensional space there are an infinite number of planes of rotation, only one of which is involved in any given rotation. In any rotation in three dimensions there is always a fixed axis, the axis of rotation. The rotation can be described by giving this axis, with the angle through which the rotation turns about it; this is the axis angle representation of a rotation. The plane of rotation is the plane orthogonal to this axis, so the axis is a surface normal of the plane. The rotation then rotates this plane through the same angle as it rotates around the axis - that is everything in the plane rotates by the same angle about the origin.

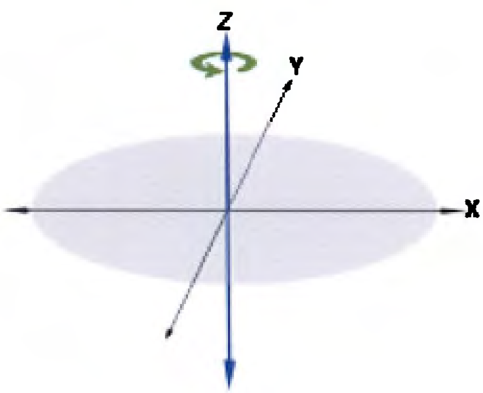


Figure 79: Plane of rotation

In Sketch up, while moving with orbit, first you define the height (up-down) of rotation and then you drag towards an orbitary direction. If the height remains fixed then a plane of rotation is created. However, usually the height changes because of the use of the unsteady mouse.

Perspective: is the way in which objects appear to the eye based on their spatial attributes, their dimensions and the position of the eye relative to the objects. In SketchUp, the orbit tool changes the position of the eye relative to the object and thus, the user sees the object from different perspectives.

11.4.8 Different Shades tool (grey, light grey and white)

Description: As soon as a shape is created (a closed co-planar shape if it is made by lines), its surface gets a shade according to its position in terms of the light source. There are 3 colours of shades that the shape can take and these are light grey, dark grey and white. The following picture shows the different colours that the shapes can take:

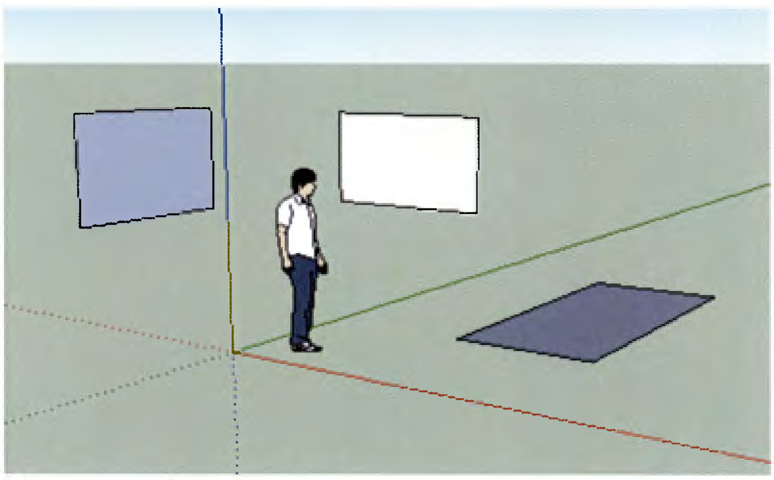


Figure 80: Dark grey, light grey and white shapes

Although at first a relationship between the three default planes (green-blue, blue-red and red-green) and the three colours can be assumed, this is not the case, at least not officially, because a shape can be created on a different plane and have the same colour as one of the three above. Thus, this attribute can be restricted to the position the shape has in relation to the source of light.

Utility: Students could notice the difference in colour throughout all the tasks. However, there were two specific tasks that were designed to focus on this attribute explicitly. The first one was Task B5: Rectangles orbit in which students were encouraged to discuss how the colour of the shapes was related to the direction the Push/Pull tool would go. The second one was the Task B8: Turning rectangles, in which students rotated the rectangles and saw how the colour changes.

Mathematics Embedded: The mathematical ideas embedded within this tool are position and orientation.

Position: A position vector is a vector that represents the position of a point P in space in relation to an arbitrary reference origin O. In SketchUp's case the reference point is the source of light and the change of colour depends on the position vector from that source.

Orientation: The orientation of a body is a description of how it is aligned to the space it is in. The orientation is given relative to a frame of reference, usually specified by a Cartesian coordinate system by giving the rotation that would move the body from a base or starting orientation to its current orientation. This is similar to the body's position, which can be given by a translation relative to a base position. The position and orientation together describe how the body is placed in space.

The Different Shades tool was thought to support the construction of situated abstractions for position and orientation by describing how the shapes are placed in space.

11.4.9 Supplementary tools

After describing the eight dimensional tools that were used in this study, I will refer to a couple of tools which helped in a supplementary way without focusing on any particular mathematical notions that they might have been embedded within them.

I will begin by talking about the Look around tool, which was used in one of the tasks with Orbit. The Look around tool was introduced during the Task B6: Orbit Vs. Look around tools where students looked at their neighbourhood using each tool at a time and observing differences in view. For the Look around tool, you click and hold your mouse button to look around. You can stay in a stationary location and can explore your model from there. Your eye height is always positioned relative to the surface you click on. The Look around tool was not used for introducing a new mathematical notion. On the contrary, the comparison of the Look around and the Orbit tools in Task B6, was thought to encourage the students to express their views regarding the Orbit tool, and more specifically to support the construction of situated abstractions for vision and perspective in 3D.

The second supplementary 'tool' was the ability of the software to make inferences. Inferencing allows you to inferencing axes or other geometry while you are drawing in SketchUp. It works by inferring points from other points as you draw while also providing you with visual cues. In other words, the software tries to 'guess' what your next move would be. While drawing lines, for instance, inferencing will draw a temporary dotted line to the more recent geometry you draw. This ability of the software was very helpful for the students especially for drawing accurately, more easily and more quickly shapes with same lengths.

11.5 Limitations of SketchUp

As any other 3D modelling software, Google SketchUp had some limitations that were considered during the research process of this particular situation. To begin with, in SketchUp, you view items in a perspective projection. Perspective, however, distorts the view such that it represents the model as though the lines were vanishing to a horizon. (Certain items appear closer while other items appear to be far away; entities are not to scale.) The following image shows a perspective projection:

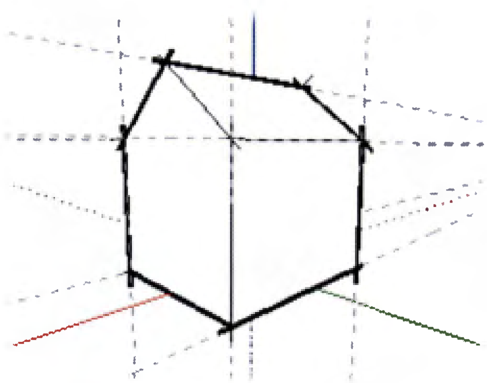


Figure 81: Perspective view of SketchUp

Although perspective projection is effective for representing 3D objects on 2D screen, it has some limitations such as having parallel lines which do not look parallel due to the perspective issue. For instance, a shaded rectangle drawn might not look as a rectangle at all:

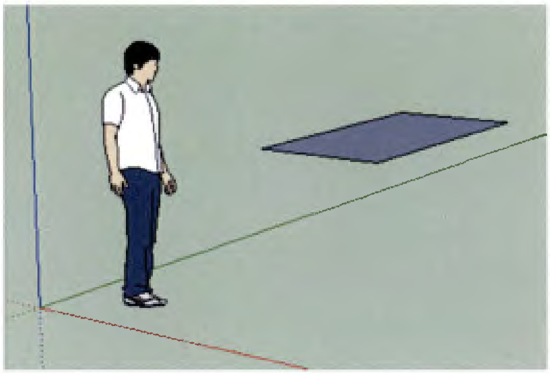


Figure 82: A shaded rectangle

However, I believe that although this is a limitation, it did not influence the outcomes of the results to any significant degree. On the contrary, students felt comfortable in working within the specific digital 3D environment and this was probably because of their familiarity to the particular perspective which was similar to some other computer software and gaming programs that they had already experienced.

I would also like to mention here that I did not expect the students to draw accurate drawings of shapes. On the contrary, I tried to avoid any measurements and therefore I did not include in purpose the measurement tool of the software during the tasks. The purpose was to focus more on the drawing process and the exploration of the 3D space and not so much on the measuring of lengths that could shift students' focus. One of the limitations of this approach was that the students did not draw shapes accurately and thus there were cases where students wanted to draw a square but drew a rectangle instead. However, it is believed that this did not influence the result outcomes as I did not focus so much on the formal definition of a square but on the drawing procedure of a 2D surface, therefore any rectangle or other quadrilateral was acceptable.

11.6 Summary

This chapter followed the ideas proposed by the designing and modelling theories to choose Google SketchUp for designing Situation IV. Subsequently, the tasks designed to accompany the software were presented, pointing to the principles followed for designing them in order to embrace the various 'missing' elements of our orientation of dimensional experience. Special attention was given to the dimensional tools of the software, which had as a purpose to embed mathematical ideas. These tools were presented in detail pointing to their description, their utility and the mathematics they embedded.

Considering the sophisticated nature of the mathematics embedded in the tools, this study did not suggest that such young students would learn the mathematical notions noted. Rather it was argued that the analysis would reveal various ways in which the students' experiences might be interpreted as situated accounts of mathematical experience.

Thus, the focus of the analysis of the data, in the next chapter, was be my interpretation of whether the dimensional tools used in the interview plan reflected my initial conjecture, which was to raise opportunities for mathematical experiences of dimension. It involved an analysis of whether students realised these potentials or not and if they did, what kind of experiences they acquired.

Chapter 12: Situation IV Data Analysis

12.1 Overview

The data analysis is focused on exploring the relationship between the design of the tasks along with the opportunities raised by the dimensional tools, and the students' articulations of dimensional experience. The data analysis of Situation IV was a phenomenographic analysis as described in Chapter 4 (p. 111). After re-reading the transcripts, the excerpts that were thought to be describing experiences of dimension according to the orientation (Table 3, p. 133) were selected to enter the Pool of meanings. 233 meanings qualified to enter the pool and that was considered sufficiently large to warrant comparisons among the meanings. Subsequently, the pool was organised according to similarities and the differences of the meanings. As an initial classification of the categories of description was made during Phase 1, I placed the excerpts which were similar to these categories to the corresponding categories, and as a next step I challenged the initial categories of description by looking at the different meanings gathered. After revisiting the categories, a final version was created.

The data analysis is presented in five parts. Section 12.2 is a chronological account describing the process of Nosakhare and Mya's interview (Pair D). The purpose of this account is for the reader to create an idea of how the research was conducted. The selection was based on which couple could best illustrate the issues raised by the analysis. Section 12.3 is a comparative account showing the similar experiences of this situation and the categories of description as extracted from the previous situations. The purpose of this account is to confirm that similar experiences were replicated and also to reassure that the categories of description created before do occur regularly across the pairs. However, two more dimensional ideas were extracted from the data, which thus brought about a reconsideration of the existing categories.

The next two parts of this analysis, Sections 12.4 and 12.5 are two accounts, which present an interpretation of the interview focusing on the two new ideas introduced during this situation. The first account is more focused on the vectorial ideas such as position, direction and orientation, while the second gives emphasis to the idea of capacity. I present these two accounts in order for the reader to form a clearer idea of how the same data can be interpreted differently by having a different focus of analysis. The last part of this chapter, Section 12.6, is a reflection on what has been argued in the previous sections by presenting the final version of the categories of description of students' dimensional experiences.

In order for the reader to get a clearer idea of the data, excerpts taken from the transcripts are included in the form of cross-reference. The coding of the cross-references was conducted according to the pair from which the excerpt was extracted (A, B, C, D, E, F) and the line number on the transcript. More specifically the letters of the alphabet represent the pairs as follows: 'A' for Joel and Laura, 'B' for Reine and Nicholas, 'C' for Kevin and Charlie, 'D' for Nosakhare and Mya, 'E' for Alexia and Beverly, and 'F' for Nicole and Nataly.

For example, the cross-reference [D32] corresponds to line number 32 on the transcript of Nosakhare and Mya and the cross-reference [E32-35] corresponds to the lines 32-35 on the transcript of Alexia and Beverly. My own words/questions in the excerpts are coded having the letter R (Researcher) at the beginning of the sentence.

The children's sentences are coded with their name initials. In addition, in some cases I place my own intervention on the excerpts in order to give extra clarifications for the reader. My own insertions are coded [*in square brackets and in italics*].

12.2 Descriptive Report on Nosakhare & Mya (Pair D)

This account has a descriptive nature illustrating how the interview of Pair D progressed. I have attempted to avoid interpretation though I have selected to emphasise in detail certain episodes while maintaining the chronology of the account with a relatively superficial gloss elsewhere. This happened because:

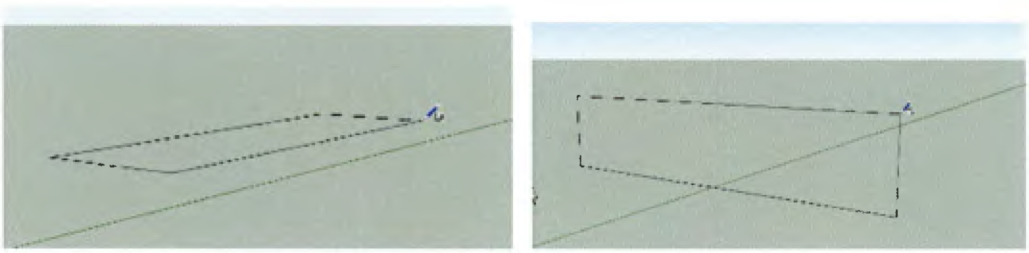
(i) some episodes continued the narrative of exploration but did not seem on subsequent analysis to reveal dimensional experience and so were not described in detail and,

(ii) my interest in the analysis later is in new articulations of dimension and so those episodes which seem to indicate new issues were given more emphasis than episodes which seemed to reiterate issues discussed previously in the thesis.

This illustrative description as described in the following paragraphs was the basis on which the interpretation accounts were built on.

12.2.1 Initial explorations

When I showed the two examples of the ready-made neighbourhoods (2D and 3D) to the Nosakhare and Mya, their initial thoughts were of the type "*this is 2D and this is 3D*", "*this one is like a plan (2D)*", "*2D is something flat*" expressing some materialistic attributes such as "*3D you can hold it*" and "*you can feel the edges*". The restriction of vision was also a characteristic that they used to talk about in relation to dimension while differentiating between 2D and 3D. For instance, when they were asked to define dimension they argued that "*it is something to do with your eyes*". They also added that 2D is when your eyes see it flat compared to 3D when your eyes see all of it. After this short discussion, the students moved forward to construct the neighbourhood. The pair used the Rectangle tool to construct their first shape. From their first attempts they distinguished what they meant by flat shapes compared to the non-flat ones (Figure 83):



Flat

Not flat

Figure 83: Flat and non-flat shapes

Adding to this, a bit later they argued that the difference between the two rectangles in their neighbourhood and the new circle they created is that the former were flat while the latter was 'wrong' because "*it is coming towards you*" (Figure 84).

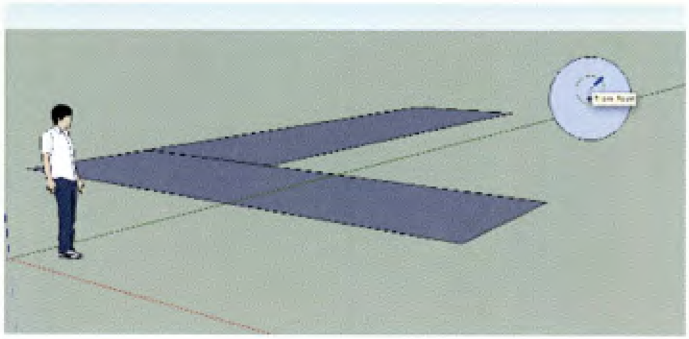
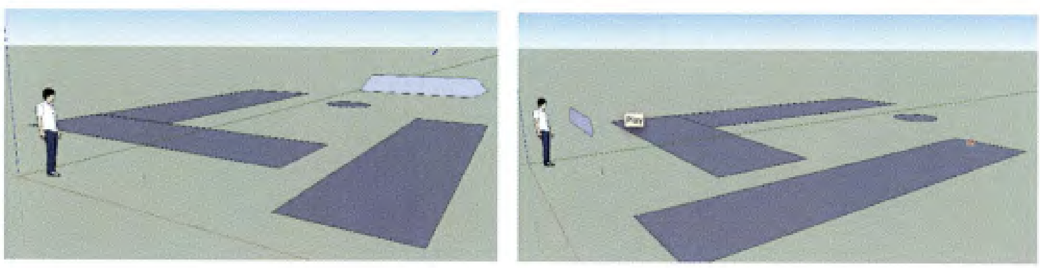


Figure 84: Comparing shapes in different planes

I then asked about the difference in colour between the shapes and the pair pointed out that it depended on the amount of light that comes to it. This interpretation remained the same until the end. After 'fixing' their circle in order to be 'flat' like the rest of the shapes, I challenged them to use the Line tool for their additional shapes. The next shape they did was a pentagon (Figure 85: Picture before orbit). They talked about it as a shape that is "*bending*" which is "*a kind of 3D but flat*". When they used the orbit tool they were surprised to see that the shape they created was not in the place they thought it was (Figure 85: Picture after orbit):



Before orbit After orbit

Figure 85: Pentagon before and after orbit

Nevertheless, they decided to keep the shape as it is and use it as a sign. Thus, they tried to create a post under it by using the Line tool. After creating the line-post they used the Orbit tool to see that the line was not in the place they thought it was. The same procedure continued for a couple of more trials of creating a line vertical to the red axis having the same result. Here I present some examples of 'wrong' lines (Figure 86):

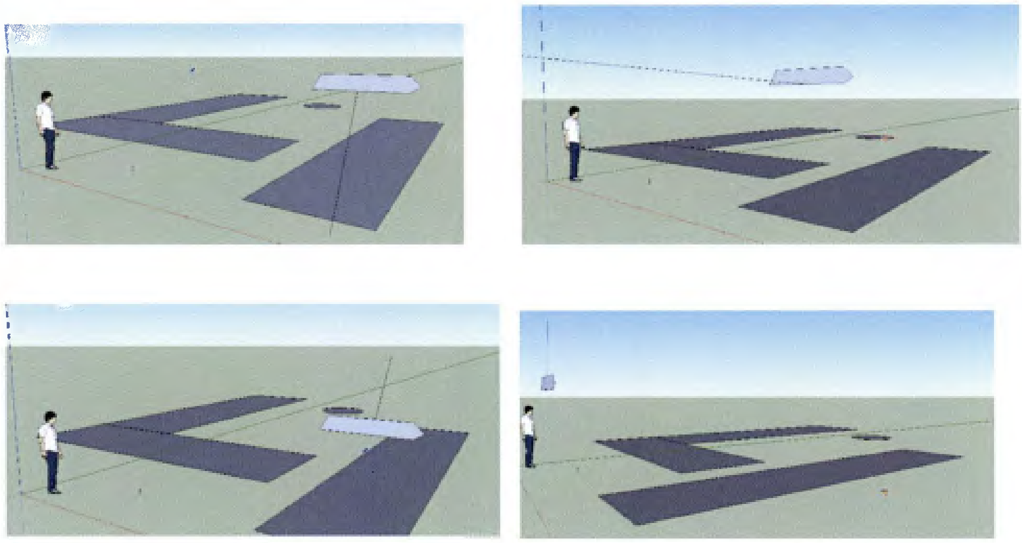


Figure 86: Trials for creating a post

After many trials, they used the Orbit tool to change the perspective into a side view and they drew a line connecting the bottom side of the sign and the red axis (Figure 87):

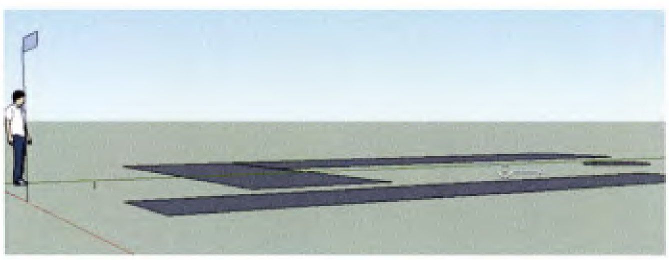


Figure 87: The final picture of the pillar

Following the construction of the 'sign', I asked the students to try and draw one more shape by using the Line tool. While drawing, the students noticed the colour of the lines drawn:

N: What is happening? Why are they green?

R: Yeah, why are they green?

N: I don't know.

They continued drawing more lines.

R: What colour is it now?

M: Blue.

Both: Red.

N: Oh I know because it is in different directions. Oh it's black again.

M: Like when it goes north, south, east, west, when it goes in these directions it has a colour, when it is going diagonal it is black. [D121-129]

The students reasoned that if the line drawn is coloured then that means that it is going in one of the cardinal directions (north, south, east, west) but if it is black it means that it is 'diagonal'. In other words, if the line is parallel to any of the axes of any of the cardinal directions then it has a colour.

Another attribute of SketchUp is that it shades the closed shapes if they belong to only one plane (see Shaded surfaces tool, p. 256). As mentioned before, the students noticed the change in colour of the lines, but they ignored it while drawing their next shape (made by lines) and thus their shape was not coloured (see the non-coloured shape in Figure 88).

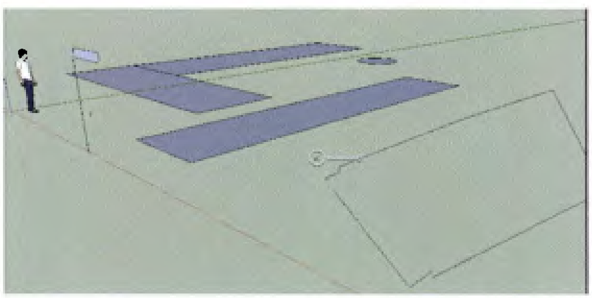


Figure 88: Shape made by lines (non-coloured)

However, students' explanations of why their shape was not shaded were based on the difference in the tools used, pointing out that shaded (coloured) shapes were the ones drawn by using the Rectangle and the Circle tools, and non-coloured shapes were shapes drawn by the Line tool:

> *N: When we used a rectangle and a circle it was coloured because it was an object, because with that we just put it there and we make the shape while here we draw it. That's why [D141].*

Indeed, if the Rectangle and Circle tools were used, the shapes produced were coloured because they were ready-made rectangles and circles, drawn only in one plane. I found it useful at that point to introduce the students to the Incomplete frames task (Task B4).

12.2.2 Exploring Task B4 Incomplete Frames

Task B4 offered students a SketchUp environment in which there were many incomplete frames that students were asked to complete in order for the shapes to be created. Some of the shapes completed had a colour and that meant they were on one plane while others did not. The purpose of the task was for the students to identify this difference. When Nosakhare and Mya completed the frames, they noticed that some of the shapes were not coloured. Their initial explanations were of similar type as before, arguing that the Rectangle tool constructs colour shapes while the Line tool does not. However, when they used the orbit tool to move around, their explanations were informed by new elements. They then argued that the coloured ones are "*just great*" and "*proper*" while the non-coloured ones are 'twisting and turning' [D171, D173, D178] (Figure 89).

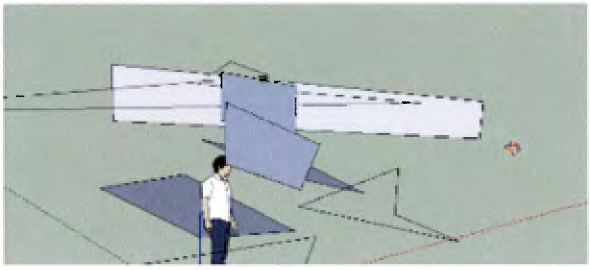


Figure 89: Incomplete shapes task

The difference in colour in the shaded ones was also noticed and thus, the students tried to classify them according to the exposure the shapes had to the light:

> *M: Because the light is coming from this way, that's why it is light and that one is dark [...] Like this is dark and this is light (top-bottom), and they are in the middle and in the middle there is no colour. [D186-192]*

Mya tried to explain why this happened further but her justifications remained incomplete: "*It has something to do with those lines as well I think*", "*I think it has to do when it is in a different...more that way or to the right it changes which way the shape turns...*", "*They are in the middle [the incomplete shape]*". To sum up, up to this stage, students' explanations were based on the tool used to create the shapes (Rectangle tool vs. Line tool), the description of the shapes themselves (proper vs. twisting), the influence of the light on the colour of the shapes (light vs. dark) and the position of the shapes in space (top, bottom vs. middle). After that, the students continued working on the neighbourhood task. Again, their attention was drawn to the non-shaded shape they drew (Figure 90):

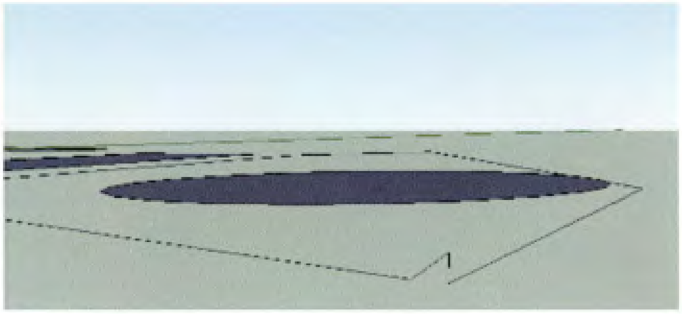


Figure 90: Non-coloured shape

They related the park (the above non-shaded shape) to the task of Incomplete frames, arguing that it was a type of a shape that changes while moving similar to the other non-shaded shapes met in the task before:

> *N: Like the other ones we saw [rectangles-orbit tasks], it changed when we moved it, so that's why [D223].*

Similarly, Mya tried to connect all the sides of the shape in order to see if it could finally get a colour but it did not, and thus she deleted it and started drawing it all over again. It is worth mentioning here, that Mya suggested that they change the view to a top view, "*If you go on top it might be easier*" (to draw it from the top view), a thought that was later ignored. While starting to draw again, the researcher drew to their attention the changes in the colour of the lines:

> *R: Why does it change colour [the line]?*
>
> *M: I think it has to do with these lines when it is going in the direction of those lines.*
>
> *R: Which lines?*
>
> *M: These lines [showing the red axis]*
>
> *R: The red?*
>
> *M: The red and the green [the axes]. [D235-240]*

However, the students ignored the above statement they made during their first attempt of drawing the shape, but they used it for their second. Mya drew the lines by first drawing a green line, then a blue, then a red and then a black one (Figure 91):

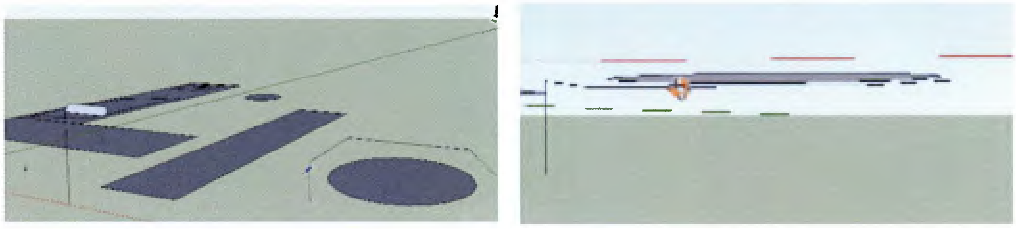


Figure 91: 'Down' line

While drawing, Mya was in the position to identify which line was 'down' without using the orbit tool (Figure 91):

> *M: Wait, wait this is down. You have to delete that line. Look [she then used orbit to show to Nosakhare that the blue line was "down"] [D254]*

The students finally managed to draw a shape that was shaded. In their explanations at the end, they used the term 'parallel':

> *N: I think I've got it. I think there's parallel, I think we've got some lines that are the same because look that's the same and that's the same [showing the various pairs of parallel lines on the shape]. Even though it is not of the same length, it is of the same type of shape that can make the same length...I am not sure about the sun...actually we can forget about the sun. [D267]*

After finishing with the drawing of shapes, the pair coloured their neighbourhood and then they used the Push/Pull tool to make it into 3D (Figure 92):

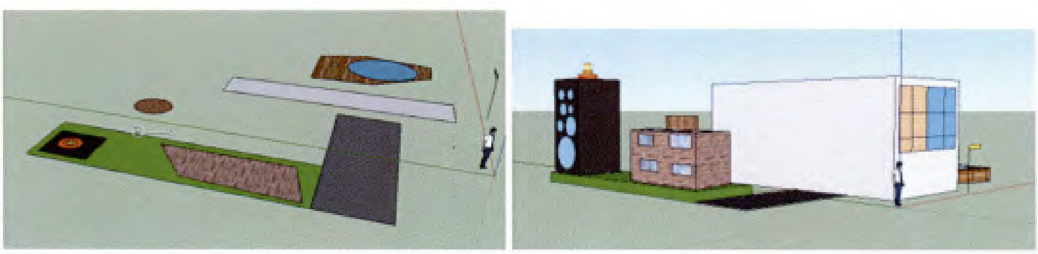


Figure 92: 2D and 3D neighbourhoods

It is worth mentioning here, that after using the Push/Pull tool, the students drew windows on the 3D shapes and they also used Orbit to turn around and drew windows on other buildings at the back. I then asked them to compare the two neighbourhoods above (2D and 3D versions), and the pair argued that the 2D is "*flat*", "*you can see the flat top*" comparing to the 3D that "*you can actually see a big piece compared to that (the 2D)*" and "*you can actually see everything inside*" [D17]. They then gave the example of the table they were sitting at, arguing that the surface of the table was 2D but the table itself was a 3D shape:

> *M: Like this table is like that [showing the table] if you look here you see just that [the surface] but if you look underneath you see the whole thing.*
>
> *R: So what is that table?*
>
> *Both: 3D.*
>
> *R: And what can be 2D in that table?*
>
> *M: That [showing the surface]*
>
> *R: Oh the surface?*
>
> *Both: Yeah the surface.*
>
> *N: The surface is always 2D*
>
> *M: But everything else is 3D. [D294-302]*

Following the example of the table and the 'top view' expressed before, the pair argued that it is similar to the case of their 2D and the 3D neighbourhoods. They pointed out that if we turn the two neighbourhoods into a top view, then they would look the same:

> *N: If you look on top it would have all of it in common but when you look at all around...*
>
> *R: You mean if I look at it from the top [I use orbit to show the top view] it will have everything in common?*
>
> *N: Yeah look at it...*

They put the top view to both the 2D and the 3D neighbourhoods. [D315-318]

Subsequently, the discussion shifted in talking about how it would be different for a man to be in a 2D or in a 3D neighbourhood. The students argued that the man is "*more likely to stay alive*" in the 3D neighbourhood. Finally, at the end of the neighbourhood task the students pointed out that the park (the last shape drawn) was the hardest shape to do because the lines they drew were going in 'different directions':

> *M: Because of the lines.*
>
> *N: Of the lines because we kept doing it, and it kept shaking and shaking and shaking?*
>
> *R: Shaking? What do you mean?*
>
> *N: Not shaking, like going down.*
>
> *M: Going different directions*
>
> *N: You wanted it to go straight, we were doing it straight and when we turned it all the way around it is just blurring against. Look at that (showing how the pond looks now) now it is perfect. I think we got symmetry line you know. [D358-363]*

12.2.3 Exploring task B5 Rectangles – Orbit

After the neighbourhood task, the students were introduced to the Rectangles-orbit task (Task B5) that involved having many rectangles on screen, in which students had to discuss how they were similar/different and how this view could change if they used orbit to move around. At the beginning of the task, the students argued that the rectangles were put in different ways (showing with their hands the vertical/horizontal orientations) and they also identified which ones would go 'up' with the push/pull:

R: Which do you think will go up with the push/pull?

Both: This one, and that one and that one [showing the dark grey rectangles]

They used the push/pull to show me that indeed they went up.

R: How do you know which ones they are?

M: Because they are flat on the ground, they are flat.

N: The other ones are not flat. [D379-385]

Then I showed them a specific rectangle, which was not on the ground but it was still going up with push/pull:

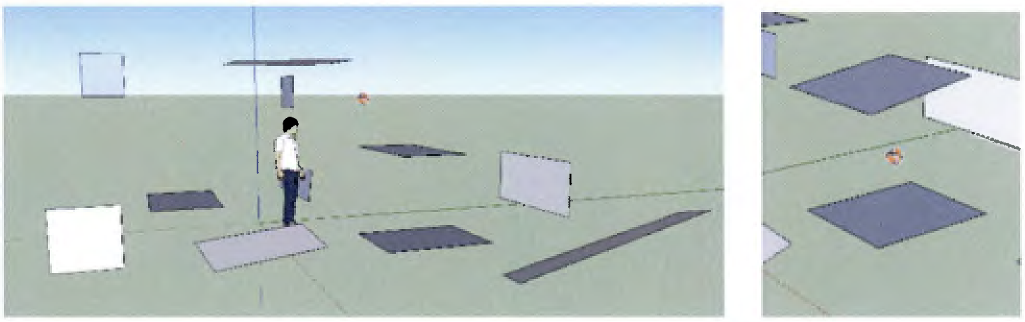


Figure 93: Incomplete frames task

R: Is this on the ground? [the dark grey rectangle on the top]

Both: No.

R: But it is going up...

M: It is because they are going up. In different angles they go in different ways.

R: What is similar between this and this? [the two rectangles above] They both go up...But this is not on the floor [the top one]

N: Oh I got it. I has to be flat, it doesn't matter if it is on the ground or not, it just has to be flat.

R: What about this? [a light grey vertical rectangle shown above] Is this not flat? It looks flat to me...[I used orbit to show them that it is flat]

N: It is flat but it is like straight like that [showing a vertical direction with his hands] Instead of being this way [showing vertical movement with a paper] it has to be that way [showing the "flat" position of a paper] [D390-398]

12.2.4 Exploring task B1 Create a rectangle

During the task of creating a rectangle in different ways (Task B1), Nosakhare and Mya had the chance to explore the Line tool further. They used green and blue lines to create a rectangle and their shape was shaded. I then asked them to explain how it was possible for a shape to be shaded, if it was drawn by using the Line tool. (Before they argued that only the Rectangle and the Circle tools could create shaded shapes.) They pointed out that this happened because they used coloured lines to draw the rectangle:

> *M: It's because the green and the blue, if you do it by using the green and the blue then...*
>
> *N: The red, the green and the blue.*
>
> *R: Try it.*
>
> *M: For example if you do that is black, and black [They drew a rectangle made out of black lines]. Now it's blue, red, red, and blue. It's coloured! [They drew a second rectangle with coloured lines]*

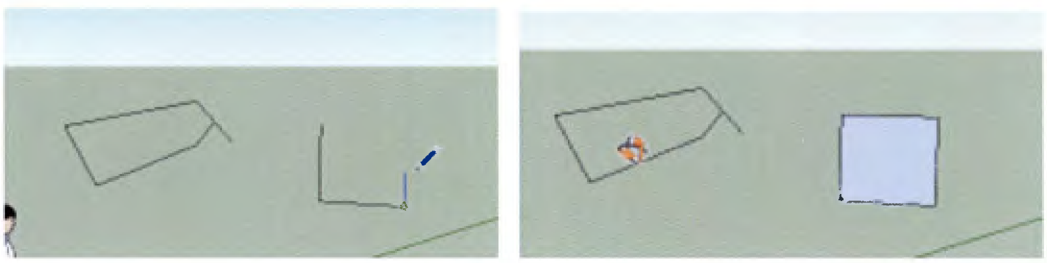


Figure 94: Designing with coloured lines Vs designing with black lines

> *R: Why?*
>
> *N: Because if it is coloured [the line] it would make a coloured shape but when... [D411-420]*

After using Orbit to look around they realised that their first shape (the one with the black lines) was not actually a proper rectangle because "*it wasn't drawn properly*". When I asked them to explain what they meant by that it was argued:

> *N: No, not wasn't drawn properly...because when I was watching while Mya was drawing, I saw that when she drew it, it was black and black and black [the lines] while when she did it here it was different colours, it was red and blue. [D433]*

Once more, when the students were asked to explain what the colours of the lines meant they pointed out that they corresponded to the cardinal directions (i.e. "*red means south*"). I then asked them to draw one more rectangle. They used red, blue, red lines and a black one, and thus the shape was not shaded. However, they identified that this happened "*because one of the lines was not coloured*" [D441]. Up to this point, students were able to find a relationship between the colour of the lines and whether the shape created was shaded or not. However, in order for a shape to be shaded it had to belong to only one plane. They did not notice that. Therefore, I decided to draw a rectangle by using all the three different colours of lines. The rectangle looked perfect, created by coloured lines but it was not shaded! The students used orbit to turn around and saw that indeed the rectangle that looked perfect was not even a rectangle:

> *N: It's not symmetry! [He then uses orbit] See! It's bending...*
>
> *R: Why? They were coloured [the lines]!*
>
> *N: I think the reason why, it's because they have to be two same colours. Because that's what happened with that [a previous coloured rectangle], it was red there and blue there.*
>
> *R: Try to do it.*
>
> *N: Blue, red, blue red! Yes! I've told you! [It turned a white colour]*
>
> *M: So it has to be two of the same colour. [D446-452]*

Thus, the students concluded that in order for a rectangle to be shaded it had to be made out of only two colours of lines. At this point, I decided to ask the students to relate the colour of the line to the colour of the axes:

R: What do the colours have to do with the three lines (the axes)? Draw a green line. [They drew a green line]. That's the relation between the green line and the line you drew?

M: Because it is going that way and that way...

R: You mean that if it was red, it would go that way? [showing the direction of the red axis]. Try.

N: Yeap

They also drew a blue one, which was going to the direction of the blue axis.

N: Yes, that's why.

They drew all the three lines: red, blue and green (Figure 95). [D456-463]

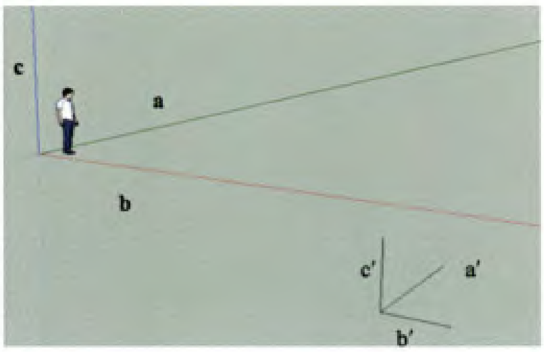


Figure 95: Relationship between colour of lines a', b', c' drawn by the student and the axes a, b, c

12.2.5 Exploring task B2 Create a cube

During the task of creating a cube (Task B2), students had to create the cube by using just the Line tool. Mya created a square with lines and then she used Orbit to turn to the side view and from there she drew the other side. However, she faced a problem when she had to estimate the new intersection point of the new sides. She changed again the view and then she managed to do it (Figure 96).

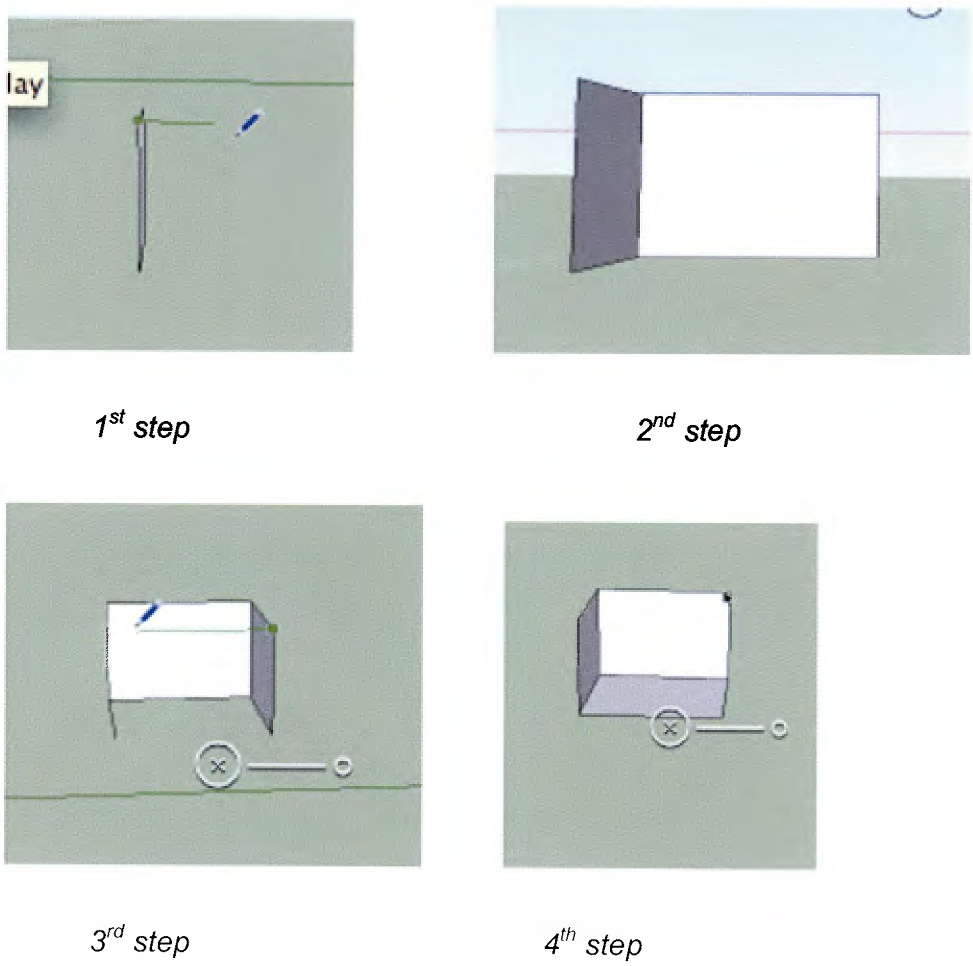


Figure 96: Procedure of creating a cube by using the line tool

Following that, the students were asked to create a cube by using only the Rectangle tool this time. They created a flat dark grey rectangle. And then they tried to bring a vertical side to it. They faced a difficulty in defining the edges of the side; therefore they could not draw a vertical face but drew horizontal ones instead. They finally drew the face and then they used Orbit to turn it around and after some attempts they managed to finally complete the other faces as well (Figure 97):

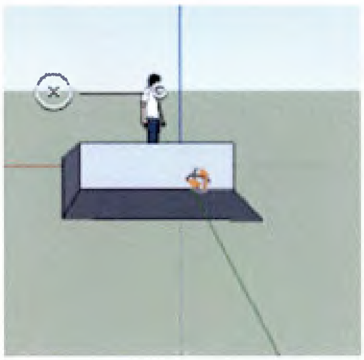


Figure 97: Creating a cube by using the rectangle tool

As in the construction of the rectangle task, I wanted the students to notice the number of the colours used in the lines drawn while creating the cube (3-dimensional shape):

R: Now, before [at the previous task] you did it with the lines. What colour of lines did you use?

M: Red and green.

R: Red and green only to make the cube?

N: No, red, green and blue to make a cube.

R: So for making a cube you need 3 colours? Because if you remember for creating a rectangle you said you need 2 colours.

M: No, I think we need 2 colours. I remember when we did the first square I only used green and red...

R: Yes but when you created the cube?

M: We used all the different colours. [D496-504]

12.2.6 Exploring tasks B3 Axes and B7 Circles

During the Task B3: Axes, the students predicted that the circle on the 'ground' would go upwards with the use of push/pull compared to the circles in the 'air' that would go sideways (one in the positive and the other in the negative direction). However, after using the tool itself they realised that their predictions were not completely correct as the two circles in the 'air' went in two different directions. They concluded that there are three or six (including the negative directions) ways of extruding circles namely, forward and backwards, left and right, and up and down.

Following this task, the next one (Task B7: Circles) acted as a heuristic for students to express their views further on the relation between the colour of the cursor while creating a circle, and the direction of the extruding circles after Push/Pull. Nosakhare and Mya were in a position to identify which circles would go 'up' or 'left and right' with Push/|Pull by just looking at them. After encouraging them to notice the colour of the cursor while hovering on the circles, they linked the colour of the cursor to the direction that the shape would go with the use of the Push/Pull tool:

R: What's the relation between blue [colour of cursor] and the axes?

M: Because the blue goes up

N: Straight up [showing the direction of the blue axis on the software]

M: And then the green...

R: Where does the green go?

M: That way [showing the green axis]

R: Where does the red go?

M: That way [showing the red axis] [D593-600]

It is worth mentioning here that during the same task, Mya expressed the idea of the three axes as forming a cube:

M: You see like...this is the bottom [showing the space between the green and the red axes]. It is not a whole cube but a part of it. This is the middle [origin point] and then you go down down, down [across the green line] that's the edge, that's like down, that's the corner [showing the angle created by the blue and the green] and that's the other line [the red axis]. [D585]

12.2.7 Exploring task C and final remarks

The final task (Task C) involved exposing views about 1D and 0D worlds/objects. After arguing that 2D is a surface and not the actual thing that is 3D, the students claimed that they have not heard of 1D before. I then asked them to define it by looking at the differences between 2D and 3D. The students argued that 1D would be "one side of a shape" [D660]. At this point I challenged the students to look back to their 2D and 3D constructions of a neighbourhood and name the number of lines they used for drawing the shapes in those respectively. They argued that they used 3 lines for the 3D ones while they used only 2 for the 2D ones. They concluded that 1D would be just one line:

> *R: And how would the shapes look like in 1D?*
> *M: Like that line. It would be that line on its own.*
> *R: Just a line?*
> *M: Just lines. [D693-696]*

At the end of the interview, the students were asked to define what 'D' in 1D, 2D, 3D etc meant. They pointed out that 'D' stands for dimension and it means "*sight*" and "*what you see*".

It is worth mentioning at this point that students also made connections to their everyday life experiences. For instance, they mentioned that a development of the 3D films would be the 4D ones that would also include "*feeling it in the chair*" [D654]. In that sense, 1D would be only one component of the film i.e. sound [D656]. Towards the end, the students also pointed out that the 3D "*you can actually see, touch, feel it, like hold*" and that shows that their materialistic expressions of dimension remained even until the end of the interview. Adding to this, in the final discussion it was also argued that "*the whole entire world*" is 3D and also "*all the planets because they are all spheres*" [D741].

This is the end of the descriptive account of one pair of students in Situation IV. The purpose of this account was to offer the reader an insight into how the interview was conducted. The subsequent sections of this chapter focus on the interpretation of the data, first by looking back to the existing categories of description, and second by presenting two interpretative accounts focusing on the two new categories that arose from the specific data.

12.3 Looking back to the Categories of Description

The first three situations showed evidence of different 'characterisations' of experiences of dimension. These characterisations were put into six categories of description. Similar experiences were noted during the fourth situation as well. This part shows how the new data could be described in relation to these six categories, leading to a discussion that challenges the initial categorisation.

12.3.1 Dimension as Action

In previous situations, *Dimension as Action* was often articulated as an algebraic term in which actions could be carried out, or as an outcome of an act. In previous situations, students tended to count the corners, edges or sides the shape had in order to find its dimension, or they were even adding or subtracting dimensions. Similar articulations were not observed in this particular situation. However, students were adding vectors instead, but this type of articulation is further described under the new category of Dimension as Vector due to its strong relationship with vector space.

12.3.2 Dimension as State (place)

In the previous situations, experiences showing dimension as a state were noted when students located an object on a dimensional domain and identified the restrictions of movement in each domain. In this situation, students talked of objects as being in different places, spaces, lanes or even angles and ways [A165, B501, C626, D376, D378]. Joel and Laura (Pair A) also added the term 'area' arguing that the shapes are in different areas:

> *L: When I say different place, I mean they might be on the same...somewhere on the floor*
>
> *J: And some of them in the other.*
>
> *L: They might be on the same floor or the same wall. [A267-269]*

According to the above, students divided the software space into two 'places': the floor and the wall. Another example was Reine and Nicholas (Pair B) who talked about the difference between moving around in the 2D and the 3D neighbourhoods respectively:

> *N: Because, that's 2D, you go only left and right and up and down, while if you are in 3D you go everywhere. For example, let's say a house if it was 2D, you wouldn't be able to go in the house [B377]*

Similarly, Nosakhare and Mya (Pair D) talked about the bug as being able to walk on anything 'he' wanted to in a 2D neighbourhood:

> *R: What if a bug walks in this neighbourhood. What does it see?*
>
> *M: He would probably think it's fun because he could walk anywhere he wanted*

N: He would walk on anything he wanted to.

M: He wouldn't have to climb, do any hard work

N: Just walk on everything. [D332-336]

Another way of talking about location was concerning position. The words 'high', 'low', 'up' and 'down' were frequently used for describing the placement in space [A287, C560, C561, D165, D517]. Students also referred to the limitation of vision for 2D compared to the 3D domain. Expressions of the type "*3D you can see all of it while 2D you only see part of it*" or "*2D you see only one side of it while 3D you see all the sides*" or "*in 3D you can look around it*" [A422, C11, C14, C783-786, C870, D19, D20, D293] were articulated presenting the 2D objects as 'parts' of the 'whole' 3D shape.

This situation showed that the particular category of description could be broken into two sub-categories, which could consist of different characteristics of dimensional experience. For instance, the first category could be 'Position' including all the experiences of locating objects in domains and the restriction of movement in each domain, and the second one could be 'Restrictions of vision' including the expressions of looking at objects or within spaces. Data drawn from the rest of the analysis helped resolve the new sub-categories as meaningful ways of partitioning the existing formation of the categories.

12.3.3 Dimension as Material

Seeing dimension as having materialistic attributes was very common among the students in this situation as well. Students tended to relate dimension to the touch of an object. For example, the students argued that "*in 3D you can feel all the sides*"

[D11] and that you can hold a 3D shape compared to 2D one [A390, A421, A672, D9, E11, E383, F220]. What is more, Pair E argued that the smaller the dimension, the thinner or flatter the object is [E11, E372-373, E405].

Another component of defining dimension, strongly supported by the students, was the ability of 'seeing' a shape arguing that the smaller dimension requires a greater difficulty of seeing the shape [A422, B23, C11, C14, C786, C888, D19, D20, D292, D294, D644, D654, E407]. For instance, Pair D argued that in 3D you could see all the edges of the shape [D11] and Pair F pointed out that in 3D you could see the corners [F222, F470]. Students also distinguished 2D from 3D by arguing that 3D shapes have a shadow [F470]. Pairs C and D defined dimension as an ability that relates to sight [C879, D17, D705, D706]. It is possible that this thought of seeing higher dimensional spaces as having greater ability of vision was influenced by the 3D movies that are very popular nowadays. All the pairs of students expressed the word 'dimensional' as having something to do with 3D glasses and cinema [A645, B13, C875, C883, C302, D647, D654].

Students often related dimension to our world by giving examples of the real world, explaining dimension as if it were a tangible object or a world. For example, students argued that a man could not live in a 2D world because he "*would not be able to live in there*" and that it was more likely to stay alive in a 3D world [B23, D23]. They also argued that a 3D shape has more detail in it than a 2D one [E185].

Students also distinguished shapes according to whether they are shapes that were more likely to belong to the real world [E185, E362]. For example, students stated that the closed loop 2D shapes (the ones that belonged to only one plane) were real or human shapes [A15, B23]. Indeed, these types of shapes were the ones that students used in schools while learning about 2D shapes, thus these shapes were more familiar to them.

12.3.4 Dimension as Abstraction

Most of the students generalisations were implanted into the specific context they initially took place. Students created these abstractions in order to make sense of the environment they were working on. Here I present some of the situated abstractions expressed by Pair D:

> *'Flat' are only the shapes, which belong to the horizontal plane.*
>
> *'Kind of 3D but flat' are the shapes, which belong to other planes.*
>
> *When a line goes to any of the four cardinal directions then it has a colour while black colour only shows up with the diagonal lines.*
>
> *When a shape is drawn with the use of the rectangle or the circle tool then it has a colour. When the line tool draws it, it doesn't.*
>
> *The coloured shapes are 'proper' and 'great' while the non-coloured ones are 'twisting and turning'.*
>
> *The colour on the shapes changes according to the position of the shape in relation to the source of light.*
>
> *When the shape is on the top of the screen it is dark, when it is on the bottom it is light and when it is in the middle it has no colour at all.*
>
> *The non-coloured shapes change with the orbit tool.*
>
> *The colour of the lines depend on the direction of the axes they are going to (red line→ red axis, blue line → blue axis etc)*
>
> *The top view of a 3D shape and a 2D shape is the same.*
>
> *The way the shapes are extruded with the push/pull depends on the angle they are at (position).*
>
> *In order for a shape to go up with the push/pull, it has to be 'flat' (see definition of flat before)*

These abstractions were very helpful for creating the neighbourhood in SketchUp, and although they were so situated and automatic, some of them were just situated simple versions of some sophisticated mathematical ideas such as plane and vector space. The category 'Dimension as Abstraction' was considered useful to be created during the previous situations because it gathered all the abstractions into one integrated category showing the significance of distinguishing them from the rest of the data. However, the situated abstractions represented ideas that have been included in other categories of description. Thus, after looking at the data from all the four different situations, I noticed that 'Dimension as Abstraction' as an independent category did not explain much by itself, and therefore it could be distributed to the rest of the categories.

12.3.5 Dimension as Cross-dimensional

During the fourth situation, the students talked of dimension as a component in cross-dimensional situations. Similar to the discussion of *Dimension as a State* above, students argued that there are more options for moving in space when we have a greater dimension, and this relates to the notion of 'degrees of freedom' of moving in space. For example, Pair B argued, "*in 2D you go only left and right and up and down, while if you are in 3D you go everywhere*" [B23]. Similar to the relation of dimension and the ability of vision discussed during the *Dimension as Material*, students argued that in higher dimensions the visual ability is greater showing again this part-whole relationship between 2D and 3D objects/spaces respectively [C11, C783-786, C888, D19, D20, D292-293, E447].

Another aspect referring to the cross-dimensional situations was what stays invariant and what changes in a set of transformations. The Push/Pull tool in SketchUp helped the students create links for how a 3D shape can be created from a 2D one. The students talked of pulling and pushing 2D shapes in order to create 3D ones [A4, C17, D2].

What is more, cross-dimensionality was a notion that was strongly articulated among students when creating relationships. Students referred to the relation between 2D and 3D. For instance, Joel and Laura (Pair A) added that a similarity between 2D and 3D shapes was that they both had sides and corners and a base, and they argued that 3D shapes had a net compared to the 2D ones that did not [A9 - 10].

After reconsidering the components that constituted the *Dimension as Cross-dimensional* category, I noticed that most of them were already included in previous categories. For instance, the restrictions of movement and vision in various dimensions were defined as 'Position' and 'Restrictions of vision' respectively in the category of *Dimension as a State*. Furthermore, the characterisations of dimension based on the creation of relationships were embedded in almost all the other categories of description. The fact that the students tried to define the relationship between 2D and 3D helped them to talk about the notion of dimension and at that point I considered that this specific reference to the creation of relationships should not be an independent section of its own. The only part that was not actually mentioned as autonomous which I believed that it should, was the generation of shapes such as from 2D to 3D. Transformations, as it was named in this section, were of great importance while creating shapes and they showed a different characterisation than the one of the relationships, and thus I thought it should act as a separate section.

12.3.6 Dimension as Hierarchy

Dimension was seen as a property for defining or separating shapes creating in that way hierarchical relationships. For example, as aforementioned in the *Dimension as Action* section, Pair B argued that because for drawing their 3D version of a neighbourhood they used all three axes, and for 2D they used only 2 axes, then for 1D it would be reasonable to use only one axis to create a shape [B43].

Similarly, Pair D, after considering the differences between 3D and 2D, argued that 1D would be a line on its own, having just lines on a line [D42]. The same pair also argued that for creating a square they used the red and the green axis but for creating the cube they used the red, the green and the blue axes [D33]. Seeing dimension as hierarchy required doing some logical operations with dimension, and thus these expressions were also mentioned in the *Dimension as Action* section. Therefore, at this stage I had to reconsider whether this category needed a separate section on its own or whether it could be incorporated into the rest of the categories.

12.3.7 Other categories of description

After challenging the existing categories of description, the next step was to look deeper into the data in order to find evidence for finalising my conjectures. In the previous paragraphs, I decided to reconsider the necessity of having the categories of *Dimension as State*, *Dimension as Abstract*, *Dimension as cross-dimensional* and

Dimension as Hierarchy as separate and independent sections of the analysis. It was noted that most of their components could be incorporated into other categories. However, at this stage this was just a suggestion that needed further consideration by looking again at the data.

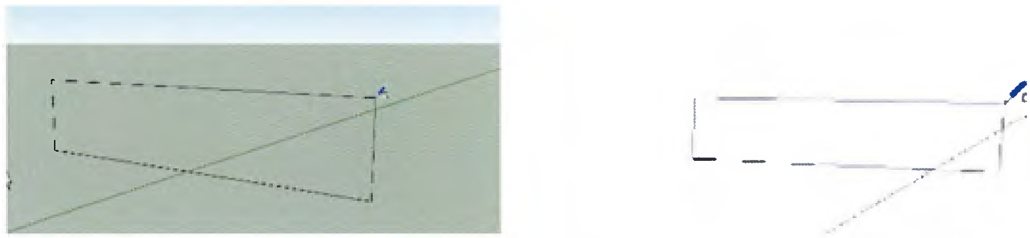
In this situation, experiences of dimension other than the above were noted due to the different setting in which the interview took place. The most frequent articulations of experiences were grouped into two new categories – *Dimension as vector* and *Dimension as capacity* - that are discussed in detail in Sections 12.4 and 12.5 of this chapter.

12.4 Interpretative account: Focus on the vectorial ideas

This section is an interpretation based on the previous descriptive account of Pair D. This account explores how students articulated dimensional experiences relating to the vectorial ideas such as position, direction, and orientation. These notions were present through all the interviews of this situation, but the specific pair was considered to illustrate these notions adequately, and thus it was chosen to be more representative for the presentation of this account. Although the account is based on Pair D, further data from the other couples are added later to report more fully the range of the vectorial ideas expressed by the children.

At the beginning of Task A: Designing of neighbourhood, Nosakhare and Mya distinguished between the 'flat' shapes that were the shapes that belonged to the green-red axes plane, from the ones which were 'coming out' that were the ones that did not belong to that plane. After removing the background colour of the SketchUp environment, the pictures looked like Figure 98:

(a) Flat



(b) Coming out

Figure 98: Shapes without the SketchUp background

Taking schooling into account, students were more familiar with the (b) case of shape for representing 2D shapes because that is the one used in pencil and paper situations (see Figure 98). Thus, it would have been expected that students' examples of 'flat' or 2D shapes would be case (b) and not (a). However, SketchUp was not a pencil and paper situation or any other similar 2D environment. On the contrary, it is a simulation of 3D space. Thus, the shape type (a) looked 'flat' because it was on the ground, while shape (b) was not flat because it was 'standing' or 'coming out' (see **Error! Reference source not found.**). As a result, it can be argued here that SketchUp offered students an environment similar to their everyday experience where flat was defined by an area of length x width, and height as the extra dimension that 3D shapes had, compared to the 2D ones.

Figure 99: Flat and coming out shape

Concerning the above, students positioned shapes in more than one plane and they were able to distinguish their orientation in space by using the characterisations of 'flat', and 'coming out' or 'standing up'. Thus, I would argue here that these characterisations were actually expressing the different orientations of surfaces in space. The students' next shape was a circle, which because at first was not 'flat', the students deleted it and reconstructed it in order to fit their definition of a 'right' shape. The distinction between 'right' and 'wrong' shapes was based on the orientation of the shapes, i.e. 'right is flat', 'coming out is wrong'.

I considered it useful at that time to prompt the students to become aware of the difference in colour between the 'right' shapes and the 'wrong' ones. Students' justifications were based on the combination of position and lighting. The circle was white because it was *up* and that was where the light is. On the contrary, the rectangles were dark grey because they were *on the ground* and thus there was not enough light. This argument confirmed the assumption made before about seeing the (b) case shape as height. The expressions of 'up' and 'on the ground' refer to the positioning of shapes in space.

After creating one more 'flat' rectangle, I asked the students to use the Line tool to construct their next shape. The Circle and the Rectangle tools involve a fixed 2D spanning in space where the variables are the length of sides for the rectangle and the length of radius for the circle. However, the Line tool acts differently by allowing the construction of lines in any possible plane in space. Using the Circle or the Rectangle tool involved a procedure of placing ready-made shapes in space and according to students, orientation ('flat' vs. 'coming out' shapes, 'right' vs. 'wrong shapes') and position ('up' vs. 'on the ground' shapes) were the variables. However, the Line tool involved creating lines in space, which was different from having a ready-made circle or rectangle to place. The students used the lines to create a pentagon. The pentagon created had a light grey colour and students argued that this is because the shape "*is bending*" and "*it is kind of 3D but it is flat*" similar to their ideas of orientation expressed before. However, after the use of the Orbit tool, they realised that the shape was not in the position they thought it was: "*It is a flying sign!*" (see Figure 85: Pentagon before and after orbit). They decided to keep it and to draw a post under it. Many unsuccessful trials of creating the post (line) followed having the students realising after the use of Orbit, that the post was not in the place they wanted it to be (see Figure 86: Trials for creating a post). The creation of the pentagon shape and the post introduced the students to 3D space and the orientation of lines in that space. The students finally managed to create a vertical to the red axis post by connecting the bottom edge of the pentagon to the red axis. It is worth mentioning here that in order to do that, the students used orbit to switch into a side view so that the side of the pentagon and the red axis looked vertical to each other and it was easier to connect them. The Line tool together with Orbit gave them a new insight of moving in 3D space adding therefore the third dimension to their drawing.

Following the construction of the pentagon and the post, I considered it helpful at the time to ask the students to construct one more shape with the use of the Line tool. While drawing the lines of the shape, I prompted the students to notice the change of the line colour. They argued that this happened "*because it is in different directions*" and that the colour depended on the direction the line was following: "*Like when it goes north, south, east, west, when it goes to these directions it has a colour, when it is going diagonal it is black*". These expressions show articulations of dimension relating to direction vectors. As aforementioned in the previous chapter, the colour of the line when the colour is drawn goes to the direction of the axis with the same colour. The notion of the cardinal directions (north, south, east, west) was probably introduced here to explain the direction of the three axes. It was a familiar notion to them, which they used to justify the change in colour. The use of the cardinal directions by these young students was probably influenced by the work in geography. However, the cardinal directions are used to describe only a point on a plane not in space and this is something that the students did not distinguish. Nevertheless, it was significant that they were able to make this link with SketchUp.

To sum up, up to this stage, students' explanations involved:

- *Shapes can be 'flat' or 'standing up' or 'coming out' or 'kind of 3D but flat'.*
- *Shapes can be 'right' or 'wrong' depending on their orientation.*
- *Shapes can be positioned 'up' and 'on the ground'.*
- *The lines go in different directions.*
- *When a line goes in any of the four cardinal directions then it has a colour.*

- *When a shape is drawn with the use of the Rectangle or the Circle tool then it has a colour. When the Line tool draws it, it doesn't.*
- *The colour of the shading on the shapes changes according to the position of the shape in relation to the source of light.*

It can be argued here that up to this point the students' explanations included situated versions of the ideas of:

(a) direction (lines going in different directions; cardinal directions)

(b) position (up or on the ground shapes)

(c) orientation (2D vs standing up/coming out shapes/kind of 3D but flat; right vs. wrong shapes; familiarity with 3D space-drawing lines on various planes)

As aforementioned, the students noticed the change of the line colour while drawing lines. However, they did not use this knowledge while drawing their next shape. They failed to extend this knowledge from the context of 'creating lines' to 'creating a shape with lines'. The shape created was not shaded:

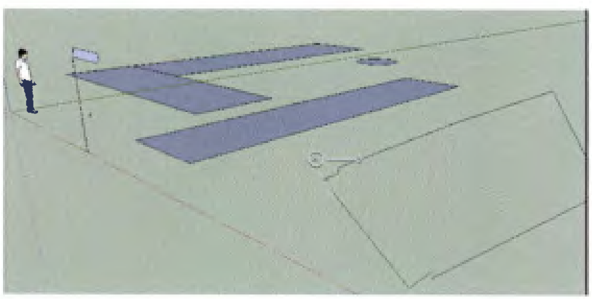


Figure 100: Non-shaded shape created by lines

The difference in the shapes' colour (different shades) was noticed again and the students argued that it varied according to the position of the shape in relation to the source of light: "*Because the light is coming from this way, that's why it is light and* *that one is dark*". Mya also tried to relate the colour of the shapes to the position and the orientation of the shapes, referring also to the axes (lines), but her thoughts were not explained further: "*It has something to do with the lines as well I think*" and "*I think it has to do when it is in a different…more that way or to the right it changes which way the shape turns…*". For instance, talking of the "way the shape turns" relates to the orientation of shapes although Mya did not explore it further.

Another explanation given for the colour difference in shading was that the shapes at the *bottom* were dark grey, the ones at the *top* light grey and the *middle* shapes had no colour. These (bottom, top, middle) are expressions of position similar to the ones mentioned before ('up' vs. 'on the ground').

The students also stated that they had to draw diagonal lines in order for the non-shaded shape to get a colour. I would argue here that diagonal lines were not the case themselves, but because the perspective of the axes was diagonal, their direction vectors had to be diagonal as well. Students also pointed out that these lines were parallel and that even though they did not have the same length, they were of the "*same type of shape*". These three expressions – 'diagonal', 'parallel' and 'same type of shape' – are referring to the orientation of the lines after being drawn.

While drawing a shape with the use of Line tool, I considered it useful again to prompt the students to the change of the line colour. They then pointed out that "*it has to do with these lines (the axes) when it is going to the direction of the lines*". This can be considered as a situated version of the definition of direction vectors. The students gradually became more fluent in identifying which lines were on the same plane and which were not, by reference to their colour while being drawn.

That shape was the last the students drew for their neighbourhood. After that, they used the Paint Bucket tool to colour their shapes and then used the Push/Pull tool to make them into 3D. The use of the Orbit tool to turn their 3D buildings around in order to be able to draw windows at the back was a sign of familiarity of moving in 3D space, showing the flexibility of positioning shapes in 3D.

To sum up, until this point, students' explanations included:

- *When the shape is on the top of the screen it is dark, when it is on the bottom it is light and when it is in the middle it has no colour at all.*
- *Diagonal lines are needed in order for the shape created to get a colour*
- *Parallel lines are of the 'same type of shape' regardless their length*
- *The colour of the lines depends on the direction of the axes they are going to (red line→ red axis, blue line → blue axis etc)*
- *Windows can be drawn at the back of the shapes*

Further to our aforementioned mathematical notions, students' explanations showed situated versions of the ideas of:

(a) direction (lines in the direction of the axes)

(b) position (bottom, top, middle of screen; drawing windows at the back of the 3D shapes)

(c) orientation ('the way the shape turns'; diagonal lines; parallel lines; 'same kind of shape' lines)

The Rectangles-Orbit task (Task B5) gave the opportunity to the students to express themselves in terms of orientation. They argued that the rectangles presented on screen were put in different ways, showing with their hands the vertical, the horizontal etc. These views related both to the positioning of shapes and to their orientation. I then asked the students to show which of the rectangles would go up with the Push/Pull tool, and surprisingly they showed the correct rectangles. When I asked for an explanation, they argued that they were the ones, which were flat on the ground, in other words, the ones that belonged to the red-green plane. I then prompt them to look at one of the rectangles, which was not on the ground but it would go up (it was parallel to the others). They then rephrased their justification by pointing out that the shape just had to be flat, not necessarily on the ground. I then challenged them again, by showing them another rectangle that was flat but not parallel to the others. They then explained that by 'flat' they meant for the shape to face a specific orientation: "*it has to be that way*" (showing the flat position of a paper). As a general comment, the students stated that shapes "*in different angles go in different ways*" and by 'angles' here we can assume the orientation of the shape.

During the task B1: Create a rectangle, the students explored the Line tool further. They used the Line tool to create a rectangle by using green and blue lines and the rectangle created was shaded. According to their previous explanations, only the rectangles created by the Rectangle or the Circle tools could have a shade. Thus, I challenged them to try and give an explanation for that. They stated that their rectangle was shaded because they used coloured lines to create it. They also drew two examples of rectangles, one with coloured lines and one with black lines in order to show me that they were right. They argued that the colours of the lines corresponded to the cardinal directions i.e. "*red means south*". However, when they were asked before to explain the same thing, they argued that the lines went in the direction of the axes, and that was how the colour was determined. If we take all these into account, it can be argued that students think of the lines as direction vectors going in the same direction as the axes; and in order to define these directions they used the cardinal ones that they were familiar with. Adding to this, the students' statement that coloured lines are needed for creating a shaded shape, also relates to the linear combination of vectors and spanning.

During the Task B2: Create a cube, while students were creating the cube by using the Line tool, they first created a square with lines and then they used Orbit to turn to the side view and from there they drew the other side of the cube. This procedure of drawing sides and turning was repetitive until the cube was created. This again shows the familiarity of working and moving around in 3D space and the flexibility of positioning shapes in that space.

During the same task, the students argued that two different colours of lines were needed for creating a 2D shape compared to the use of all three colours of lines for creating a 3D shape. Although this refers to the generation of shapes, which will be further discussed in the following section of Capacity, the 'different colour' of each line also shows a different direction and students acknowledged that.

After the Task B3: Axes, students concluded that there were three or six (including the negative directions) ways of extruding circles, which were: forward and backwards, left and right and up and down. This also refers to direction but it can also be considered as an introduction to the notion of normal vectors. The Task B7: Circles that followed, gave a clearer insight into the notion of normal vectors. It acted as a heuristic for the students to express their views further on the relation between the colour of the cursor while creating the circle, and the direction of the extruding circles after the Push/Pull tool. At first, the students were in a position to identify which circles would go 'up' or 'left and right' with Push/Pull by just looking at them. This was an expansion of their previous statement arguing that the flat shapes would go up. After encouraging them to notice the colour of the cursor while hovering on the circles, they pointed out that the blue cursor meant that the circle would go up, the green and the red cursor meant that the circle would go towards the direction of the green and the red axes respectively.

To sum up, until this point, students' explanations included:

- *The shape created by coloured lines is shaded.*
- *The rectangles were put in different ways and angles*
- *In order for a shape to go up with the push/pull, it has to be 'flat' (see definition of flat before)*
- *The colour of the lines is the same as the colour of the axis that has the same direction*
- *There are three or six ways of extruding circles*
- *The way the circles are extruded with the push/pull depend on the colour of the cursor and the corresponding colour of the axis*
- *2/3 different colour lines are needed for creating 2D/3D shapes.*

The above show situated accounts of the ideas of:

(a) direction (colour of line same as colour of axis with same direction; two/three different lines needed for creating a 2D/3D shape; flat shapes go up; 3 or 6 ways of extruding circles; relation of colour of cursor and the axes)

(b) position (shapes put in different ways)

(c) orientation (in different angles the shapes go in different ways)

To conclude, I would argue here that the tasks helped the students to infer situated versions of some mathematical ideas relating to vectors and space. Even though their expressions of these notions remained embedded in the setting where they first took place, these situated abstractions can form a basis on which a more developed or sophisticated thinking can be built on. The following sections gather all the above three notions explaining each one of them separately by reinforcing them with extra data from the other pairs of Situation IV.

12.4.1 Direction

Some students expressed dimension by referring to the idea of direction. More specifically it was extracted:

- *The direction of the lines goes in the cardinal directions.*
- *The lines follow the direction of the three axes.*
- *The colour of line is the same of the colour of axis with same direction*
- *There are 3 or 6 (adding the negative directions) ways of extruding circles.*
- *Two/three different coloured lines were needed for creating a 2D/3D shape.*
- *The 2D shapes that are flat (to the ground) go up with the use of Push/Pull tool.*
- *The colour of cursor on the circles shows the direction of the 3D shape created with the Push/Pull.*

The other pairs of students also expressed dimension in similar terms. Students talked about the different colours of the lines and that they went to different directions [A613-617, A591, C536, C754, E342]. Particularly, Pair C tried to *add* the different lines drawn, showing an operation similar to the addition of vectors. In Sketch Up, the pink line shows a line which is parallel or perpendicular to another line that is not parallel to any of the three default ones (see Figure 101: The pink line).

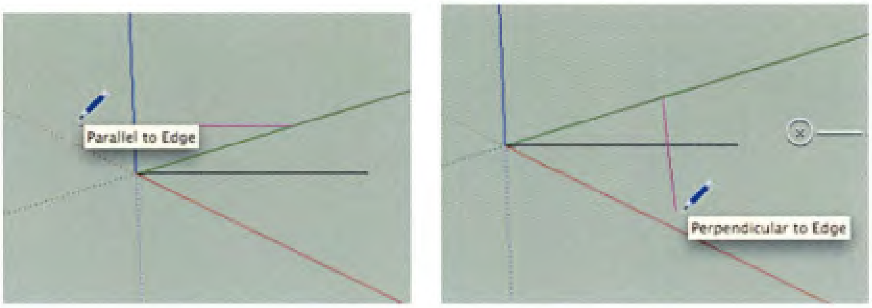


Figure 101: The pink line

Pair C argued that if we added the red line (axis) to the green one (axis), the pink line was created [C564-571]. This shows that students recognised that the pink line did not belong to any of the default axes but it was different. However, they tried to relate it to the ones they knew and thus made the addition. This could be seen as a situated account of the more sophisticated notion of adding vectors.

Logical operations, similar to arithmetic sequences, were also carried out among the pairs of students. For instance, when Pair B was asked to explain how many axes were going to be used for creating a 1D space/object, they argued: "*Because if you realise for 3D we used all of them, but for 2D only 2 of them, and 1D it would be good if we use one*" [B796]. Similarly, Pairs E and F expressed this generalisation in terms of how many 'lines' did shapes in various dimensions have [E416, F431, F502]. This was perhaps a situated way of thinking about the more sophisticated idea of the number of linearly independent vectors in the basis of the vector space:

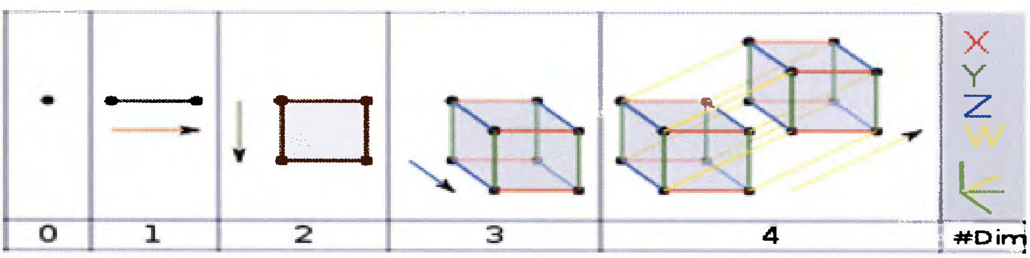


Figure 102: Dimensions, Source: en.wikipedia.org (Dimension)

This idea relates to direction but it also refers to the generation of shapes, which is discussed further in the interpretative account.

Students also related the colour of the circle's cursor to the direction they would go with the Push/Pull tool [A320, A326-346, B622-627, B661, C611, E341, E355, E362]. Students also acknowledged the three ways of movement in space [F356-359]. These expressions as well as the last two in bullet point expressions above, show articulations of the idea of the normal vector. The colour of the cursor on the circle shows the direction of the normal vector, which has the same direction as the 3D object created by the Push/Pull tool. The idea of the extrusion was noted as a transformation from 2D to 3D. Surfaces could be extruded along normal vectors parallel to the red, the green and the blue axes and the above expressions are situated versions of this idea.

Each of the above articulations expressed ideas of direction. The mathematical notion of direction vector is directly related to these ideas. In order for the students to express the direction of the lines (=vectors) they used their familiar notions of cardinal directions. A more sophisticated version of the above could be the articulation of the lines (=direction vectors) following one of the three axes. In other words, every blue line drawn is a multiple of the blue axis. The students noted the three different directions, which described the direction of the three axes. This was again a situated version of the three directions needed for 3D spanning. Adding to this, the extruding of circles helped the students to also notice the negative directions and thus exploring the notion of direction further.

12.4.2 Position

Students' articulations of dimension were also referring to the idea of position and location. In particular, students' situated abstractions included:

- *Shapes can be positioned up or on the ground.*
- *Shapes can be positioned on the top, bottom and, middle of the screen.*
- *Windows (2D shapes) were drawn at the back of the 3D shapes.*
- *Shapes were put in different ways.*

The expressions above relate to the ideas of position and placement in space. The expressions of 'top', 'bottom' and 'middle' are the standard familiar articulations of position. However after exploring the software and the tasks further, students realised that these expressions were not enough. Many shapes, which could have been considered as being 'on the top' of the screen were placed in different planes and thus looked in different positions. Thus, a situated version of the expression of 'in different planes' can be 'shapes were put in different ways'.

Looking back to our initial categorisation of the categories of description, and pointing to our aforementioned discussion on the components of *Dimension as State*, I would argue that the articulations of dimension referring to (i) locating objects in domains and (ii) pointing to the restrictions of movement in each domain (see *Dimension as State*) could be incorporated within this category. Consequently, I refer to the specific articulations from the *Dimension as State* category in order to support my argument here.

Looking across the overall data of this situation, students talked of objects as located in different places, spaces, lanes, angles and ways [A165, B501, C626, D376, D378, E150]. The shapes were also in different areas, the floor and the wall or the sky

[A267-269, B591, E23, E39]. I will mention here, Pair A's expression of non-shaded shapes (shapes which belonged to more than one plane) as shapes that were half on the floor and half in the air [A216-219]. Furthermore, the words 'high', 'low', 'up', 'down' and 'forward' were frequently used for describing the placement in space [A287, C560, C561, D165, D517, E94]. It is also worth mentioning here that Pair E argued that dimension is position and how things were placed [E465, E487-488]. They talked of dimension as positioning, arguing that in 2 dimensions there are 2 ways of positioning things while in 3 dimensions there are 3 ways [E479-481].

The restrictions of movement in a particular domain were also articulations that could be incorporated to this category [B377, F169-170]. For instance, Reine and Nicholas (Pair B) pointed out the difference in moving around in the 2D and 3D neighbourhoods respectively:

> *N: Because, that's 2D, you go only left and right and up and down, while if you are in 3D you go everywhere. For example, let's say a house if it was 2D, you wouldn't be able to go into the house [B377]*

Similarly, Nosakhare and Mya (Pair D) talked about the bug as being able to walk on anything 'he' wanted to in a 2D neighbourhood [D332-336].

Although the idea of the restrictions of movement in a dimensional domain could also refer to base vectors, I chose to include it in this category because it resembled characteristics of dimension as a dynamic version of the placement in space, including all the possible positions that an object could occupy in a specific space.

12.4.3 Orientation

Students referred to orientation while exploring dimension during the fourth situation. Students' situated accounts included:

- *Diagonal lines needed to be used in order for the shape created to get a colour*
- *Parallel lines are of the 'same type of shape' regardless of their length*
- *Lines can be drawn on various surfaces.*
- *Shapes can be distinguished according to their orientation (2D vs. standing up, coming out, kind of 3D but flat shapes; right or wrong shapes)*
- *The shapes are put in different angles.*

The other pairs of students expressed ideas of orientation using similar phrasing. Students distinguished 'flat' from 'facing up' shapes [E64] and they even matched the colour of the shapes to their dimension i.e. the white shapes are 3D [E146]. The students also acknowledged that shapes could face in different ways [F248] and they also referred to 'angles' as Pair D did [A487]. Students also judged whether a place/position of an object was 'right' or 'wrong' according to their orientation as Pair D [B118, E217, E249]. What is more, students also related the colour of the shape's shade to the way it would go with the Push/Pull i.e. "*Maybe white (surface colour) is going like the red (axis), the light grey like the green and those like the blue*" [C743]. They also defined parallel shapes arguing that if they used the Push/Pull they would go to the same direction [C934].

The orientation together with position describes how the object is placed in space. The expressions of orientation above show that students got familiar with working in a 3D environment and more specifically, positioning both 2D and 3D objects in both 2D and 3D spaces.

12.5 Interpretative account – Focus on the idea of capacity

Similarly to the previous account, this section is an interpretation of the interviews, but this time, it focuses on the idea of dimension as capacity. Although Pair D is again the center of attention for presenting the analysis, excerpts from other pairs were necessary to be used in order to elaborate the argument. This account explores how the students articulated dimensional experiences in the form of situated accounts of capacity.

As mentioned before, one of the attributes of SketchUp was that it shaded the closed shapes if they belonged to only one plane. If they lay in more than one plane then they were not shaded. Due to the fact that the students did not pay attention to the colours of the lines while drawing, Nosakhare and Mya (Pair D) created a shape that did not belong to only one plane and thus, was not shaded (Figure 100: Non-shaded shape created by lines). When I asked the students to explain why this happened, they stated that it was because of the type of software tools used to create it. They argued that when they used the Rectangle tool and the Circle tool to create shapes, the shapes created were shaded. This was not the case with the Line tool. Indeed, if the Rectangle and Circle tools are used the shapes created are shaded because ready-made rectangles and circles were drawn in only one plane. On the contrary, with the Line tool, a line could be drawn on any possible plane in space and thus a shape that did not belong to only one plane could be created. Although the students' explanation about generating shaded shapes was reasonable, it was still situated in the software tools, without pointing out the potentials that each tool had.

Therefore, in order for the students to actually realise the difference between the shaded and the non-shaded shapes, I introduced them to Task B4: Incomplete Frames. After using the Orbit tool, the students' explanations regarding the shading on shapes changed. They argued that the coloured (shaded) shapes were 'just great' and 'proper' while the non-coloured (non-shaded) ones were 'twisting and turning'. In other words, the shapes that belonged to one plane were proper because they were similar to their familiar 2D shapes. On the other hand, the shapes that belonged to more than one planes looked as if they were twisting and messed up. This is an idea of containment, showing what shapes could be incorporated into a 2D plane and what shapes could be incorporated into 3D space. This shows that students realised that shapes that looked flat in 3D can be twisted. Although this classification was not as sophisticated as a Maths expert would explain the difference of the shapes if lying on one or more planes, it can be very useful as an intuition or situated cognition expressing the idea of containment that can be developed further at a later stage.

After Task B4, the students continued working on their neighbourhood. They related the non-shaded shape they created to the twisting and turning shapes of the Incomplete frames task arguing that they were of the "same category that changed form with the use of Orbit". This is again a characterisation of the shapes that although they looked flat in 3D, they could be twisted. Similarly, students talked of the difficulty they faced while creating the shapes, arguing that the lines they drew were 'shaking', and by 'shaking' they referred to the lines in going in different directions from the ones they wanted them to go in. This shows the freedom of movement in 3D space compared to the 2D and expresses an idea of containment similar to the 'twisting' and 'proper' shapes above.

Mya then argued that they should try and connect the edges of their shape in order to get a shade. The connection of edges refers to how shaded shapes could be generated and it is similar to the distinction between using the Line tool compared to the Circle or Rectangle tools for creating shapes. Although the students tried to connect the edges of their shape, their shape did not get a shade and thus, they deleted the whole thing and started all over again.

The distinction between the shapes that belonged to one plane from the ones that did not, was also evident among the other pairs as well. For instance, Pair B characterised the shapes, which belonged to more than one plane as 'wrong', 'not completed', 'not done properly', and 'wonky' [B70, B74] and when they finally created a shaded surface, they argued that it was because it was 'good', 'right', and it was done 'neatly' 'it is perfect' and 'it does not change with orbit' [B97, B100, B102, B130, B133, B137, B148, B383]. Likewise, Pair A characterised the shaded surfaces as 'more shapish', 'real shapes', 'more like shapes' while the non-shaded ones (shapes created by lines on more than one plane) as 'just lines' and 'not real shapes' [A232, A233, A238, A249-251, A253]. Pair C also distinguished between 'proper' shapes and not proper shapes [C178]. Similarly to Pair D, students realised that shapes which look flat in 3D could be disconnected and it was supported that a shape had a shade only when its edges were connected [B223-225, C153].

Up to this point, students' expressions showed that although a shape would look flat in 3D, it could be disconnected or twisted. The additional direction of movement allows the possibility of illusory effects and also the additional freedom throws up the possibility of 2D shapes that are twisted. Thus, a key idea about dimension seems to be that in some sense depicts the level of capacity of the space and more specifically the containment of that space (edges could be twisted) and the generation of shapes (shapes could be disconnected).

The difference between shapes, which belonged to only one plane and the ones, which belonged to more than one, can be considered as expressing a situated version of spanning as well. Students identified the direction vectors and realised that if, for instance, a direction vector was red then all the red direction vectors were parallel to each other. They mentioned that these parallel shapes did not need to have the same length to be the *same kind of shape*, and *same kind of shapes* were considered as the lines that have the same orientation (see Orientation section of this chapter p. 317). It was argued that 'same kind of shapes' lines were needed for creating shaded surfaces. This expression can be an initial thought about plane. In other words, all the lines drawn had to be on the same plane in order for the shape to be shaded regardless the length.

I can mention at this point, that the students started identifying which lines were on the same plane or not intuitively. For instance, they were in a position to identify that a line drawn was 'down' (on another plane) without using the Orbit tool. Students argued that the lines could combine and result to a proper shape. This is similar to the combination of direction vectors to create a surface. Students noticed the symmetry of the colour of lines within this combination:

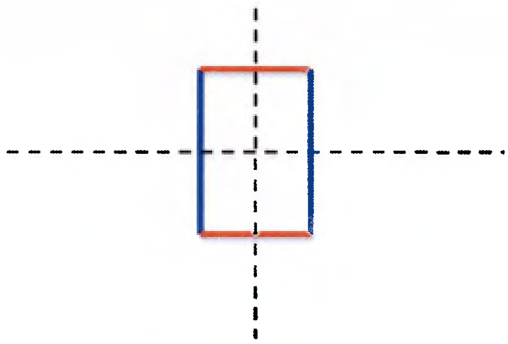


Figure 103: Symmetry in the colour of lines that create a surface

Furthermore, students felt the need to define the planes they were working with. For example, Pair A distinguished between the 'floor level' shapes which were created with the use of red and green lines from the 'red-blue' lines shapes, and this distinction helped them in creating the cube using only lines [A629-635]. Similarly, Pair F argued that "*the floor can be the green thing*" [F41-43]. These expressions show that students see the various planes (red-blue plane, floor plane) as domains that incorporate shapes (red-blue lines shapes, floor level shapes) and thus, illustrating an idea of dimension as containment, where higher dimensional domains incorporate lower dimensional shapes. Last but not least, they connected the SketchUp domain to their familiar coordinate plane, arguing that the red line could be the Y-axis and the green the X-axis [C491].

The above generalisations on surfaces express situated versions of the mathematical ideas of vectors, their linear combination and spanning. The students distinguished between surfaces in various planes (red-blue shapes, floor level shapes) showing an idea of containment, but they were also in a position to talk about the generation of surfaces. The closed and shaded surface acted as a representation of the spanning of the two base vectors and students inferred that by arguing that in order for a surface to be created it needed to have closed and connected edges and also the lines used had to be of two different colours (2 base vectors).

To sum up, by this point, students' explanations included:

- *'Proper' and 'neatly' as the surfaces which belong to one plane and 'twisting' and 'turning' as the shapes which belong in more than one planes.*
- *The 'proper' and 'neat' surfaces are shaded compared to the un-shaded ones that are 'twisting' and 'not properly done'.*
- *Shapes that look flat in 3D can be twisted or disconnected.*
- *There are shapes that can change with Orbit.*
- *In order for a surface to be coloured, its edges have to be connected.*
- *"Same type of shape" lines and diagonal lines create a shaded surface.*
- *Line could be shaking after Orbit.*
- *The colour of the lines used should be symmetrical in order for a 'proper shape' (surface) to be created.*
- *In order to create a 'proper shape' two different colours of lines are needed.*
- *The shapes made by the use of red and blue lines were the 'red-blue lines shapes'.*
- *The shapes which belonged to the plane where the floor was, were the 'floor-level shapes'.*

The above show situated versions of the ideas of:

(a) Containment ('proper and neatly' vs. 'twisting and turning'; shaded vs. non-shaded shapes; coordinate plane; floor-level shapes vs. red-blue lines shapes; shapes that change with Orbit; shaking lines)

(b) Generation (connected edges; symmetry; 2 different colours of lines for 2D shapes; diagonal lines; 'same type of shape' lines)

Up to this point, the students were able to find a relationship between the colour of the lines and whether the shape created was shaded or not. However, in order for a shape to be shaded it had to belong to only one plane, something they failed to clarify. Therefore, while having the students watching, I drew one rectangle by using all the three different colours of the lines. The rectangle looked perfect, created by coloured lines but it was not shaded. The students used Orbit to turn around and see that indeed the rectangle that looked perfect was not even a rectangle. Their first reaction was that it was not symmetrical but they also added that in order for a rectangle to be shaded it had to be created by using only two colours of lines and not three. I considered it helpful at the time to also ask them to relate the colour of the lines once again to the colour of the axes. They pointed out that a green line would go towards the direction of the green axis, a red towards the direction of the red axis and the blue towards the blue axis. They also created lines, one of each colour, to show me the relationship:

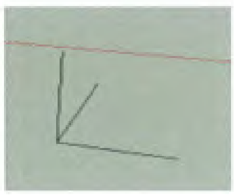


Figure 104: Red, Green and Blue lines

Likewise during task B2, Nosakhare and Mya (Pair D) used the Line tool to create a cube and after creating it, I asked them to name and number the colours of lines they used for doing that. They replied that they used two different colours of lines for creating a rectangle; while for creating the cube they used all three colours. The two/three *different colours* of lines showed the different directions that the line could go. At the same time, it expressed a generalisation of how lines could generate 2D and 3D shapes respectively. This can be considered as a situated version of spanning. Two direction vectors are needed in order to span a plane while three direction vectors are needed for spanning in space.

The specific task (Task B3) was very helpful for creating relationships between 2D and 3D objects/spaces. For instance, Joel and Laura (Pair A) tried to create a cube by using only rectangles but they found it hard. So they argued:

L: We should use lines to make it like standing up [A518]

So, lines (1D) could be used for creating 3D and other pairs noticed that as well [i.e. F85, F244]. Joel and Laura faced a difficulty in doing it by using only rectangles, thus they decided to create the rectangles independently and stick them on in such a way as to create a cube. This attempt, however, failed as their drawing got mixed up while connecting. Then they thought of creating the net of the cube:

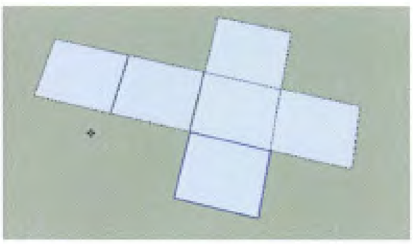


Figure 105: Creating the net of a cube

J: Now we flip it up. [A533]

However while flipping their drawing got mixed up again. So they decided that nets did not work with this software and they tried again using the rectangle tool but this time connecting it and not being left independent from the initial shape. They accidentally made it:

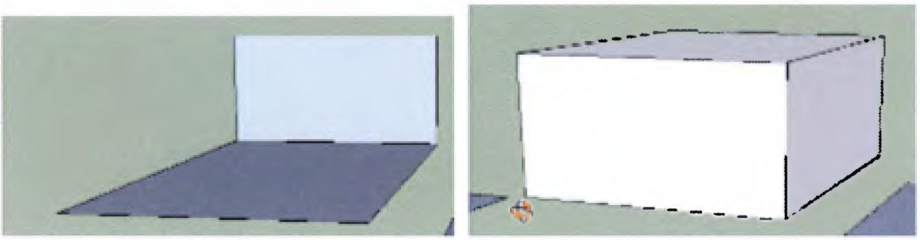


Figure 106: Creating a cube made of rectangles

R: Why did it work out now?

J: Because we just put them on. We started them off 3D.[A564]

After creating a cube by using rectangles I asked them to find another way and they thought of using lines. They created a cube but it was not shaded:

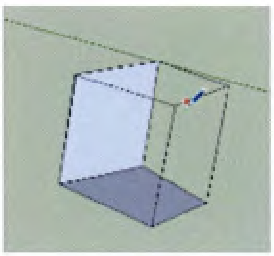


Figure 107: Creating a cube made of lines

During the same task, Reine and Nicholas (Pair C) drew two rectangles, parallel to each other and they tried to draw lines to connect them.

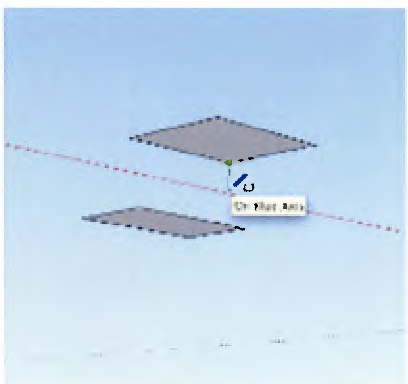


Figure 108: Creating a cube using both rectangles and lines

However, they did not complete their task because Nicholas remembered that he should only use rectangles and not lines, because he used lines before. Nevertheless, the above example showed that the students acknowledged that a 3D cube could consist of both 2D rectangles and 1D lines. Similarly, Kelvin and Charlie (Pair C) during the Task B2: Create a cube, used the rectangle tool to create two rectangles and then they used the line tool to draw lines to connect them [C406-407]:

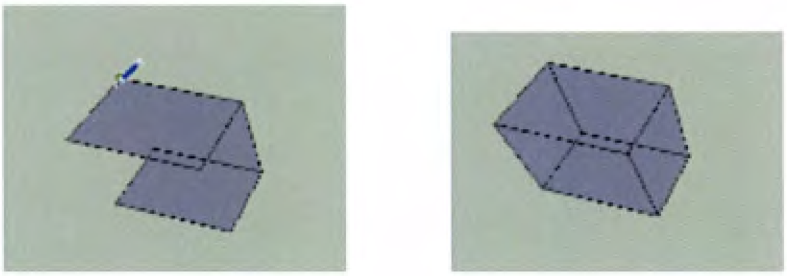


Figure 109 : Creating a cube by using both rectangles and lines II

Looking back to Nosakhare and Mya's interview (Pair D), during the Task B7: Circles, Mya expressed the idea of the three axes as forming a cube:

> *M: You see like...this is the bottom (showing the space between the green and the red axes). It is not a whole cube but a part of it. This is the middle (origin point) and then you go down down, down (across the green line) that's the edge, that's like down, that's the corner (showing the angle created by the blue and the green) and that's the other line (the red axis). [D585]*

Mya thought of the axes as forming a cube (object), which included many other objects and of course the whole design of their neighbourhood (space). The idea of the axes forming a cube was also noted by Reine and Nicholas [B544-552], who argued that some shapes were out of the cube and some in the cube [B591-598]. Thus, these statements were a way of expressing this duality of dimension both as a quality of object and space, bringing out the idea of capacity of both containment and generation, as lower dimensional objects create higher dimensional objects/spaces (axes creating a cube) but at the same time showing that higher dimensional objects/spaces (a cube) can incorporate lower dimensional objects/spaces (1D/2D/3D objects).

Task C involved exposing views about 1D and 0D worlds/objects. After arguing that 2D was a surface and not the actual thing that was 3D, the students claimed that they had not heard of 1D before. I then asked them to define it by looking at the differences between 2D and 3D. The students argued that 1D would be "*one side of a shape*" [D660] and that shapes in 1D would be just lines. This is again a way of expressing this idea of containment within various dimensions.

By this point, students' expressions of dimension included:

- *Extra lines can make a 2D shape to stand up*
- *2/3 different colours of lines are needed for creating 2D/3D shapes respectively*
- *Lines and rectangles can be used for creating a cube*
- *The three axes form a cube which is a domain that includes shapes*
- *1D is one side of a shape*
- *1D includes only lines*

The above show situated versions of the ideas of:

(a) Containment (the three axes forming a cube, 1D is one side of a shape, 1D includes only lines)

(b) Generation (the three axes forming a cube, 2/3 different colours of lines creating 2D/3D shapes, using lines and rectangles to create a cube)

After pair D finished their neighbourhood design, I presented them with the 2D and the 3D versions of their neighbourhood as follows:

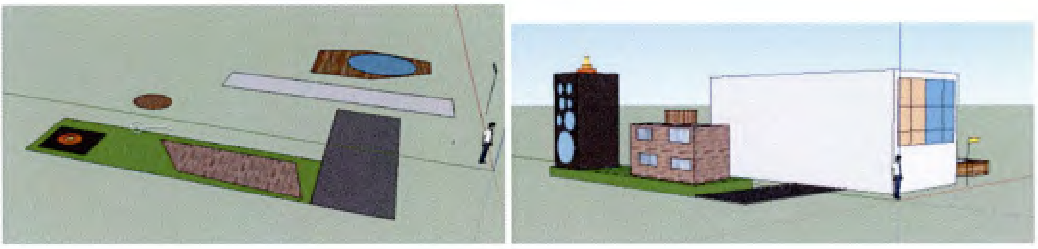


Figure 110: 2D and 3D neighbourhoods

The students were asked to compare the two versions and they argued that 2D is "*flat*", "*you can see the flat top*", comparing to the 3D that "*you can actually see a big piece compared to that (the 2D)*" and "*you can actually see everything inside*" [D292-293]. They then decided to give the example of the table they were sitting at, arguing that the surface of the table is 2D but the table itself is a 3D shape:

> *M: Like this table is like that (showing the table) if you look here you see just that (the surface) but if you look underneath you see the whole thing.*
>
> *R: So what is that table?*
>
> *Both: 3D.*
>
> *R: And what can be 2D in that table?*
>
> *M: That (showing the surface)*
>
> *R: Oh the surface?*
>
> *Both: Yeah the surface.*
>
> *N: The surface is always 2D*
>
> *M: But everything else is 3D. [D294-302]*

The students thought of 3 dimensions as offering something more than the 2 dimensions, like showing the whole picture of something. In their experiences, 2D was incorporated into 3D; in other words, 2D was 'housed' by 3D i.e. surface-whole table, flat top - everything inside. These are expressions showing the idea of containment and how higher dimensional domains could incorporate lower dimensional ones. Adding to this, the example with the tabletop they gave was also from a context outside the SketchUp environment and this showed their ability to link what they observed on the computer screen with some of their experiences of the material world.

Similar experiences of showing the restrictions of vision between 2D and 3D objects/spaces were acquired by using the Orbit tool. The use of the Orbit tool, changed students' pen and paper thinking that they had until then, showing them the affordances of moving in 3D space [E280-281, F105-107]. Students used the Orbit tool in order to see 'the whole thing' of their constructions or a 'better view' compared with looking at 2D object/space where vision is partial [A77, A146-148, B200, B289, B353-354, F293-294, F237-239]. The restrictions of vision and movement within 2D and a 3D space were expressed as a part-whole relationship, where the 2D offered a limited view of the 'whole 3D thing'. This again shows an articulation of dimension as containment similar to the ideas mentioned in the previous paragraph.

At the end of Task A, Pair D argued that the versions of their 2D and the 3D neighbourhood respectively were similar. They pointed out that if we turned the two neighbourhoods into a top view, then they would look the same.

> *N: If you look on top it would have all of it in common but when you look all around...*
>
> *R: You mean if I look at it from the top (I use orbit to show the top view) it will have everything in common?*
>
> *N: Yeah look at it...*
>
> *They put the top view to both the 2D and the 3D neighbourhoods. [D315-318]*

This excerpt shows that students thought of 2D (top view) as part of the 2D and the 3D objects. In other words, 2D could be included in both 2D and 3D objects/spaces. Likewise, Pair E argued that 2D was included into the 3D by giving the example of the 3D computer having a 2D screen [E388-393]. At the end of the interview, the same pair also argued:

> *B: You know like opening a book. It's like a big book, if you take the first page is flat, but the book is big like a volume. If you close it, it goes back to normal as a big volume. [E493]*

Pair E argued that while pushing a 2D shape with the use of the Push/Pull tool, more rectangles were created [E208-210]. Similarly, pair F argued that after the use of the Push/Pull they had a *full* version of their neighbourhood [F193]. The Push/Pull tool helped the students to get an idea about transformations and more specifically about extruding shapes. Using that tool, the students formed articulations of dimension relating to containment as mentioned before.

Nevertheless, after using the specific tool, students also referred to the generation of shapes. Although among the students' expressions were in the form of "*the position (of the 2D shape) changed like we raised it to make it 3D*" or "*Pull means it stretches it*" or "*when it is flat 2D and then you push it you make it, it will turn 3D*", these were still expressions of how a 3D shape could be generated from a 2D one [B37, D38, E196]. Plane was also characterised as 'frame' arguing that after using the Push/Pull it did not look like a frame anymore [C304-306]. What is more, students acknowledged the fact that these transformations were more likely to happen through a computer and not so often with concrete materials [E485-493].

To sum up, by this point, students' explanations included:

- *2D is top view, 3D is the whole object.*
- *2D is a surface, 3D is the whole object.*
- *If we put both 2D and 3D into a top view, they look the same.*
- *By using the Push/Pull a 2D shape more of this shape or a full version of the shape is created.*
- *2D is included into the 3D.*
- *Push/Pull stretches or pushes the 2D shape and turns it into a 3D one.*

The above show situated versions of the ideas of:

(a) Containment (2D as the top view of 3D; 2D surface and 3D whole object; 3D as having something more than 2D; 2D is included into the 3D)

(b) Generation (if we push/stretch a 2D shape we create a 3D one)

Expressing dimension as containment was also noticed when students created lower dimensional objects on higher dimensional ones. For instance, students made use of lines in order to create 2D windows on 3D shapes. Reine and Nicholas (Pair B) created a pattern by using rectangular windows on the cube and circular windows on the cylinder. Likewise, Nosakhare and Mya (Pair D) created windows on their buildings both rectangular and circles [D283].

Up to this point, students' expressions of dimension were dominated by talking about how higher dimensional objects/spaces could incorporate lower dimensional spaces/objects, expressing ideas of containment, i.e. 1D/2D/3D objects in 3D space/objects; and how lower dimensional objects/spaces could create higher dimensional objects/spaces, expressing ideas of generation, i.e. 2 different colour lines create a rectangle. However, students also talked about the reverse situations of whether it was possible to have higher dimensional spaces/objects incorporated into lower dimensional spaces/objects or having higher dimensional spaces/objects generating lower dimensional spaces/objects. For example, Charlie and Kelvin (Pair C) argued that it was impossible for a 2D shape to create a 1D shape:

> *C: Because if you use a rectangle to create a line you can't because it can't make a line that straight but if you use the line it would just go straight.*
>
> *R: So you say we cannot create a line by a rectangle?*
>
> *C: Yeah.*
>
> *R: Why?*
>
> *C: Because it is kind of impossible because if you go to rectangle and I want to draw a straight line it is going to open (to a rectangle). [C907- 911]*

This shows expresses an idea of generation and what shapes can or cannot generate other shapes. Looking at the ideas of containment, the students talked of how a 3D object would behave in a 3D world. This discussion was mostly triggered when students were asked to describe what would a man (3D) be like in the 2D neighbourhood. Students argued that a man (3D) could walk on a 2D world but he could not go inside any buildings. This showed the inability of lower dimensional spaces to incorporate higher dimensional objects. When I asked the students to imagine a man walking in a 2D neighbourhood they replied:

> *N: He sees flats, we would just walk on the thing...he can't go inside because it is flat, he would just walk on the floor. [B832]*
>
> *N: He would just step onto things [B834]*
>
> *P: He wouldn't be able to get inside because it is flat. [B835]*

Whilst if the man was in the 3D neighbourhood:

> *J: And he could actually go in the pool (Pair A) [A398]*
>
> *P: He would be able to get in... (Pair B) [B837]*

The same was noticed during the interview of Pair C:

C: You can go in there (3D) neighbourhood but you can't go in there (2D).

R: Oh…you mean inside the buildings?

Both: Yeah.

R: Whereas here (2D)?

K: He can't go inside, he can only stand on it. [C283-287]

[…]

C: These you could go inside (3D neighbourhood) [C785]

To sum up, by this point, students' explanations were enriched by:

- *2D surfaces can be created on 3D shapes*
- *A 2D shape cannot create a 1D shape*
- *A 3D object can go inside the 3D neighbourhood and its buildings but it can't go in the 2D one.*

The above show an experience of:

(a) Containment (Creating lower dimensional object in higher dimensional ones; A 3D object can go inside the 3D neighbourhood and its buildings but it can't go in the 2D one)

(b) Generation (It is impossible for a 2D shape to create a 1D shape)

To conclude, the idea of capacity lies within the thought that higher dimensional spaces/objects can incorporate spaces/objects with lower dimensions and also that lower dimensional spaces/objects can generate higher dimensional objects/spaces. I would argue here that the tasks helped the students infer the ideas of seeing dimension as containment and generation of shapes/spaces bringing the notions of vector space into play.

Looking back to the categories of description, the notion of transformations as described during the *Dimension as Cross-dimensional* category, was incorporated in this section. As aforementioned, the use of the Push/Pull tool helped the students to talk about pulling and pushing 2D shapes in order to create 3D ones [A4, C17, D2]. This was thought to be an aspect of generation, in other words, how 3D shapes can be generated by using 2D ones.

Likewise, the 'Restrictions of vision' sub-category of the *Dimension as State* was merged in the Containment part as it reflected the part-whole relationship of 2D and 3D. Similar was the component of *Dimension as Cross-dimensional* category in which it was argued that in higher dimensions the visual ability is greater showing again 2D as part of 3D. For instance, expressions of the type "*3D you can see all of it while 2D you only see part of it*" or "*2D you see only one side of it while 3D you see all the sides*" showed this part-whole relationship [A422, C11, C14, C303, C783-786, D19, D20, D292-293, F193, F201-202]. Other expressions showed the ability to see 'inside' of a shape [C783-786, C888, D292-293]. This showed again a part-whole relationship; students would not be able to move around the inner object, if the latter was not contained with another object (vector space). Similar was the expression of 'looking around' the 3D shapes [C870, C303, E447].

As aforementioned in the previous account of the vectorial ideas, students' expressions of these notions remain embedded in the setting they first took place in. However, these situated generalisations can form a basis on which a more developed or sophisticated thinking can be built on. The following table gathers all the notions discussed within this account, explaining each one of them separately:

12.5.1 Containment

Students' articulations included seeing dimension in terms of containment. More specifically, their expressions included:

- *A top view is 2D while the whole object (whose top view belongs to) is 3D.*
- *A surface is 2D while the whole object (whose surface belongs to) is 3D.*
- *The three axes in SketchUp form a cube which is a domain that includes shapes*
- *By pushing/pulling a 2D shape, more of this shape is created*
- *2D is included into the 3D*
- *If we put both 2D and 3D into a top view, they look the same*
- *1D is one side of a shape.*
- *1D is just lines*
- *A 3D object can go inside the 3D neighbourhood and its buildings but it can't go inside the 2D one.*
- *The shapes which belonged to the plane where the floor was, were the 'floor-level shapes'*
- *Shapes that look flat in 3D can be twisted and disconnected.*
- *The 'proper' and 'neat' surfaces are shaded compared to the un-shaded ones that are 'twisting' and 'not properly done'*

The expressions above show an articulation of the notion of containment and relate to the mathematical idea of the dimension of a vector space. The dimension of a vector space V is the number of vectors on a basis of V, and all the bases of a vector space have an equal number of linearly independent vectors. The idea of containment reflects on the idea of dimension as greater dimensional objects/spaces can incorporate equal and lower dimensional objects/spaces.

12.5.2 Generation

The other aspect of capacity noted in the interview is the idea of generation. Students' expressions included:

- *A 2D shape cannot create a 1D shape.*
- *The three axes in SketchUp form a cube.*
- *A rectangle can be created by the use of lines.*
- *A cube can be created by the use of lines and rectangles.*
- *Cubes are created from the use of three differently coloured lines*
- *2D shapes are created from the use of two differently coloured lines*
- *3D shapes can be created by pushing/pulling 2D shapes.*
- *Push/Pull stretches or pushes the 2D shape and turns it into a 3D one.*
- *Extra lines can make a 2D shape to stand up*
- *The shapes made by the use of red and blue lines were the 'red-blue lines shapes'*
- *The colour of the lines used should be symmetrical in order for a 'proper shape' (surface) to be created*
- *In order for a surface to be shaded, its edges have to be connected*
- *In order to create a 'proper shape' two different colours of lines are needed.*

The above expressions of generation of objects/spaces relate to the mathematical idea of how vectors can create a vector space. According to theorems regarding linear spanning, a basis of a vector V is the minimal spanning set when V is finite dimensional. For example, 1D objects can create a 2D space but not the other way around. Similarly, students recognised that only lower dimensional objects can generate higher dimensional ones, and this is not a two way process.

12.5.3 Containment, Generation and the relation to the Vectorial idea

The Vectorial idea refers directly to the properties of the vectors themselves. For example, the vectorial idea refers to direction, position and orientation. These properties do not relate directly to the vector space as an entity. On the contrary, the experiences of capacity seem to integrate two different ideas which both relate to vector space: Containment and Generation. By containment one can see the space as incorporating objects; in this sense the vector space contains the vectors. At the same time, the space can be thought of as generated by the vectors through the way that they span it.

12.6 Summary

In this chapter, an analysis of the data from Situation IV was presented. Firstly, a descriptive account of one pair (Pair D) was presented. The purpose of this account was first for the reader to get an idea of how the interview in this situation progressed and second, for them to understand how the interpretation came about. Secondly, this situation's data were compared to the categories of description as extracted from the previous three situations. The purpose of this comparison was to challenge the previous phenomenographic analysis pointing to the validity and the reliability of the results. It was noted that although similar experiences of dimension were also present in this situation, after challenging the categories by adding the new data, some modifications arose regarding the characteristics of experience in some of the categories. After the two interpretative accounts that followed, two new categories of description –the Vectorial ideas and the idea of Capacity- were added and some of the existing ones were merged.

The *Dimension as State* category was divided into two parts namely 'locating objects in dimensional domains' and 'talking about the limitation of vision of various domains' and each part merged into a subsection of the *Dimension as Vector* and *Dimension as Capacity* categories respectively. The experiences of 'locating objects in dimensional domains' were moved to the *Position* section of the *Dimension as Vector* because they referred to the notion of position and placement in space. On the other hand, the experiences showing the 'restrictions of vision in various domains' were placed under the *Containment* section of the *Dimension as Capacity* because they represented part-whole relationship.

Furthermore, the situated abstractions of the *Dimension as Abstraction* category were present in the other categories that they were directly representing and thus, it was decided the *Dimension as Abstraction* components could be spread across the categories they reflected as ways of expressing experience within those categories.

What is more, the *Dimension as Cross-dimensional* category was reconsidered and its components were transferred into two other categories. Its experiences relating to the restriction of movement and vision in cross-dimensional situations were moved to the sections of *Position* of the *Dimension as Vector* and *Containment* in *Dimension as Capacity* respectively, and its part relating to what stays invariant and what changes in a set of transformations between 2D and 3D was placed under the *Generation* section of the idea of *Capacity*, to which it directly relates.

Last but not least, the idea of having a separate category for *Dimension as Hierarchy* was re-evaluated, and it was decided that its components are perfectly matched to the components of seeing *Dimension as Action* because both types of experiences reflect expressions of dimension followed by logical operations.

In general, this chapter described each category's main features by referring to excerpts as examples. The basic components of each category as described before are presented in the next table, showing the new version of the categories of description:

Dimension as Action

Experiencing Dimension as Action includes expressing dimension as an operation term such as adding, subtracting, combining or separating. It is conjectured that such experience might be a root of the more sophisticated idea of the addition of vectors. This category also included expressions resulting from actions or operations.

Dimension as Material

Material Dimension includes all the expressions of dimension as an object having materialistic attributes. These attributes could be either related to touch, vision, thickness or number of corners/edges the shape has. There was also a reference in relating dimension to everyday experience and distinguishing between what shapes are real or not.

Dimension as Vector

Dimension was articulated as a vectorial idea, pointing to the notions of Direction, Position, and Orientation. Direction, Position and Orientation show articulations of dimension relating to placement in space. I would argue here, that experiencing dimension as a Vector relates to the experiences of dimension within any domain. Direction, Position, Perspective and Orientation can be considered as essential components characterising the vectors of a particular vector space.

Dimension as Capacity

Dimension was expressed as capacity by pointing out the ideas of containment and generation. Articulations of containment included expressions of having lower dimensional objects/spaces incorporated in higher dimensional objects/spaces, i.e. a square could 'house' a line. On the other hand, articulations of dimension as generation consisted of the expressions of having higher dimensional objects generated by lower dimensional objects, i.e. a square can be created by using lines.

This list can be considered as the revised form of the categories of description. The two new categories – Vector and Capacity – seem to be complementary. By that I mean that the Vectorial ideas present expressions within a dimension, while the Capacity ones correspond to expressions about looking across dimensions. For instance, the way a 3D shape is generated involves examining more than only one dimension. On the contrary, the notion of direction is incorporated in each dimensional domain.

It can be noted here that this list is not exclusive; other categories of description might have resulted from the use of different methods of data collection or settings. The next chapter moves a step forward by finding relationships between the different categories of description and thus forming the outcome space of dimensional experience. Subsequently, emphasis will be given on how and why the dimensional tools in the final situation facilitated experiences of dimension that were novel. Taking into account the variation of the setting in all four situations, the organising idea of the next chapter is the relationship between the articulations of dimensional experiences and the setting in which they took place.

Chapter 13: Discussion

13.1 Overview

At the end of Phase 2, the revised version of the categories of describing dimension was presented, consisting of the categories *Dimension as Action* (articulations of dimension as a type of measurement), *Dimension as Material* (having materialistic attributes), *Dimension as Vector* (expressing vectorial ideas such as direction, position and orientation) and, *Dimension as Capacity* (expressing ideas of containment and generation). This chapter begins by discussing how the categories of description extracted from this study could be used for talking about the capability of dimensional experience as the total range of variation of meanings that could be inferred for dimension. Based on that, the outcome space of the study is constructed, which involves a deeper examination of the four categories aiming to form relationships between them. The purpose of outcome space is to present a more generalised form of dimensional experience, informing in that way the mathematical definition of dimension.

After the construction of the outcome space, this chapter presents the final remarks of the role of the setting on the formation of experience, which was evident throughout the research process. More specifically, it gathers evidence from both Phases 1 and 2 by looking within and between the various situations and examines them in terms of their quality as an expressive 'window' for dimensional experience.

13.2 Outcome Space: Experience as Capability

In Phase 1 it was found that some of the characterisations of our orientation towards dimensional experience and the definition of dimension as described in Chapter 2 were 'missing' or not clearly articulated in the data. After considering the role of designing and modelling for mathematical abstraction, the fourth situation in Phase 2 was designed in order to generate experiences of the absent elements, which were:

Table 10: Reproduction of the missing elements of the orientation as extracted from Phase 1

MISSING ELEMENTS FROM THE ORIENTATION OF DIMENSIONAL EXPERIENCE
1. The development of situations expressing an informed background of many aspects of the world relating to dimension that might be used to stimulate and challenge the students.
2 The identification of what stays invariant and what changes in a set of transformations
3. The construction of various dimensional figures in various dimensional domains.
4. An expression of dimension as a quality of both object and space.
5. The development of active situations for expressing dimension as quality of freedom of movement.
6. The development of active situations for expressing dimension as quality of capacity.

The above elements were considered as a basis for the designing of Situation IV. With the use of Google SketchUp, students were actively involved in the situation, as they had to build their neighbourhood, which was a task that acted as a stimulus for the children and formed a connection to their everyday lives. Through the designing, the students were engaged in constructing their neighbourhood, which consisted of creating numerous dimensional shapes in different dimensional domains.

Complementary to the designing of the tasks, were the dimensional tools presented in the software, which embedded mathematical ideas of transformations as well as vector space, ideas that were considered relevant for affording the required missing experiences.

After the conduct of situation IV, the new articulations of dimensional experience were merged to the existing data of the previous situations. Thus, the categories of describing dimensional experience were revised reaching their final form, which consists of four different categories, namely: Dimension as Action, Dimension as Material, Dimension as Vector and Dimension as Capacity. Although each category was described by having different characteristics from the others, this did not imply that the excerpts were exclusive to just one category. On the contrary, this research showed that the same excerpt could display elements of more than one category depending on the variety of meanings that could be inferred.

The data analysis showed that indeed students experienced dimension in new ways not observed before, or at least not expressed in the same way as before. For instance, students articulated experiences explicitly relating to vectors and capacity. Students' articulations referred to the attributes of vectors such as direction, position and orientation. For instance, they identified the various directions that a particular domain might have and also talked about placing objects within that domain pointing to their orientation. In addition, students' expressions showed the duality of the idea of capacity as expressing generation or containment of a vector space. Students talked in their situated language of how vectors could combine to span a surface in a dimensional domain, and also talked of what objects could be included in a particular dimensional domain.

The categories of description can be considered as showing not only the dimensional experiences students articulated, but also the *capabilities* of experiencing the phenomenon of dimension. By capability, I refer to the Oxford's dictionary definition of '*the power or ability to do something*' together with Princeton University Wordnet dictionary's characterisation of '*the susceptibility of something to a particular treatment*' (Oxford Dictionaries Online, 2010; WordNet, 2010). In our case, capability can be thought of as the total range of variation of meanings of the particular phenomenon of dimension. As far as the categories of description showed, students were capable of expressing dimension in these four ways. Of course, other ways/meanings of dimension are not excluded, but so far we have evidence illustrating these specific four categories. Phenomenographic studies in general show variation and change not only in experiencing a phenomenon but also the capabilities of experiencing the specific phenomenon:

> These capabilities can, as a rule, be hierarchically ordered. Some capabilities can, from a point of view adopted in each case, be seen as more advanced, more complex, or more powerful than other capabilities. Differences between them are educationally critical differences, and changes between them we consider to be the most important kind of learning (Marton and Booth, 1997, p. 111)

This study showed that children were capable of experiencing dimension in the above ways as long as the nature of the setting offered them the affordances to do so. In contrast to other phenomenographic studies (Lybeck et al., 1988; Marton and Saljo, 1984; Neuman, 1987; Neuman, 1999; Ramsden et al., 1993; Reid and Petocz, 2002), this study did not seek to find which of the above four categories of description were more advanced or more complex in order to create a 'less powerful to most powerful' hierarchy to represent the outcome space. On the contrary, it looked for connections between these categories in order to find ways of describing the overall capability of experiencing dimension in a more general sense and thus creating the outcome space:

> As we can find different combinations of the very same set of constituent parts in different ways of experiencing the phenomenon, we can identify logical relations – such as complementarity and inclusion – between them, within the complex they together constitute, which we call outcome space.(Marton, 1996, p. 164)

These connections became apparent after the final creation of the categories of description, as aspects that were previously considered evident changed after the interaction of the students with the software:

> It may be useful to separate the features built into a computer microworld by its designer from aspects of the microworld which emerge when it is placed in front of a learner; that is, from characteristics that become apparent only when the microworld is actually used (Edwards and Benedickt, 1995, p. 143)

After looking through the four categories of description, and the four settings designed, three distinct features of capability became apparent, which comprised the outcome space: (a) a revision of the definition of dimension as a quality of vector and a quality of vector space, (b) a passive perspective compared to a more dynamic perspective of experiencing dimension and, (b) experiencing within a dimension or looking across dimensions. More about how these components emerged is described in the next paragraphs.

13.2.1 Passive vs. Dynamic qualities

One of the missing elements of the orientation, which was explored during the fourth situation, was the articulation of dimension as a quality both of object and of space. Dimension as a quality of space was articulated in both Dimension as Vector, showing the attributes of a vector space and in Dimension as Capacity showing the generation and the containment of a vector space. However, after forming the categories for describing dimensional experience, the articulations of dimension as a quality of 'object', as defined in Chapter 2, was leading to something fixed and constant.

Looking at dimension as a quality of something fixed, such as an object, was articulated when students considered themselves as passive observers describing a fixed situation. For example, the expression "*The more the dimension, the thicker the object*" is a statement someone made as a passive observer of reality. Although thickness as an attribute of shapes can be measured, students showed that they could name the dimension of a shape with just a glance of the shape as a whole i.e. if it is thin→ then it is flat→ so it is 2D, while if it is thick is 3D. On the contrary, the (incorrect) observation that "*the more corners a shape has, the more its dimension*" shows a more active involvement towards the phenomenon where the observer must count the corners of a shape in order to find its dimension. This duality of looking at dimension as a passive or dynamic quality was evident within each category of description. The examples noted above were both parts of the category *Dimension as Material*. Similarly, the latter expression was also part of the *Dimension as Action*. However, *Dimension as Action* did not only describe dimension as an act but also as an outcome of an action, which depicts a more passive stance. For instance, the expression belonging to the Dimension as Action category that "*a shape can be both 2D and 3D at the same time*" was an articulation of dimension expressing a static situation.

The duality of looking at dimension as a passive or a dynamic quality was expressed during the category *Dimension as Capacity* as well. The containment component of capacity shows a static picture of vector space as a fixed object that incorporates various shapes. On the contrary, the generation component of capacity shows a more dynamic expression by imagining vectors forming a basis that generates the vector space. Thus, in Figure 111, the red frame of the cube can be thought of as 3D vector space with the capacity to contain objects such as vectors. However, also in Figure 111, the green vectors can be seen as activity creating the 2D vector space on the right hand edge of the cube.

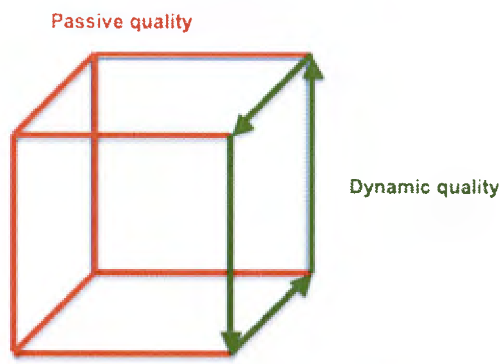


Figure 111: Experiencing dimension as a passive or dynamic quality

Likewise, the *Dimension as Vector* category incorporates components of dimension that could be both considered as static or dynamic depending on the observer's stance. For example, an observer can describe a vector by referring to its fixed attributes such as direction, position and orientation. On the other hand, an active observer could construct a vector in space by drawing on a particular direction, position and orientation.

Considering these static and active qualities of dimension, the term 'object' did not seem to incorporate both qualities. Therefore, I have adopted the language of vectors rather than objects as the vectorial idea seems to embrace both passive and dynamic ways of looking at dimension. Having that in mind, as well as considering the four categories of description, instead of arguing that dimension is a quality of object/space, it is argued that dimension was articulated both as a quality of vector and as a quality of vector space. (Of course, the connection with vector is an inference by the researcher. The children's articulations, though apparently far removed from the mathematical constructs of vectors and vector spaces, are seen in this study as situated roots for those sophisticated ideas.)

Consequently, dimension as a quality of vector space can be experienced both as a dynamic process of generating/spanning vectors or, as a static vector space that incorporates equal or lower dimensional objects. Likewise, dimension as a quality of vector can be seen as describing static attributes of vectors or dynamic attributes that generate vectors (see Figure 112).

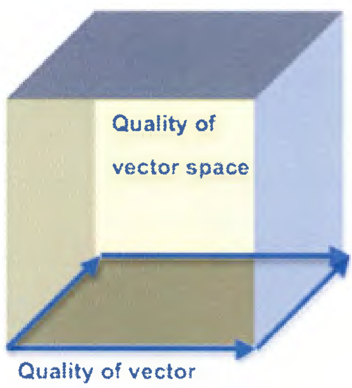


Figure 112: Dimension as a quality of vector or vector space

Considering the above, Situation IV can be considered as a situation that incorporated both of these notions. However, this comparison between object and space gave an insight into another idea of dimensional experience: the capability of experiencing dimension as passive and dynamic qualities of experience.

13.2.2 Passive-Dynamic vs. Object-Process

Expressing dimension as having both a passive and a dynamic quality of experience suggested the possible value of relating its outcomes to the object/process theories developed in the mathematics education literature during the 1990s (Dubinsky, 1991; Gray and Tall, 1994; Piaget, 1985; Sfard, 1991). The articulations of dimension as both a dynamic process and a static object leads us to the relationship of object and process as described by these studies. Starting from Piaget's (1985) focus on how "actions and operations become thematized objects of thought or assimilation" (p. 49), Dubinsky (1991) and Sfard (1991) followed on exploring how a process can be transformed into an object.

The notion of 'object' was elaborated differently among these researchers. For instance, Piaget defined it as the "thematized object of thought", while Dubinsky (1991) and Sfard (1991) referred to the "encapsulated" and "reified" object respectively. The common idea among these theories is that they all have the notion of the 'object' as the highest developmental goal of concept formation. In search for a meaningful relation between these theories and our dynamic process and static object experiences, a deeper examination of the transition between process and object was needed.

To begin with, Dubinsky (1991) defined as 'objects' all the mathematical objects, and argued that individuals are involved in 'actions' on these objects in order to make calculations with these objects. When the individual creates an internal construction related to one of these '*actions*' as it is performed, then we can talk of the '*interiorization*', which is the transformation of an action into a '*process*':

> For example, the computation of the dual of a particular vector space is an action on that object. The idea, independent of any particular vector space, that it may have a dual and it can often be computed, is the process that results from interiorizing the data (Dubinsky, 1991, p. 107).

The act of performing calculations on the objects relates to numerical calculations, although Dubinsky (1991) argued that it goes beyond that by giving the examples of the computation of the homotopy group of a topological space and the determination of the dual of a vector space. Nevertheless, this distinction between objects and process does not fit with our characterisation of passive object and dynamic process, which in our sense are parallel and not graded in hierarchical order.

Looking for a clearer relation between object-process theories and our outcomes, the work of Sfard (1991) was also explored. Sfard (1991) talked of the duality of a concept as being both an object and a process. She talked of objects as:

> [...] being capable of referring to it as if it was a real thing – a static structure, existing somewhere in space and time. It also means to be able to recognise the idea "at a glance" and to manipulate it as a whole, without going into details" (p. 4)

The above definition captures our characterisation of expressing dimension as a static/passive object. Following Sfard's (1991) definitions, a process was expressed "as a potential rather than actual entity, which comes into existence upon request in a sequence of actions" (p. 4). Sfard (1991) distinguished between the ideas of seeing a concept as an object (structural conception), and describing it through processes (operational conception). In her question of "*How can anything be a process and an object at the same time?*" (p. 4), she replied that the two are complementary:

> The term "complementarity" is used here in much the same sense as in physics, where entities at subatomic level must be regarded both as particles and as waves to enable full description and explanation of the observed phenomena.(Sfard, 1991, pp. 4-5)

This idea of complementarity could be used to explain our relation between the passive and the dynamic aspects of dimension. Sfard (1991) gave the example of symmetry to distinguish the structural and the operational aspects of the concept. It was noted that a structural description of symmetry could relate to the concept as a property of a geometrical shape, while an operational description could be as a transformation of a geometrical shape. Relating this to the experiences of dimension, a structural aspect of dimension could relate to its 'passive' description, of looking at dimension as a property of shapes and space 'at a glance', while the operational structure could express this 'dynamic' nature of dimension noted where the children were using dimension for performing actions (in this case processes) i.e. generating shapes. Although, Sfard (1991) argued that the operational and the structural activities of a concept are complementary notions considered as 'two sides of the same coin', she then created a hierarchy between them:

> It seems, therefore, that the structural approach should be regarded as the more advanced stage of concept development. In other words, we have good reasons to expect that *in the process of concept formation, operational conceptions would precede the structural* (p. 10)

The difference between our approach and Sfard's lies in the relationship between the structural and the operational aspects of a concept. She pointed out that the structural conception evolves from the operational one and she defined *reification* as the point where a process would be reified into an object. Our approach differs in that the two 'dynamic' and 'passive' capabilities of experience are not only complementary but they also have the same weight of experience. In other words, this study showed that the two might happen simultaneously without having a hierarchical relationship between them.

In looking for an approach that links the two notions of 'passive' and 'dynamic' in both a complementary and a simultaneous way, Gray and Tall's (1994) 'amalgam of concept and process' (p. 121) seemed to embrace this idea. Gray and Tall (1994) talked of the symbol as incorporating the concept-process idea and thus becoming a 'procept'. Although their definition seemed to incorporate both our aspects of 'static' and 'dynamic', Gray and Tall (Gray and Tall, 1994; Tall et al., 1999) created the 'procept' in order to explain the ambiguity of symbolism in arithmetic and algebra. As our study does not involve any symbols of this kind, the notion of procept cannot fully elaborate the experiences of dimension and vector space in this study. What is more, similarly to Sfard (1991), Tall and Gray (2001) created a spectrum of sophistication of concepts ranging from procedure to process, and then to procept. In general, the early work of Sfard, and of Gray and Tall, was largely based in arithmetic and algebra rather than geometry. This study showed that it seemed possible for the children to gain an appreciation of attributes of the object (as vector or vector space) without needing to encapsulate the idea from processes.

An appreciation of dimension would be incomplete without experiences relating to both objects and processes, or to both structural and operational structures. The notion of capacity, for instance, seemed to allow the construction of lower dimensional objects within higher dimensional objects (containment) as well as the construction of higher dimensional objects from lower dimensional objects (generation).

13.2.3 Within a dimension vs. Across dimensions

Additional to the capabilities of experiencing dimension as a quality of vector or vector space, and having a passive or dynamic quality, students' articulations of experience were also divided between looking within a particular dimension and looking across dimensions. Thus in Figure 113, the mauve coloured base of the cube represents experience that seemed located within a specific space, whereas the vertical arrow depicts experience and transcended different dimensional space (here, across the 2D base and the 3D cube).

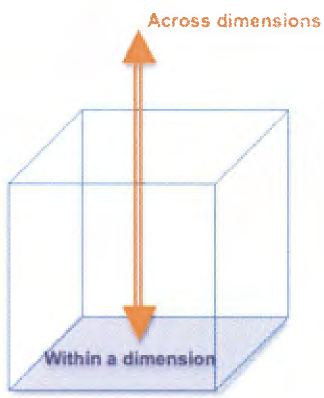


Figure 113: Within a dimension vs. Across dimensions

In *Dimension as Action* students' experiences could be distinguished between expressing articulations within a particular dimension or of looking across various dimensions. For instance, the articulation "*the more the corners, edges and sides of a shape, the greater its dimension*" relates to a quality within a particular dimension, while the expression "*the number of angles of 2D shapes remain the same even after they transform into 3D*" shows how a property stays invariant across dimensions.

Similarly, in *Dimension as Material* the statement "2D people move like fishes and see like a caterpillar" describes a property within a particular dimension compared to the argument "the less the dimension, the more flexible the shape is" that expresses a relationship between the property and the various dimensions.

In contrast to the clear illustration of this dual nature within the previous categories, the *Dimension as Vector* category includes mostly experiences of looking within a particular dimension and more specifically to the attributes of the vectors themselves such as their direction, position and orientation. These attributes can be considered as essential components for characterising the vectors of a particular vector space. Even though within this category there are statements showing experience across dimensions such as "*two/three different lines were needed for creating a 2D/3D shape*" which illustrates the various directions across a 2D or a 3D domain respectively, they might well describe the attributes of just the 3D space the students were working at. In other words, working within this virtual 3D space, students described 2D shapes by 2 'lines' while the 3D ones by three. Although, at times it is not clear which dimensional domain students were focused on, this does not undermine the idea that there is variation in the experiences when looking within a dimension compared to those when looking across dimensions.

Following on, the *Dimension as Capacity* category incorporates ideas of generation and containment, which involve a view across dimensions. For instance, within the concept of generation, dimension is expressed as the number of linearly independent vectors in the basis of the vector space, where 1-dimensional vectors combine to span/generate a 2D plane or a 3D space. As containment, dimension captures the scope for a space to contain equal or lower dimensional objects. Thus, looking across dimensions is expressed in both generation and containment aspects of capacity. Similarly to the *Dimension as Vector* category, there were articulations of dimension that were difficult to categorise as 'within' or 'across' dimensions. For example, the statement *"the three axes in SketchUp could be seen as forming a cube"* is an idea of capacity that describes a particular dimensional domain. Likewise, the statement "*in order to create a 2D shape, lines of two different colours should be used*" shows an attribute of a particular dimensional shape generated by two linearly independent vectors, but it could be also seen as incorporating lower-dimensional shapes (lines).

It should be acknowledged that the capabilities of dimensional experiences could include this blurring of looking within a dimension or looking across dimensions. Looking back to the other two dualities of dimensional experienced discussed before (quality of vector/vector space, passive/dynamic quality), this blurring or ambiguity (Tall might call it *flexibility*) tends to occur in those as well. For instance, dimension as a quality of vector space can be experienced both as a dynamic process of generating/spanning vectors or, as a static vector space that incorporates equal or lower dimensional objects. At the same time, the dynamic process, for instance, could be seen as taking place within a particular dimension but also involving other dimensions (vector as 1D shape generating higher dimensional shapes). On the other hand, the static quality of the vector space may as well involve a description of a particular vector space generated (within a dimension) or a description of various dimensional shapes included in this space (across dimensions). Likewise, dimension as a quality of vector can be seen as describing static attributes of vectors or dynamic attributes that generate vectors. Vectors could be described 'passively' within a particular dimensional domain, or across other dimensional domains.

Consequently, the outcome space of this study is the variation of the capability of dimensional experience, which includes all the three dualities, potentially simultaneously whilst having a complementary relationship between them (see Figure 114).

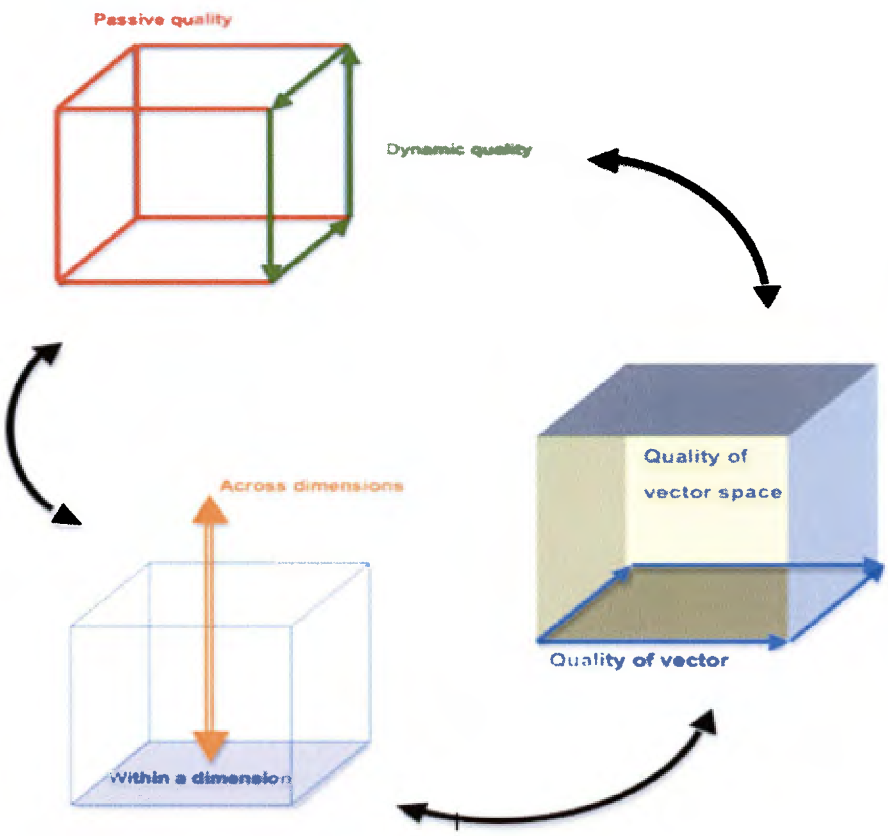


Figure 114: Outcome space

Considering the different dualities of dimensional experience forming the outcome space, dimension as a notion does not just express a theoretical abstract knowledge of geometry but it acts as a tool for modelling relationships between its various qualities. This study showed that children were capable of experiencing dimension in the above ways as long as the nature of the setting offered them the affordances to do so. These qualities were connected through the use of the dimensional tools and demonstrated

Noss and Hoyles' (1996) argument that tools create "a way to reconnect geometry with itself, a way to model by enriching relationships within shape and space, rather than denuding them of structure" (p. 235).

The data analysis showed how students interacted with the tools, how they formed conjectures to solve problems that appeared because of the tools' restrictions and how they constructed patterns to accomplish their goals. As Noss and Hoyles (1996) argued:

> A microworld comprises tools to construct objects. But these tools are themselves objects which encapsulate relationships. This process/object duality is at the root of mathematical activity, and we have shown how interaction in microworlds can transform this duality into a dialectic (p. 227).

The tools are objects themselves in the sense that they embed mathematical ideas within them. At the same time, these ideas cannot be inferred unless the tools are used. The use of the tools, therefore, describes a process through which the students articulate experiences and create relationships. Consequently, any mathematical activity consists of an object/process dialectical relationship, which is mostly promoted by the software tools and this study showed that in a novel settings and in a new mathematical domain.

Indeed, after looking *within a situation* and *between situations* in Phases 1 and 2 of this study respectively, the role of the tools, and in general the role of the 'window', was found to be significant in generating experiences of dimension. Therefore, as a next step, this study follows a different stance, by taking a micro-perspective in looking across the four designed situations and considering their quality to act as expressive windows into students' experiences of dimension. This exploration of the four situations aims to highlight the features that make a window expressive.

13.3 Within a situation vs. Between situations

As previously discussed, this study was divided into two phases. Phase 1 consisted of three different situations designed to explore the variation in the students' experience of the situation. The data showed that the students were largely oriented toward the phenomenon of dimension as it appeared within each situation. Therefore, as a next step, a fourth situation was created by taking a deeper consideration of the role of designing and modelling, having the intention to generate experiences of the phenomenon of dimension not observed before. Subsequently, Phase 2 aimed to explore the variation in the students' experience of the phenomenon of dimension by looking between situations.

The purpose of the four situations designed was to act as windows on the students' experiences of dimension. As it was argued in Chapter 10, the window as a notion itself has a dual nature: first a window can be considered as the medium through which children could articulate experiences of the phenomenon of dimension; second, a window can be considered as the medium for communication between the researcher and the student in order for the former to 'glimpse traces' of the latter's experiences of dimension (Noss and Hoyles, 1996). In other words, windows are created for looking into two relationships: the student-phenomenon relationship and the researcher-student relationship. What is critical to the effectiveness of both aspects of the window is *expressiveness* and *feedback.*

Feedback is considered as one of the most significant aspects of the notion of window (Edwards and Benedickt, 1995; Healy and Hoyles, 2002; Sarama and Clements, 2009). Although most of the times feedback is provided in a visual form, it can be of great importance for students' debugging process while working with the tools:

> Guided by the visual feedback resulting from their programming activity, users can edit or debug their symbolic constructions until they are satisfied with their final product(Healy and Hoyles, 2002, p. 236)

Together with feedback, characterising an effective medium as *expressive* or having an 'expressive power' is another component of the evaluation of windows (diSessa, 2000; diSessa, Hoyles and Noss, 1995; Hoyles and Noss, 2008). A window can be expressive in the sense that it can present ideas (abstract mathematical ideas) in a concrete form. Noss and Hoyles (1996) talked of an autoexpressive medium, which "can contain the elements of a language to talk about oneself" (p. 69). The window needs to provide an environment through which the student can express their thoughts and experiences:

> We have thought of learning to participate in mathematical activity as learning to articulate structure and relationships, and we have sought to describe conceptual worlds and possess this expressive power. In these domains, abstraction is situated in a system of signs and practices, which, as in official mathematics, play a critical part in structuring which relationships are expressed, and how (Noss and Hoyles, 1996, p. 227).

Connecting the ideas of effectiveness and feedback to the dual nature of the window mentioned before, expressiveness allows the children to try out their ideas and use feedback to develop their ideas, but at the same time, expressiveness allows the researchers to 'observe' the students' experiences of dimension. The following paragraphs look across the four situations designed and examine their quality to act as expressive windows of dimensional experience.

Students externalised their dimensional experiences mostly through the formation of situated abstractions within the particular setting they were working on and this showed that students were largely oriented toward the situation in which the phenomenon was embedded. Looking between situations, and exploring the corresponding situated abstractions noted, there was reason to believe that some situations provided more expressive windows than others as was apparent in the nature of those situated abstractions expressed. Similarly, looking from the perspective of the window as an expressive vehicle for the researcher to observe students' dimensional experiences, there was evidence to show that some situations acted as more expressive windows than others in illustrating students' experiences.

Situation I offered students an environment in which they could manipulate objects on a screen and create other objects. Both Cubix Editor and Math Wheel applications presented ideas of moving between dimensions and transformations between dimensions. Within both environments, students could actively construct such experiences of dimension; however the artefacts had some limitations that restricted them from acting as very expressive windows. To begin with, the mathematics embedded within the Cubix Editor application related to representations and perspective, which were similar to students' school experiences of dimension. Except from the task of constructing the 3D object on screen, the rest of the tasks involved activities that students were already familiar with, and thus completed without any difficulty. Therefore, it did not offer the opportunity for the students to be challenged by their experiences and to develop further. Looking from the researcher's perspective, Cubix Editor gave a familiarity to the students' environment and tasks that did not challenge their experiences and subsequently their explanations were restricted to familiar mathematical terms such as top, side and front views. Therefore, I would argue here that it did not offer an expressive window into students' experiences of dimension.

Math Wheel, on the other hand, offered a completely new environment, which as far as I know the students had never experienced before, at least in this way (rotating 2D shapes to create 3D ones). However, this environment proved to be very hard for the children to make sense of and therefore, their actions were restricted to trial and error procedures. Students sought patterns and made some generalisations, which were helpful for completing the task, but they did not relate these to the notion of dimension and they were not applicable in other settings i.e. *The more to the left (the 2D shape), the smaller on the top (the 3D shape).* Due to this, Math Wheel did not act as an expressive window for the researcher either.

Looking now at Situation II, it did not show strong evidence of influencing the experiences of dimension that students inferred. In contrast to the other three situations, Situation II did not have an artefact to act as a stimulus for students to talk about dimension. On the contrary, it consisted of general questions about dimension which often were too obscure for the students as the questions were not well-connected to the students' experiences, i.e. *What does the notion of 'dimension' mean to you?* The students did not have a basis for their explanations. Thus, the students needed to create their own basis resulting in most students discussing real objects and their materialistic attributes. I would like to add here that although most of these abstractions could be true in everyday life, i.e. "*you feel more a 3D object than a 2D one*", they were often mathematically vague or worse still incorrect. Furthermore, although these articulations follow a logical hierarchy in classifying and identifying shapes and could form the basis on which more sophisticated mathematically correct meanings could be built, Situation II did not offer them primitive ideas, or starting points for further development. However, although Situation II did not have a stimulus for the students, it gave an insight for the researcher into students' materialistic way of looking at dimension.

Moving to Situation III, the story of Flatland was offered as a stimulus for the students to connect to their prior experiences of dimension and also to extend those experiences in dimensions higher or lower than their familiar 2D and 3D domains. Looking from a square's perspective was something that perturbed students' experiences and students were introduced to the notion of multidimensionality. Situation III gave an insight into students' experiences because it offered them an unusual stimulus to talk about: Flatland. Thus, their experiences were challenged and that was evident through the language of their articulations. However, the film as a resource had its limitations. First, it did not relate to students' everyday lives, as it had an imaginary plot which students failed to connect to reality. Second, the students acted mostly as passive viewers of the film and passive recipients of its information; therefore, their articulations of experience showed evidence of repeating and recalling quotes from the plot without allowing any action on the situation. It seemed too much like students recalling the teacher's words in a traditional lesson.

In contrast to the previous artifacts, Google SketchUp showed evidence of a more expressive window into students' dimensional experiences. To begin with, the main interview task led the students to see a purpose in acquiring their goal and that was to design a neighbourhood, which helped to sustain activity. What is more, the dimensional tools provided opportunities by embedding the mathematical ideas of vector space, which were directly relevant to the concept of dimension in a form with which students could connect. The constraint of working with the dimensional tools perturbed students' experiences and directed them towards looking for patterns and creating generalisations that were in the form of abstractions created *in situ*. The dimensional tools acted as see-through medium to the children's experiences and therefore, the researcher could identify instances of students' thinking. The students were actively participating in the designing and they were not just passive recipients of information as in Situation II and III. The data analysis showed that students referred to experiences of dimension not observed during the other three situations and that confirms the significant impact that the tools had on the formation of experience.

However, Situation IV had some limitations as well. For example, a 3D virtual environment would be more realistic if it allowed the user to 'enter' the buildings drawn. Although there was a tool that could have been used for this facility, it was considered difficult for the students of that age to use and thus it was removed from the tools menu. However, this might have had an effect on how students experienced dimension as capacity. Adding to this, although the main task was quite open (to create a neighbourhood) leaving space for the students to explore the various tools, some of the students had in mind a very complicated version of a neighbourhood that they wanted to construct, which would have led into more advanced mathematical ideas. Consequently, the researcher tried to restrict students' extensive imagination to the basic shapes of primary geometry.

To conclude, looking between situations in search of an expressive window on students' dimensional experience, Situation II was considered as having limited potential of generating experiences of dimension while Situation I was not considered as a transparent window for the researcher to look through the students' experiences of dimension. Situation IV, however, seemed to have incorporated both qualities of a window and that was mostly promoted through the use of the dimensional tools that both helped the students to generate experiences of dimension but also acted as a communication medium between the researcher and the student.

13.4 Summary

In this chapter the categories of description of dimensional experience have been used to develop the idea of the students' capabilities of experiencing dimension. Developing this analysis in terms of capabilities suggested the need to further explore the categories of description in order to find relationships between them. This was how the outcome space was formed. Three dualities were noted as a result of this exploration, which focused on expressing dimension as a quality of vector/vector space, experiencing dimension as a dynamic/passive quality, and articulating an experience within a dimension or across dimensions. These dualities were noted as happening simultaneously and as being complementary with one another in order to capture the idea of dimensional experience adequately.

Subsequently, the discussion followed a different stance by taking a micro-perspective in looking across the four designed situations by considering their quality to act as expressive windows into students' experiences of dimension. The exploration of the four situations highlighted the features that made a window expressive. To begin with, an expressive window offers the mathematical ideas embedded in a transparent way and it presents them in a communicative language through its tools. It offers a stimulus for the students to talk about and an environment which they can connect with their everyday and school experiences. However, this environment and these mathematical ideas are not offered in a familiar way but on the contrary, they seek to perturb students' existing experiences. This can happen through the embedded mathematical language, but also through the way it is represented within the tools. The constraints of the tools help significantly in this debugging process, in which through active involvement students' express their experiences. What is more, the window has to be designed at the level of the students so that they would be able to use it effectively. The Google SketchUp used in Situation IV together with its dimensional tools were considered as the most expressive window both for exposing students to the phenomenon of dimension and for revealing to the researcher the students' articulations of dimensional experience.

Chapter 14: Conclusion
A retrospective analysis

14.1 Summary of thesis

This study aimed to explore the experiences of dimension among 10-year old children. From the exploration of the literature, a definition of dimension and an orientation of what might be considered as an experience of dimension were formed. The children's experiences on dimension were examined by designing a phenomenographic study.

The study was divided into two phases. Phase 1 explored the variation in the children's experience of the situation in each of the three different situations which acted as windows on dimensional experience. More specifically, clinical interviews were the method for data collection accompanied by the design of tasks using the Elica applications in Situation I, using real objects as an introductory task in Situation II and using Flatland the film in Situation III. Data were collected from 12 children and subsequently, six categories of describing dimensional experience were extracted from the analysis of the data namely, *Dimension as Action, Dimension as State, Dimension as Material, Dimension as Abstraction, Dimension as Cross-dimensional* and *Dimension as Hierarchy*. Comparing the categories of description to the orientation and the definition of dimension as formed by the literature, some elements were 'missing' from the categories.

The analysis of Phase 1 showed that the children were largely oriented towards the way the phenomenon of dimension was present in each situation. Consequently, the role of designing and modelling for abstraction was further explored leading to Phase 2, in which a fourth situation was designed aiming to generate the 'missing' elements of dimensional experience. Situation IV used clinical interviews for the collection of data accompanied by the design of tasks using the software Google SketchUp. The specific software was chosen because it had many affordances and especially because of its dimensional tools which embedded mathematical ideas about vector space, which was considered very relevant for bridging the gap between the categories of description and the missing elements of dimensional experience. Data were collected from 12 children and the categories of description were revised by merging the new experiences extracted by Situation IV.

Phase 2 aimed to explore the variation in the children's experience of the phenomenon of dimension. The revised categories of describing dimension were *Dimension as Action, Dimension as Material, Dimension as Vector* and, *Dimension as Capacity*. Subsequently, these categories were further explored pointing to the duality of the passive or the dynamic way of experiencing dimension as well as looking *within* and *between* dimensions. These characteristics of the dimensional experiences informed the notion of dimension in general as incorporating a dual nature, as an object but also as a process. Subsequently, the outcome space of Phase 2 looked across the four situations examining their quality to act as expressive windows into dimensional experience. Through this procedure, the software Google SketchUp together with its dimensional tools was considered as the most expressive window of the four situations. This comparison of situations gave an insight into what makes a window expressive both to the student-phenomenon relationship as well as to the researcher-student one.

This concluding chapter discusses how the conducted research elaborated the research questions posed at the beginning and how it contributes to knowledge.

14.2 Research questions

The aim of the study was to explore children's experiences of geometric dimension and I will discuss each in turn:

14.2.1 How is dimension experienced in school and outside of school?

In order to examine the above question, this study followed a phenomenographic approach trying to map out the various experiences children have with regards to the phenomenon of dimension. Windows on dimensional meanings were the four situations designed in order to get a glimpse of students' dimensional experiences. The four situations, including the tasks and the tools used, were designed in such a way as to offer environments for students to expose their experiences and also to allow the researcher to infer new experiences of dimension relating to our orientation of dimensional experience and definition of dimension as formulated by the relevant literature. After analysing the meanings extracted from both Phases 1 and 2, this study proposed four categories of describing dimensional experience:

Dimension as Action: Children experienced dimension as an act or as an outcome of an action. Dimension was used as an arithmetic term (adding/subtracting dimensions or properties of shapes to find the dimension).

Dimension as Material: Children experienced dimension as being a real object having materialistic attributes such as thickness, and related dimension to their everyday experiences of objects and space.

Dimension as Vector: Children experienced dimension as a vectorial idea that included position, direction, and orientation. In other words, they implicitly expressed in situated ways dimension as relating to the properties of vectors.

Dimension as Capacity: Children experienced dimension as containment, by having greater dimensional objects/spaces incorporating lower dimensional ones, and also as generation, by having lower dimensional objects/spaces generating higher dimensional ones.

As aforementioned, other experiences of dimension are not excluded from this categorisation. On the contrary, this study demonstrated that the children were largely oriented towards the way the phenomenon of dimension was presented in each situation and this showed that the setting in which these experiences take place influences their type. Thus, I would argue here that it is possible in other settings to have more types of dimensional experience though, since after Phase 2 the original definition and orientation towards dimension are now well represented, it may be more difficult to find other articulations that fall outside of these four categories.

14.2.2 How are experiences of dimension structured?

After exploring the qualities of the four categories of description further, it was noticed that within each category, students' articulations showed two different ways of experiencing dimension: passive and dynamic. Looking at dimension as a passive quality was articulated when children were noted as passive observers describing dimension as a quality of a fixed and static object or situation. On the contrary, looking at dimension in a dynamic way was articulated when children were actively engaged within the phenomenon by for instance counting corners, constructing shapes or generating vector spaces.

Subsequently, the definition of dimension as a quality of object/space was reconsidered as the notion of 'object' did not seem to incorporate both the dynamic and the passive qualities of dimension. Therefore, it was argued that, instead, we should be talking about dimension as a vectorial quality.

This duality in the experiences of dimension extracted was justified by looking at the dual nature of dimension as both an object and a process. Dimension as a quality of vector space can be experienced both as a dynamic process of spanning vectors or, as a static vector space that incorporates equal or lower dimensional objects. Likewise, dimension as a quality of vector can be seen as describing static attributes of vectors or dynamic attributes that generate vectors.

Nevertheless, this dynamic and passive structure of dimensional experience is present both within a dimension and across dimensions. For example, the *Dimension as Vector* category involves experiencing the attributes of the vectors themselves within a specific dimensional domain, while the 'generation' aspect of the *Dimension as Capacity* category involves looking across dimensions for generating vector spaces.

This structure of the experiences of dimension was discovered by looking at the situated abstractions students created within the various situations. Abstract mathematical ideas of dimension were embedded in the tasks and the tools offered to students, and this study was interested in finding how students articulated experiences of dimension by abstracting knowledge within the specific settings. This study considers situated abstractions as having the potential of growing into something more complex and eventually mathematically abstract:

> We have pointed to the evidence that suggests that the situated, the activity-based, the experiential can contain within it the seeds of something more general. The corollary is that we need to focus on the issue of representation of mathematical objects and how these are expressed; and more fundamentally, how the resources of a setting mediate that expression (Noss and Hoyles, 1996, p. 49).

Indeed as a next step, this study looked into the settings in which these abstractions took place, and looked for evidence to reconsider the components of a design that make it possible for these abstractions to occur. The following section presents a discussion around the factors that shaped these experiences of dimension.

14.2.3 What factors shape the experiences of dimension?

This study designed windows on mathematical meanings in order to explore students' experiences of dimension. Evidence showed that the windows created influenced the formation of the experiences of dimension exposed by the children. Thus, although in Phase 1 of this study, the settings were designed in order to find a way to expose students' experiences, in Phase 2 the setting acted as a dominant ingredient in the articulation of these experiences.

From the analysis of the first three situations, it was noticed that some experiences were more likely to appear in specific settings than others and, thus, the role of the design followed a different stance by trying to stimulate new experiences of dimension not previously observed. New experiences of dimension were in fact stimulated during Situation IV through the use of the software SketchUp and its dimensional tools.

After considering the four situations designed, their quality of acting as windows into dimensional experience was re-examined. The quality of a window was considered as being the medium for communication both between the children and the phenomenon of dimension but also between the researcher and the articulations of students' experiences. Furthermore, the characteristics of a window such as expressiveness and feedback were considered as dominant factors for evaluating the artifacts used. Situation I was considered as the least expressive window from the researcher's perspective while Situation II was thought of as offering a limited medium for the students to externalise their experiences. In contrast, Situation IV was considered to be the most expressive window for both the children-researcher and the students-phenomenon relationship although some of its limitations were noted.

Searching for the widest and most expressive window of dimensional experience and finding Situation IV as the most effective in this respect situation designed, showed evidence of the significance of the designing for mathematical abstraction and the modeling of dimension to the formation of experience. Looking back to the designing aspects of Situation IV, the designing of the software and the tasks were based on a sequence of characteristics. To begin with, the tasks offered something for the students to talk about; in other words, it had a purposeful outcome that mattered to the children cared to reach. The outcome, which was the building of their neighbourhood, related to their everyday life phenomena and this was a stimulus for the students. Furthermore, the tasks were based on active participation of students in constructing shapes in space. What is more, nothing from the above would have worked properly if it were not for the availability of the dimensional tools in the software. The dimensional tools embedded abstract mathematical ideas, such as vector space, which students could experience in situated ways through their use (utility aspect). The restrictions offered by the dimensional tools and the SketchUp environment perturbed students' experiences and, through the process of 'debugging', the mathematical ideas became transparent for the students to connect and abstract.

This study showed the significance of the designing of the setting in the formation of experience. The setting was considered to be the tasks, the software and the tools offered to students. Adding to the importance of the tasks/software/tools, in a macro-perspective, although I do not have systematic research evidence, I presume that the school setting and the everyday experience also influenced students' dimensional experiences.

The phenomenographic study showed that children experienced dimension in some cases as a type of a measurement unit, leading to the performance of actions (see Category of description – *Dimension as Action*). So as reasonable as it is to have an object 10cm in length, it might seem reasonable to have a 2 as a dimension. It was also noted that actions could be carried out with dimensions. So as an object with three sides is a triangle, then similarly a shape with 4 points can be considered as 4-dimensional (see Category of description – *Dimension as Action*). The curriculum promotes geometry as a type of measurement and thus children's view of dimension as an action is not surprising. Searching The National Numeracy Strategy (DfEE, 1999) revealed that dimension is either presented under the section of measures in which the metric units are taught or under the section of Shape and Space whose targets include again a type of measurement.

Measuring was an action occurring both in the same dimension and in moving between dimensions during all the situations. For instance, during Elica students measured the corners of the 3D shape in order to construct the 2D cross-section, during the interviews students defined dimension according to the number of corners the object had, and during Flatland there was a discussion about the number of sides of the circle and whether the decagon is the shape with the most sides. These articulations are what we know about students' measuring and they were probably formed by their experiences of dimension in the school context.

Adding to students' experiences of measuring, another experience, which was probably developed through schooling, is the ability to work in a 3D environment. For instance, students talked of 'proper' shapes as the ones that are on one plane compared to the twisted ones, which belonged to more than one plane. School promotes the traditional pencil and paper setting for solving tasks, thus it is difficult for the students to escape this 2D domain and act in 3D simulated space. For instance, students found it difficult to visualise the depth of the 3D object from its 2D representations in Cubix Editor.

Together with schooling, the everyday experience of students could have influenced their articulations of experience. A category of description, which was present during all the situations, was the *Dimension as Material*. This included articulations of dimension as an object having materialistic attributes or expressing relation to everyday experience. Children grow up in a 3D world and that is where they have first experienced dimension. As this study noted, the everyday experiences students have regarding dimension were expressed through their abstractions. For instance, students talked in terms of tangible objects and they even distinguished between real shapes and not. It seems that students found it difficult to accept anything other than their everyday tangible experience. For example, students were surprised by the way the 2nd dimension was represented in Flatland the film, and they mentioned that they had not seen a film in 2D before. Although this triggered their thinking and exposed them into new experiences of dimension, there were times that students related these 'new' experiences to similar objects in their world such as 'It's like a fish', 'It's like a caterpillar' (see Category of description – Dimension as Material).

To sum up, this study illustrated how the design of the setting in which students' activity takes place influences the formation of abstraction. The affordances that the tasks, the software and its tools need to have were discussed, in order to act as wide windows for the students to the phenomenon of dimension as well as to act as transparent windows for the researcher to the students' experiences.

The following sections of this chapter consist of an exploration of this study's research and pedagogical implications as well a statement of how the study contributed to knowledge.

14.3 Research implications

This study followed a phenomenographic approach for exploring students' experiences of dimension. Within this overall methodology, there was a need to structure that experience. Whereas in some phenomenographic studies, the aim was to consider the depth of knowledge by forming a hierarchy between surface and deep approaches of a phenomenon (Lybeck et al., 1988; Marton and Saljo, 1984; Neuman, 1987; Neuman, 1999; Ramsden et al., 1993; Reid and Petocz, 2002), in this study the connection between setting and meaning became apparent. It therefore became appropriate to analyse that setting from the perspective of it being a 'window' on the children's experience of dimension and to see that experience as articulated in terms of 'situated abstractions' (Noss and Hoyles, 1996).

Considering the role of designing and modeling for abstraction as well as the experiences of dimension and their structure, this study argues that in order for a setting (tasks/software/tools) to be successful in shaping the experiences of students on a specific phenomenon relating to dimension the following characteristics of a window seem especially felicitous. The task and tools might:

- have a purposeful outcome that guides and stimulates student discussion;
- relate with prior experiences (i.e. a real situation) that would act as a stimulus for the students to make connections
- promote the active participation of the students in designing or constructing;
- involve students both in situations within a dimension but also in cross-dimensional situations;
- allow expression of dimension's dual nature: both as an object and as a process;
- bridge different ideas of the categories of description of dimensional experience offering both a dynamic and a passive quality;
- incorporate dimensional tools that present mathematical ideas in a transparent way in which students' could extract and connect;

•incoorporate dimensional tools that have restrict actions and lead students towards a 'debugging' process.

This study considered the significance of the setting for acting as an expressive window into students' experiences of dimension, drawing its attention to the situated abstractions formed by the children within the four situations designed. The question raised now is *how can we make use of these situated abstractions?* Considering the expansion of students' 'contextual neighbourhood' of dimension (Pratt and Noss, 2010), these situated abstractions could be considered as the 'seeds' on which something more general or more sophisticated abstract could be built (Noss and Hoyles, 1996).

This phenomenographic study offers a starting point for further studies of dimensional geometry and vector space and it can be the basis for other research on students' experiences of shape and space. Such studies might aim to develop the categories of description by probably adding data from other situations designed.

14.4 **Pedagogical implications**

There was a debate in the literature chapter whether the teaching of 2D geometry should precede the teaching of 3D or vice versa (Bell, Hughes and Rogers, 1975; Cooke, 2007; Ghali, 1999). This study's use of tools such as the Elica applications, the Flatland the film, and SketchUp demonstrated that both 2D and 3D dimensional geometry could be embraced. The findings extracted from the study did not show which dimension is *better* to be taught first, but on the contrary, it illustrated some interesting experiences students formed about dimension as a construct that traverses

2D and 3D. The categories of description of dimensional experience as identified through the phenomenographic study could act as *roots and routes* (Mason et al., 1985) for the children to develop more expertise ideas of dimensional geometry and vector space. Talking about Algebra, Mason et al. (1985) used the terms ROOTS to describe "these underlying strands as primitive basic ideas from which algebra is derived" and ROUTES as "these basic strands can be thought of as signposts on a mathematical journey" (p. 1). Connecting this to geometry, it can be argued that these early stage experiences students acquired regarding dimension could be used first as 'roots' in the sense that they could grow into more expert abstract ideas and second, as 'routes' in the sense that with some extra help could eventually open the way for sophisticated ideas such as Vector space. These 'roots' could constitute the basis for the teacher to design a 'route' for the children building on their strengths and weaknesses.

Thus, the question raised now is how we, as teachers and educators, could *extract* similar types of experiences from students and how we could *make use* of students' dimensional experiences. The elements for designing for abstraction mentioned in the previous section on *Research Implications* could be applied in a teaching setting as well. For instance, the teacher can make use of the above elements in order to create tasks for students not only for teaching dimensional geometry but also for any other topic of mathematics. The quality of the expressive 'window' could be applied in schools as well, where teachers could use artefacts for promoting the communication between their students and a specific phenomenon, and for having a 'glimpse' into their students' thinking.

I believe that primary schooling has a need for further accessible resources for teaching mathematical concepts such as shape and space. The tools used in this study - Elica applications, Flatland the film, Google SketchUp - could be of great use for primary school teachers for the teaching geometry, either it is about the attributes of shapes and spaces or about geometric transformations or about developing representation and visualisation skills. The next paragraphs describe how each tool can be used in teaching and learning.

14.4.1 Elica applications

The Cubix Editor from the Elica application can be used for getting children to become familiar with 3D virtual space and the idea of depth compared to the 2D representations on paper. However, the tasks should be designed in such a way as to promote the exploration of 3D shape. One of Cubix Editor most significant tools is the rotation of the whole platform of the construction, giving the student the chance to visualise the front, side and top views in a dynamic way (Boytchev, Chehlarova and Sendova, 2007). Two tasks suggested by Boytchev et al. (2007) which I consider suitable for primary school students are:

Task 1: Change the place of one of the unit cubes in Fig. 1 so that the new construction could take the position of Fig. 2 in the space.

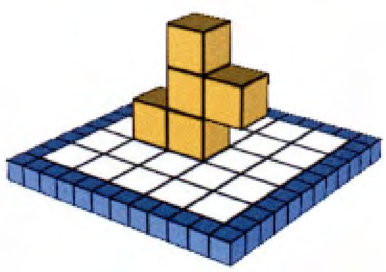


Fig. 1 Initial configuration

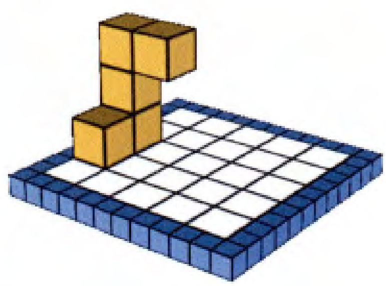


Fig. 2 Final configuration

Task 2: Build a composition in the style of a famous building.

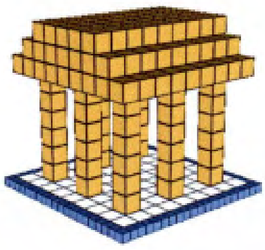
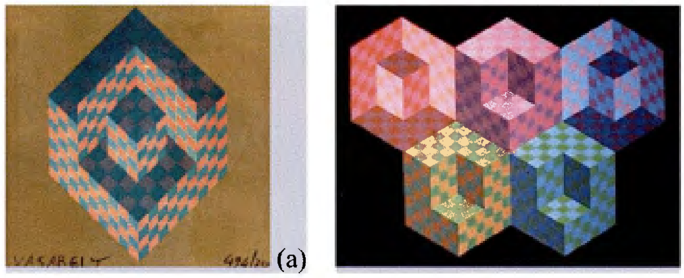


Fig. 4 The Parthenon



(a) (b)

Fig. 5 Compositions by Vasarely

The above tasks promoted the movement in a 3D space by rotating the platform and especially the second engages the mathematical ideas of space and shape to architecture and art.

14.4.2 Flatland the film

Flatland the film may trigger children's imagination by introducing new dimensional worlds that they have not experienced before such as Flatland, Lineland and Pointland. Although it is an imaginary film, it offers students the opportunity to escape their 3D reality they take for granted and to experience dimensions beyond the three, and to get familiar with mathematical abstraction. Nicol and Crespo (2005) used an abbreviated version of the book of Flatland to research students' interaction with imaginative tasks, in an opportunity to escape the traditional curriculum activities of measuring the area and the volume of shapes in order to explore their properties. Their findings showed that students expressed a personally motivated engagement towards Flatland that continued even at the end of the lesson:

> They experienced the abstractness of mathematics and the role imagination plays in creating objects to think with, such as points with zero dimension or lines with one dimension, and the challenges of physically representing these

> abstract objects. The story of Flatland stressed the students to consider their familiar space differently, and it engaged them in an imaginative and extreme (rather than everyday) experience with mathematics (Nicol and Crespo, 2005, p. 247)

Flatland the film can be used for introducing children to multidimensionality and to the abstraction of shape and space. However, in order for the children to be more actively engaged, the film can be accompanied by extra tasks having children to build or create something i.e. drawing the worlds of Flatland, Lineland, Pointland, or guessing shapes by just feeling them.

14.4.3 Google SketchUp

I would like to emphasise here the qualities of Google SketchUp's dimensional tools for embedding abstract mathematical ideas accompanied by restrictions that involve the students in a debugging process leading them to abstraction. These dimensional tools and the abstractions formed through them could be seen as forming intuitions that students could use at a later stage in order to make sense of more advanced mathematical ideas. These specific tools seem similar to be what Papert (1980a) meant by the 'gears' of his childhood.

> The gear can be used to illustrate many powerful "advanced" mathematical ideas, such as groups or relative motion. But it does more than this. As well as connecting with the formal knowledge of mathematics, it also connects with the "body knowledge," the sensorimotor schemata of a child. You can *be* the gear, you can understand how it turns by projecting yourself into its place and turning with it. It is this double relationship--both abstract and sensory--that gives the gear the power to carry powerful mathematics into the mind. (Papert, 1980a, p. viii)

Papert's relationship with gears was deep and emotional. According to Papert, Logo, and more specifically the turtle, had the potential to act similarly to his gears. In this study the children's time with the dimensional tools was transient, not therefore allowing the scope for the children to develop the sort of deep relationship with the dimensional tools that Papert has with gears and envisaged with the turtle. Although Papert's relationship with gears was built as a long-term natural construct from everyday experience, there is reason to believe that the dimensional tools were an artificial construct (in the sense that the researcher created them) that also worked in similar ways. Thus, I would argue that the software tools could act as 'artificial' gears as long as they reflect the principles formed for designing situations of abstraction by triggering affective responses at the same time as shaping conceptual change.

Taking a different perspective, if we look more closely at the teacher and his/her role in the designing of mathematical situations, Google SketchUp can act as a powerful resource not only for the notion of dimension but also for teaching specific areas of Shape and Space in particular. Google created the *Google SketchUp for Educators,* which offers tutorials and lesson plans for the teacher and also a discussion forum through which the teachers could exchange ideas of using SketchUp in the classroom. According to Google SketchUp for Educators, the use of SketchUp in the classroom could help students to "visualise geometry and other mathematical concepts, create model buildings and learn about architecture, design full-scale 3D environments and easily share designs with others via the Web".

What is more, Google SketchUp version 8 has just released, and among its additional tools is the Animation tool through which the user can create animations as a series of scenes that are displayed in succession to give a hands-free tour of a model. Drawing on the importance of the tools on the formation of experience, this tool could be used by the students to demonstrate their work to their peers. In reviewing Google SketchUp's use in education, Clark (2007) pointed out the intuitive use of its modelling tools and the significance of using it for teaching a variety of geometry-oriented courses and activities. In general Google SketchUp offers students the opportunity to become designers and modelers of shapes and space themselves. Such software and tasks should be more promoted as they are more richly designed for abstraction.

14.5 Limitations of this study

This study followed a phenomenographic approach, which gathered data from 24 students. The students were 10 years old and they were from a specific primary school in London. For the purposes of the first situation, the sample also included 4 students of the same age from a primary school in Cyprus. However, it was really difficult to continue having access in the schools in Cyprus so the rest of the sample was gathered from the previous school in London. Thus, the students from London were all from the same school. The sample of children was not controlled for gender or cultural background, although each of these variables might have an effect in the way a child articulates expressions of dimension.

The students were chosen to be upper-middle ability by their teacher, having the assumption that they would be more able to express their ideas during the interview. It is possible that there are further differences to be observed in how children experience dimension, particularly among children who are less competent at school.

Although the sample of the participants is small, each interview consisted of several tasks and lasted between 2-3 hours. Time was a constraint of this study as the time needed for transcribing and analysing the interview had to be considered. This not only restricted the number of students doing the interviews but also the number of tasks designed for use during the interview. Nevertheless, in phenomenographic research the sample does not consist of the participants themselves but the number of the meanings extracted from them. 401 meanings were generated from the interviews (168 from Phase 1 and 233 from Phase 2), a rich database to infer comparisons and form the relevant categories of describing students' experiences.

A limitation might be the level of interventions from the researcher. The interview plans were designed to be flexible for the researcher to interact with the participants as they worked on the tasks. Thus, the sequence of the tasks was also varied so that I could follow the children's attention as it shifted between the different ways of experiencing dimension.

Last but not least, as for most of the phenomenographic studies, the validity of the findings is not easily verified. In this study, I have tried to provide a detailed account of both the interviews and the type of interventions I made. The categories of description as well as the components of outcome space were a result of deep consideration from me, and my supervisor Dave Pratt who provided feedback on the inferences being made as a test of their inherent validity. As for the presentation of the study, I have aimed to ensure that I give a clear view to how this study was conducted by drawing attention to both examples of excerpts from students and relevant work of the literature.

14.6 Contribution to knowledge

There is evidence to support that this study contributed to knowledge in both research and mathematics pedagogy as a discipline. First, I would like to focus on the originality of researching the experience of dimension, a powerful mathematical construct that is rarely taught or researched explicitly and not normally construed as something that is 'experienced'.

Adding to the above, this study gave an insight into the different ways that children experience dimension. The categories of description created, could be of great use to studies relating to both mathematics and science education and more specifically geometry education with a focus to 2D or 3D geometry. The software, the tasks, and the tools this study used could be of a great use to both the teaching of mathematics as well as to further research in mathematics education.

What is more, this study chose a phenomenographic approach to study the experiences children have on the notion of dimension. This thesis can be considered as an exemplar for doing phenomenographic research with children as young as 10 years old. Adding to this, the idea of windows on mathematical meanings suggested by Noss and Hoyles (1996) was followed for creating situations by designing tasks in order to act as windows into students' experiences of dimension. The importance of the setting in children's experiences was most of the times ignored by phenomenographic research. However, this study showed evidence of the significance of the setting on the creation of abstraction. Therefore, it acts as a stimulus for phenomenographers to examine further its impact on children's articulations of experience during the research process. Furthermore, the principles for designing for abstraction extracted by looking at the potentials and the limitations of the four situations, laid the foundations of how to design a study from which meanings of a phenomenon could be inferred from children.

14.7 The Space of Learning Dimension

Looking back to this study's initial motivation (see Introduction Chapter), there was a discussion about the enacted object of learning a phenomenon, or what is otherwise called the 'space of learning', which depicts what is possible to learn, the variation of meanings of that phenomenon (Marton, Runesson and Tsui, 2004). This study identified four different categories of describing dimensional experience and pointed out some characteristics that dimensional meanings have, such as dynamic-passive or within-across dimensions. Looking at the patterns of variation (contrast, generalisation, separation and fusion) as discussed in the particular chapter (see p. 25), this study showed that the students experienced these patterns in order to form some abstractions about the phenomenon of dimension. In particular, students used 'contrast' while comparing different dimensional shapes or even different dimensional 'worlds' (i.e. Situation II, Flatland). They used 'generalisation' by experiencing varying appearances of dimension, i.e. as an object or as a process, and they used 'separation' when they noticed specific dimensional features out of the complex situation they were present in. What is more, students used 'fusion' when they considered different dimensional features at the same time in order to form some abstractions, i.e. two different colours of lines are needed for creating a 2D shape while 3 different colours of lines are needed for creating a 3D shape.

Through the process above, students formed their abstractions and they showed what is possible for them to learn (the capability of dimensional experience) at the specific age, within the specific situations. Students' articulations of experience showed that they were able to form situated abstractions of advanced mathematical ideas such as vector space. These abstractions could be of great use for developing the formal mathematical notions at a later stage of schooling:

> Research into language acquisition indicates that [...] if a child is not introduced early to other languages, he or she will experience much more difficulty in learning a second tongue. Might the same be true with respect to mathematical perceptions? If we wait until students have developed a great deal of arithmetic sophistication (and a great many misconceptions) before we encourage them to think about solid objects and the interaction between different dimensions, we may be depriving them of the chance to appreciate the full power and scope of geometry (Banchoff, 1990, p. 14).

The articulations of dimensional experience showed that students were capable of experiencing dimension in these various ways as long as the situation allowed them to do so. As the role of situation was considered important on the formation of that experience, it is significant to reconsider the role of geometry in the primary curriculum. There is a need to offer 'gears' (Papert, 1980a) to children from the time of their childhood in order to help them acquire deeper understandings of advanced mathematical ideas in secondary or higher education.

Bibliography

Elica applications. [Online]. Available at: http://www.elica.net/site/. Last accessed 28th December 2010.

Google SketchUp for Educators. [Online]. Available at: http://sitescontent.google.com/google-sketchup-for-educators/Home.

Abbott, E. A. (1884), *Flatland*: Seeley & Co.

Abelson, H. and DiSessa, A. (1986), *Turtle geometry: The computer as a medium for exploring mathematics*: The MIT Press.

Abrahamson, D. and Wilensky, U. (2007), 'Learning axes and bridging tools in a technology-based design for statistics'. *International Journal of Computers for Mathematical Learning,* 12 (1), 23-55.

Ainley, J., Pratt, D. and Hansen, A. (2006), 'Connecting Engagement and Focus in Pedagogic Task Design'. *British Educational Research Journal,* 32 (1), 23-38.

Alberti, M. and Marini, D. (1995), 'Knowledge Representation in a Learning Environment for Euclidean Geometry'. In A. diSessa, C. Hoyles and R. Noss (eds), *Computers and exploratory learning* (Vol. 146, pp. 109-126). Berlin: Springer.

Armitage, J. (2003), 'The Place of Geometry in a Mathematical Education'. In C. Pritchard (ed.), *The Changing Shape of Geometry: Celebrating a Century of Geometry and Geometry Teaching* (pp. 515-526). Cambridge: Cambridge University Press.

Banchoff, T. (1990), 'Dimension'. In A. L. Steen (ed.), *On the shoulders of giants: new approaches to numeracy* (pp. 11-59). Washington: National Academy Press.

Barrantes, M. and Blanco, L. J. (2006), 'A Study of Prospective Primary Teachersí Conceptions of Teaching and Learning School Geometry'. *Journal of Mathematics Teacher Education,* 9 (5), 411-436.

Battista, M. T. and Clements, D. H. (1988), 'A Case for a Logo-based Elementary School Geometry Curriculum'. *Arithmetic Teacher,* 36 (3), 11-17.

Bell, D., Hughes, E. and Rogers, J. (1975), *Area, Weight, and Volume: Monitoring and Encouraging Children's Conceptual Development*. London: Nelson for the Schools Council.

Bishop, A. J. (1980), 'Spatial abilities and mathematics education—A review'. *Educational Studies in Mathematics,* 11 (3), 257-269.

Bowden, J. (1996), 'Phenomenographic research: some methodological issues'. In G. Dall'Alba and B. Hasselgren (eds), *Reflections on phenomenography-Toward a methodology?* (pp. 49-66). Göteborg: Acta Universitatis Gothoburgensis.

Boytchev, P. H., Chehlarova, T. K. and Sendova, E. Y. (2007), *Enhancing spatial imagination of young students by activities in 3D Elica applications.* Paper presented at the 36th Spring Conference of the Union of Bulgarian Mathematicians, Varna, Bulgaria.

Champoux, J. E. (1999), 'Film as a Teaching Resource'. *Journal of Management Inquiry* 8,206-218.

Chazan, D. and Yerushalmy, M. (1998), 'Charting a course for secondary geometry'. In R. Lehrer and D. Chazan (eds), *Designing learning environments for developing understanding of geometry and space* (pp. 67-90). New Jersey: Lawrence Erlbaum Associates.

Christou, C., Jones, K., Pitta-Pantazi, D., Pittalis, M., Mousoulides, N., Matos, J. F., Sendova, E., Zachariades, T. and Boytchev, P. (2007a), Developing student spatial ability with 3-dimensional applications, *5th Congress of the European Society for Research in Mathematics Education (CERME).* Larnaca, Cyprus.

Christou, C., Pittalis, M., Mousoulides, N., Pitta, D., Jones, K., Sendova, E. and Boytchev, P. (2007b), *Developing an active learning environment for the learning of stereometry.* Paper presented at the Paper presented at the 8th International Conference on Technology in Mathematics Teaching (ICTMT8), Hradec Králové, Czech Republic, July 1-4, 2007.

Clark, P. (2007), 'Google SketchUp'. [Online]. *Mathematics and Computer Education.* Available at: http://findarticles.com/p/articles/mi_qa3950/is_200710/ai_n21100290/.

Clements, D. H., Swaminathan, S., Hannibal, M. A. Z. and Sarama, J. (1999), 'Young Children's Concepts of Shape'. *Journal for Research in Mathematics Education,* 30 (2), 192-212.

Clements, K. (1982), 'Visual imagery and school mathematics'. *For the Learning of Mathematics,* 2 (3), 33-38.

Cohen, L., Manion, L. and Morrison, K. (2000), *Research Methods in Education.* (5th Edition ed.): Routledge.

Confrey, J., Hoyles, C., Jones, D., Kahn, K., Maloney, A., Nguyen, K., Noss, R. and Pratt, D. (2010), 'Designing software for mathematical engagement through modeling'. In C. Hoyles and J.-B. Lagrange (eds), *Mathematics Education and Technology-Rethinking the Terrain: The 17th ICMI Study* (Vol. 13, pp. 19-45). New York: Springer.

Confrey, J. and Maloney, A. (2007), 'A theory of mathematical modelling in technological settings'. In W. Blum, P. L. Galbraith, H. Henn and M. Niss (eds), *Modelling and Applications in Mathematics Education* (Vol. 10, pp. 57-68).

Cooke, H. (2007), *Mathematics for Primary and Early Years: developing subject knowledge.* (2nd ed.). London: A Sage Publications Company.

Dahlgren, L. (1984), 'Learning conceptions and outcomes'. In F. Marton, D. Hounsell and N. Entwistle (eds), *The Experience of Learning: Implications for*

Teaching and Studying in Higher Education (2nd ed., pp. 23-38). Edinburgh: Scottish Academic Press.

Dall'Alba, G. (1996), 'Reflections on phenomenography–An introduction'. In G. Dall'Alba and B. Hasselgren (eds), *Reflections on phenomenography: toward a methodology*. Göteborg: Acta Universitatis Gothoburgensis.

Denscombe, M. (2010), *The good research guide: for small-scale social research projects*. (Fourth ed.). Buckingham: Open university press.

DfEE. (1999), The National Numeracy Strategy: Framework for Teaching Mathematics from Reception to Year 6: Cambridge, Cambridge University Press.

diSessa, A. (1988), 'Knowledge in pieces'. In G. Forman and P. B. Pufall (eds), *Constructivism in the computer age* (pp. 49-70). New Jersey: Lawrence Erlbaum Associates.

diSessa, A. (2000), *Changing Minds: Computers, Learning, and Literacy*. Cambridge, MA: MIT Press.

diSessa, A., Hoyles, C. and Noss, R. (eds) (1995), *Computers and exploratory learning*. Berlin: Springer in cooperation with NATO Scientific Affairs Division.

Dubinsky, E. (1991), 'Reflective Abstraction in Advanced Mathematical Thinking'. In D. Tall (ed.), *Advanced Mathematical Thinking* (pp. 95-123). Dordrecht: Kluwer Academic Publishers.

Edwards, L. and Benedickt, M. (1995), 'Microworlds as Representations'. In A. diSessa, C. Hoyles and R. Noss (eds), *Computers and exploratory learning* (Vol. 146, pp. 127-154). Berlin: Springer.

Fischbein, E. (1987), *Intuition in Science and Mathematics: An Educational Approach*: Reidel, Dordrecht.

Francis, H. (1996), 'Advancing phenomenography: Questions of method'. In G. Dall'Alba and B. Hasselgren (eds), *Reflections on Phenomenography-Toward a methodology* (pp. 35-48). Göteborg: Acta Universitatis Gothoburgensis.

French, D. (2004), *Teaching and learning geometry : issues and methods in mathematical education*. London: Continuum.

Frobisher, L., Frobisher, A., Orton, A. and Orton, J. (2007), *Learning to Teach Shape and Space: A handbook for students and teachers in the primary school*. Cheltenham: Nelson Thornes Ltd.

Fujita, T. and Jones, K. (2002), The bridge between practical and deductive geometry: developing the 'geometrical eye', *26th Conference of the International Group for the Psychology of Mathematics Education, UEA* (Vol. 2, pp. 384-391): A. D. Cockburn and E. Nardi (Eds).

Gargarian, G. (1996), 'The art of design'. In Y. Kafai and M. Resnick (eds), *Constructionism in practice. Mahwah, NJ: Lawrence Erlbaum Associates*. New Jersey: Lawrence Erlbaum Associates.

Ghali, S. (1999), *A Geometric Framework for Computer Graphics Addressing Modeling, Visibility, and Shadows.* Unpublished PhD thesis, University of Toronto.

Glenn, J. A. (1979), *Children learning geometry : foundation activities in shape (5-9); a handbook for teachers, National Association of Teachers in Further and Higher Education. Mathematical Education Section,.* London [etc.]: Harper and Row.

Goldenberg, E., Cuoco, A. and Mark, J. (1998), 'A role for geometry in general education'. In R. Lehrer and D. Chazan (eds), *Designing learning environments for developing understanding of geometry and space* (pp. 3-44). New Jersey: Lawrence Erlbaum Associates.

Gravemeijer, K. (1998), 'From a different perspective: Building on students' informal knowledge'. In R. Lehrer and D. Chazan (eds), *Designing learning environments for developing understanding of geometry and space* (pp. 45–66). Mahwah, NJ: Lawrence Erlbaum Associates.

Gray, E. and Tall, D. (1994), 'Duality, ambiguity, and flexibility: A" proceptual" view of simple arithmetic'. *Journal for Research in Mathematics Education,* 25 (2), 116-140.

Gray, E. and Tall, D. (2001), *Relationships between embodied objects and symbolic procepts: an explanatory theory of success and failure in mathematics.* Paper presented at the 25th Conference of the International Group for the Psychology of Mathematics Education Ulrecht.

Gutierrez, A. (1996a), Visualisation in 3-dimensional geometry: In search of a framework, *Proceedings of the 20th PME Conference* (Vol. 1, pp. 3-19): L. Puig and A. Gutierrez

Gutiérrez, A. (1996b), 'Children's ability for using different plane representations of space figures'. *New directions in geometry education (Centre for Math. and Sc. Education, QUT: Brisbane, Australia),* 33–42.

Hasselgren, B. (1996), 'Tytti Soila and the phenomenographic approach'. In G. Dall'Alba and B. Hasselgren (eds), *Reflections on phenomenography - Toward a methodology* (pp. 67-82). Göteborg: Acta Universitatis Gothoburgensis.

Healy, L. and Fernandes, S. (2009), Relationships between sensory activity, cultural artefacts and mathematical cognition. In M. Tzekaki, M. Kaldrimidou and H. Sakonidis (eds), *33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 137-144). Thessaloniki.

Healy, L. and Fernandes, S. H. A. A. F. (2011), 'The role of gestures in the mathematical practices of those who do not see with their eyes'. [Online]. *Educational studies in mathematics,* 1-18. Available at: http://dx.doi.org/10.1007/s10649-010-9290-1. Last accessed 5th February 2011.

Healy, L., Hoelzl, R., Hoyles, C. and Noss, R. (1994), 'Messing up'. *Micromath,* 10 (1), 14-16.

Healy, L. and Hoyles, C. (2002), 'Software tools for geometrical problem solving: Potentials and pitfalls'. *International Journal of Computers for Mathematical Learning,* 6 (3), 235-256.

Hershkowtiz, R., Schwarz, B. B. and Dreyfus, T. (2001), 'Abstraction in Context: Epistemic Actions'. *Journal for Research in Mathematics Education,* 32 (2), 195-222.

Hoyles, C. and Noss, R. (1992), 'A pedagogy for mathematical microworlds'. *Educational Studies in Mathematics,* 23 (1), 31-57.

Hoyles, C. and Noss, R. (2008), 'Next steps in implementing Kaputís research programme'. *Educational studies in mathematics,* 68 (2), 85-97.

Hunting, R. (1997), 'Clinical interview methods in mathematics education research and practice'. *Journal of Mathematical Behavior,* 16 (2), 145-165.

John, P. and Sutherland, R. (2004), 'Teaching and learning with ICT: new technology, new pedagogy?'. *Education, Communication and Information,* 4 (1), 101-107.

Johnson, J. T. D. (2007), Flatland the movie.

Jones, K. (2002), 'Issues in the Teaching and Learning of Geometry'. In L. Haggarty (ed.), *Aspects of Teaching Secondary Mathematics* (pp. 121-139). London: RoutledgeFalmer.

Jones, K. and Mooney, C. (2003), 'Making space for geometry in primary mathematics'. In I. Thompson (ed.), *Enhancing Primary Mathematics Teaching* (pp. 3-15). London: Open University Press.

Kaufmann, H., Schmalstieg, D. and Wagner, M. (2000), 'Construct3D: A Virtual Reality Application for Mathematics and Geometry Education'. *Education and Information Technologies,* 5 (4), 263-276.

Laborde, C. (1995a), 'Designing tasks for learning geometry in a computer-based environment'. *Technology in mathematics teaching,* 35-67.

Laborde, C. (1995b), 'Designing tasks for learning geometry in a computer-based environment'. In L. Burton and B. Jaworski (eds), *Technology in mathematics teaching- a bridge between teaching and learning* (pp. 35-68). Hove: Chartwell-Bratt.

Laborde, C. and Laborde, J. (1995), 'What about a learning environment where Euclidean concepts are manipulated with a mouse?'. In A. diSessa, C. Hoyles and R. Noss (eds), *Computers and exploratory learning* (Vol. 146, pp. 241). Berlin: Springer.

Lave, J. (1988), *Cognition in practice*: Cambridge University Press New York.

Lehrer, R. and Chazan, D. (1998), *Designing Learning Environments for Developing Understanding of Geometry and Space*: Lawrence Erlbaum Associates.

Lehrer, R., Jacobson, C., Thoyre, G., Kemeny, V., Strom, D., Horvath, J., Gance, S. and Koehler, M. (1998), 'Developing understanding of geometry and space in the primary grades'. In R. Lehrer and D. Chazan (eds), *Designing learning environments for developing understanding of geometry and space* (pp. 169-200). New Jersey: Lawrence Erlbaum Associates.

Lehrer, R., Jenkins, M. and Osana, H. (1998), 'Longitudinal study of childrenís reasoning about space and geometry'. In R. Lehrer and D. Chazan (eds), *Designing learning environments for developing understanding of geometry and space* (pp. 137-167). Mahwah: Lawrence Erlbaum.

Lehrer, R. and Schauble, L. (2000), 'Modeling in mathematics and science'. In R. Glaser (ed.), *Advances in Instructional Psychology: Educational Design and Cognitive Science* (Vol. 5, pp. 101- 159). New Jersey: Lawrence Erlbaum Associates.

Longman English Dictionary Online. (2009), *Experience*. [Online]. Available at: http://www.ldoceonline.com/. Last accessed 12th November 2010.

Lybeck, L., Marton, F., Stromdahl, H. and Tullberg, A. (1988), 'The phenomenography of the" mole concept" in chemistry'. In P. Ramsden (ed.), *Improving learning: New perspectives* (pp. 81-108). London: Kogan Page Ltd.

Marton, F. (1981), 'Phenomenography—Describing conceptions of the world around us'. *Instructional Science,* 10 (2), 177-200.

Marton, F. (1992), 'Phenomenography and "the art of teaching all things to all men'". *International Journal of Qualitative Studies in Education,* 5 (3), 253-267.

Marton, F. (1994), 'Phenomenography'. *The International Encyclopedia of Education,* 8, 4424-4429.

Marton, F. (1996), 'Cognosco ergo sum-Reflections on reflections'. In G. Dall'Alba and B. Hasselgren (eds), *Reflections on phenomenography - Toward a methodology* (pp. 163-188). Göteborg: Acta Universitatis Gothoburgensis.

Marton, F. and Booth, S. (1997), *Learning and awareness*. New Jersey: Lawrence Erlbaum.

Marton, F. and Pong, W. Y. (2005), 'On the unit of description in phenomenography'. *Higher Education Research & Development,* 24 (4), 335-348.

Marton, F., Runesson, U. and Tsui, A. (2004), 'The space of learning'. In F. Marton and A. Tsui (eds), *Classroom discourse and the space of learning* (pp. 3-40). New Jersey: Lawrence Erlbaum Associates.

Marton, F. and Saljo, R. (1984), 'Approaches to learning'. In F. Marton, D. Hounsell and N. Entwistle (eds), *The Experience of Learning: Implications for Teaching and Studying in Higher Education* (2nd ed., pp. 39--58). Edinburgh: Scottich Academic Press.

Mason, J., Graham, A., Pimm, D. and Gowar, N. (1985), 'Routes to/roots of algebra'. *Milton Keynes: Open University.*

Mason, J. and Johnston-Wilder, S. (2005), *Developing Thinking In Geometry.* London: The Open University in association with Paul Chapman Publishing.

Morgan, C. (2005), 'Word, Definitions and Concepts in Discourses of Mathematics, Teaching and Learning'. *Language and education,* 19 (2), 103-117.

Nemirovsky, R. (2002), 'On guessing the essential thing'. In K. Gravemeijer, R. Lehrer, B. van Oers and L. Verschaffel (eds), *Symbolizing, modeling and tool* *use in mathematics. Dordrecht, The Netherlands, Kluwer Academic* (pp. 233-256).

Neuman, D. (1987), *The origin of arithmetic skills: A phenomenographic approach.* University of Gothenburg, Gothenburg.

Neuman, D. (1999), 'Early learning and awareness of division: A phenomenographic approach'. *Educational studies in mathematics,* 40 (2), 101-128.

Nicol, C. and Crespo, S. (2005), 'Exploring mathematics in imaginative places: Rethinking what counts as meaningful contexts for learning mathematics'. *School Science and Mathematics,* 105 (5), 240-251.

Noss, R. and Hoyles, C. (1996), *Windows on Mathematical Meanings: Learning Cultures and Computers*. Netherlands: Dordrecht, Kluwer.

Noss, R. and Hoyles, C. (2006), 'Exploring mathematics through construction and collaboration'. In R. K. Sawyer (ed.), *Cambridge handbook of the learning sciences* (pp. 389-405). Cambridge: Cambridge University Press.

Ogden, R. M. (1937), 'Naive geometry in the psychology of art'. *American Journal of Psychology,* 49, 198-216.

Orhun, E. (1995), 'Design of computer-based cognitive tools'. In A. diSessa, C. Hoyles and R. Noss (eds), *Computers and Exploratory learning* (Vol. 146, pp. 305). Berlin: Springer.

Osta, I. (1998), 'CAD tools and the teaching of geometry'. *New ICMI studies series,* 5, 128-144.

Oxford Dictionaries Online. (2010), *Capability*. [Online]. Available at: http://oxforddictionaries.com/. Last accessed 10th December 2010.

Papert, S. (1980a), 'Foreword: The Gears of My Childhood'. In S. Papert (ed.), *Mindstorms* (pp. vi – viii.). Brighton: Harvester.

Papert, S. (1980b), *Mindstorms: children, computers, and powerful ideas*: Basic Books, Inc. New York, NY, USA.

Papert, S. (1988), 'The conservation of Piaget: The computer as grist for the constructivist mill'. In G. Forman and P. B. Pufall (eds), *Constructivism in the computer age* (pp. 3-13). New Jersey: Lawrence Erlbaum Associates.

Piaget, J. (1968), 'Quantification, Conservation, and Nativism'. *Science,* 162 (3857), 976-979.

Piaget, J. (1978), *Success and understanding*. Cambridge, MA: Harvard University Press.

Piaget, J. (1985), *The equilibration of cognitive structures*. Cambridge, MA: Harvard University Press.

Piaget, J., Inhelder, B. and Szeminska, A. (1960), *The Child's Conception of Geometry*: Routledge and Kegan Paul: London.

Pizlo, Z. (2008), *3D shape: its unique place in visual perception*: The MIT Press.

Pramling, I. (1996), 'Phenomenography and practice'. In G. Dall'Alba and B. Hasselgren (eds), *Reflections on phenomenography - Toward a methodology?* (pp. 83-102). Göteborg: Acta Universitatis Gothoburgensis.

Pratt, D. and Noss, R. (2002), 'The Microevolution of Mathematical Knowledge: The Case of Randomness'. *The journal of the learning sciences,* 11 (4), 455-488.

Pratt, D. and Noss, R. (2010), 'Designing for mathematical abstraction'. *International Journal of Computers for Mathematical Learning,* 15 (2), 81-97.

Presmeg, N. C. (1986), 'Visualization and mathematical giftedness'. *Educational Studies in Mathematics,* 17 (3), 297-311.

Presmeg, N. C. (1997), 'Generalizing using Imagery in Mathematics'. In L. English (ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (pp. 299-312). Mahwah, N.J: Lawrence Erlbaum.

Prodromou, T. and Pratt, D. (2006), 'The role of causality in the co-ordination of two perspectives on distribution within a virtual simulation '. *Statistics Education Research Journal,* 5 (2), 69-88.

Raghubir, P. and Krishna, A. (1999), 'Vital Dimensions in Volume Perception: Can the Eye Fool the Stomach?'. *Journal of Marketing Research,* 36 (3), 313-326.

Ramsden, P., Masters, G., Stephanou, A., Walsh, E., Martin, E., Laurillard, D. and Marton, F. (1993), 'Phenomenographic Research and the Measurement of Understanding: An Investigation of Students' Conceptions of Speed, Distance, and Time'. *International Journal of Educational Research,* 19, 301-316.

Reid, A. and Petocz, P. (2002), 'Students' conceptions of statistics: A phenomenographic study'. *Journal of Statistics Education,* 10 (2), 2-8.

Resnick, M. (1995), 'New Paradigms for Computing, New Paradigms for Thinking'. In A. diSessa, C. Hoyles and R. Noss (eds), *Computers and exploratory learning* (Vol. 146, pp. 31). Berlin: Springer.

Robson, C. (2002), *Real world research: A resource for social scientists and practitioner-researchers.* (Second ed.). Oxford: Blackwell publishing.

Saljo, R. (1988), 'Learning in educational settings: Methods of inquiry'. In P. Ramsden (ed.), *Improving learning: New perspectives* (pp. 32-48). London: Kogan Page Ltd.

Säljö, R. (1996), 'Minding action. Conceiving of the world versus participating in cultural practices'. In G. Dall'Alba and B. Hasselgren (eds), *Reflections on phenomenography-Toward a methodology?* (pp. 19-34). Göteborg: Acta Universitatis Gothoburgensis.

Sandberg, E. H., Huttenlocher, J. and Newcombe, N. (1996), 'The development of hierarchical representation of two-dimensional space'. *Child Development,* 67 (3), 721-739.

Sandberg, J. (1996), 'Are phenomenographic results reliable?'. In G. Dall'Alba and B. Hasselgren (eds), *Reflections on phenomenography - Toward a methodology?* (pp. 129-140). Göteborg: Acta Universitatis Gothoburgensis.

Sarama, J. and Clements, D. (2009), '"Concrete" Computer Manipulatives in Mathematics Education'. *Child Development Perspectives,* 3 (3), 145-150.

Sfard, A. (1991), 'On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin'. *Educational studies in mathematics,* 22 (1), 1-36.

Simpson, G., Hoyles, C. and Noss, R. (2005), 'Designing a programming based approach for modelling scientific phenomena'. *Journal of Computer Assisted Learning,* 21 (2), 143-158.

Society, R. (2001), *Teaching and learning geometry 11-19. Report of a Royal Society/Joint Mathematical council working group* (Policy document 16/01).

Stavridou, F. and Kakana, D. (2005), 'When Adolescents Represent the Third Dimension: Three Case Studies'. *The International Journal of Art & Design Education,* 24 (1), 53-70.

Strauss, A. L. and Corbin, J. (1990), *Basics of qualitative research*: Sage Newbury Park, CA.

Tall, D. (2004), 'Thinking through three worlds of mathematics'. *Proceedings of the 28th Conference of the International,* 4, 281-288.

Tall, D., Thomas, M., Davis, G., Gray, E. and Simpson, A. (1999), 'What is the object of the encapsulation of a process?'. *The Journal of Mathematical Behavior,* 18 (2), 223-241.

TechSmith, C. (2009), Camtasia: Screen Recording & Video Editing Software.

Todd, J. (2004), 'The visual perception of 3D shape'. *Trends in Cognitive Sciences,* 8 (3), 115-121.

Travis, J. (2007), Flatland, A journey of many dimensions: The movie edition (pp. 35min). U.S.A: Princeton.

Usiskin, Z. (1982), *Van Hiele Levels and Achievement in Secondary School Geometry*. Chicago: University of Chicago.

Üstün, I. and Ubuz, B. (2004), *Student's Development of Geometrical Concepts Through a Dynamic Learning Environment.* Paper presented at the 10th International Congress on Mathematical Education, Copenhagen, Denmark.

Von Helmholtz, H. and Southall, J. (2004), 'Treatise on physiological optics'. In R. Schwartz (ed.), *Perception* (pp. 42-49). Cornwall: Blackwell Publishing.

Vygotski , L. S. (1978), *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Watt, D. (1998), 'Mapping the classroom using a CAD program: Geometry as applied mathematics'. In R. Lehrer and D. Chazan (eds), *Designing learning environments for developing understanding of geometry and space* (pp. 419-438). Mahwah: Lawrence Elbaum Associates.

Weisstein, E. W. *Normal Vector.* [Online]. Available at: http://mathworld.wolfram.com/NormalVector.html. Last accessed 28th June 2010.

Wijntjes, M., Volcic, R., Pont, S., Koenderink, J. and Kappers, A. (2009), 'Haptic perception disambiguates visual perception of 3D shape'. *Experimental brain research,* 193 (4), 639-644.

Wilensky, U. (1991), 'Abstract meditations on the concrete and concrete implications for mathematics education'. In I. Harel and S. Papert (eds), *Constructionism.* Norwood N. J.: Ablex Publishing Corp.

Williams, M. E. and Shuard, H. (1994), *Primary Mathematics Today: Towards the 21st century* (Fourth edition ed.). London: Longman.

WordNet. (2010), *Capability*. [Online]. Available at: http://wordnet.princeton.edu. Last accessed 12th November 2010.

Appendices

Due to the large size of the appendices, I include a sample in paper and I have uploaded the rest.

This thesis includes:

Uploaded appendices:

Appendix 3: Pool of Meanings for Situations I, II, and III

http://rapidshare.com/files/450159250/Appendix_3.zip

Appendix 4: Transcripts of all pairs from Situation IV

http://rapidshare.com/files/450159440/Appendix_4.zip

Appendix 1: Pool of Meanings for Situation III

Codes	Pool of meanings
	Quotes added from SITUATION I
Kai & Paige **F1**	P: normally in movies and films nowadays that we've got they've always got 3D people and objects, and that's weird because I haven't seen a film that is 2D and it's like the house that they are in and we get view from the top, and it is not like we are standing up, the ground is here (showing surface of the table) and it is like house lays from the ground. Like they are on the ground, they don't have stairs or anything,
F2	Ka: It's like a fish. They are just moving in and out of....
F3	**R: What did you like and what did you dislike from this world?** Ka: I like the invention that they go round, and they are flat and they are showing us different kind of dimensions so learning the same time and we are having a little bit of fun while watching it.
F4	**R: How does this world differ from our world?** P: It is weird because most of it is it not the same as our world because we are 3D and they are 2D, they move around like fishes and it is just weird really
F5	**R: What shape would a child of a square be?** P: Maybe a circle K: It has to be a circle but I am not sure P: It goes in order like how many sides it has. K: Then the nonagon should have a decagon and then the decagon should have a circle. P: And then it goes on and on until it becomes a decagon again. K: Until it goes back.
F6	**R: How is the social class of people formed?** K: The circles because they have less sides they are the most important and then the triangles and then I thinks it's the squares

	and then it's the pentagons
F7	**While discussing the shapes of Flatland:** K: No, that's not a triangle, that's a pyramid the main shape of Flatland.
F8	**R: Do you want to add anything on the brainstorming of dimension? What is a dimension?** K: I think it was Mr Teacher that told us this earlier on in the year that if it is 3D it has edges I think and if it is 2D it has sides or points. P: Yeah, if it is 2D it has sides and if it is 3D it has edges. P: Or was it the other way around? I can't remember.
F9	**R: What are you adding? (In the brainstorming of dimension)** K: Possibilities of more dimensions R: Other than? K: Other than 2D and 3D.
F10	K: Oh yeah, there was this line with all these little units and then she spread them round and they were nine squares and then they went all together and they made a mega square.
F11	**R: If we move the unit, it would be a line, if we move the line, it would be a square, if we move the square?** P: A 3D cube, you need like, if you want to make a square from this piece of paper all you have to do is claps (she folds the paper to make a square), like to fold it, and it is like a triangle and then you cut that part off and it is like a square left, you need 4 of them to make a whole cube so you need nine squares in that cube mega thing K: Or you can draw some flat squares and keep doing them around and... P: And then folding them K: Yeah, when you have enough, you can cut it out, you can glue these little bits together and you would have...
F12	**R: What did the Square dream of?** P: That he went into like another world that it was 0-dimensional.

F13	**R: How is the world of Pointland?** K: Just one dot.
F14	P: And the lineland is like two or three lines on a big line so it is like a line (She takes a piece of paper and draws a line).
F15	**R: Why did the king of Pointland not give any attention to the Square?** P: And I think he couldn't see him that's why maybe he thought he was imagining it, because he was like a little dot compared to this big square, it might be like, because he is so small he used to see things that are very tiny but when Arthur, is used to see quite big things, so Arthur can see him but he couldn't see Arthur. K: The line was kind of the same. They couldn't see above or below because the line was here and Arthur actually got himself there and his eye was here so he could actually see him. But when he was above or below he couldn't see Arthur.
F16	**R: Why was the King of Lineland not able to see the Square?** K: Because the king of Lineland can't see up, his eyes can only see left and right P: It is like a caterpillar. His eyes are in front of him, when he turns over here his eyes are here and when he turns over there his eyes are there.
F17	**R: Why did the King of Lineland know only the directions 'left' and 'right', and why did he not understand 'above' and 'below'?** P: Because his eyes aren't like above him or below him, they are to the left and to the right.
F18	**R: Why did the King of Lineland become aggressive with the Square?** P: Maybe because it was blocking his way to see his friends and stuff or It was annoying him because it scared him and he got quite annoyed and Arthur kept annoying him and he said 'I am coming for you' but Arthur went above and the line couldn't see him and Arthur just went away.
F19	**R: How did the Square become visible to the King of Lineland?** K: He came down so the line could see him. He came to the line's level.

F20	**Chapter 6: A visitor (14:51-17:11)** P: Look! It's just a line! (That was when Arthur went to Space and it is shown really thin as a line)
F21	**R: What happened when the Sphere appeared?** K: He didn't know there's anything above...like the line before when Arthur came kind of the same thing is happening for Arthur, because the Sphere was round and it had much of itself, Arthur couldn't see her because he was like 3-dimensional so the Sphere had to be like that to speak to him (showing the Sphere on the table) so it was half of it, half of it was above and half of it was below so she could speak to Arthur. P: It is like Arthur couldn't see him if it was left or right but couldn't see it if it was above or below. P: What I mean by below is that below (takes the paper up the table and shows the space under the paper) and not next to him.
F22	**R: What did A Square see when he looked at the Sphere?** K: He saw a circle. P: Yeah, a big circle.
F23	**R: Why do you think A Square saw the Sphere getting smaller and bigger and changing its size?** K: I think he was turning around a lot and he was also moving side to side going up and down P: Because he was rotating and rolling, like a rolling ball, rolling down the hill but it looks as if it was rolling forwards and backwards strolling and getting smaller quickly and that is why it was getting smaller and bigger.
F24	**R: How, do you think, the Sphere was able to 'see' through everything in Flatland?** K: He was looking down and maybe he could go down and look upwards.
	R: Why did the Square see the Sphere as a line/circle? P: Because he couldn't see above and below him. He could only see the middle part K: It like it is chopped in half, like a semi-circle and he could only see the top side of the face of the part of it kind like a semi-circle, it

	can't see below it, it is like it is chopped in half, he can only see the middle K: This is flat but if it was a cube, it would have some more here and some more here. (She shows a paper in the air, saying it is flat but if it was a cube it would have thickness down and up the paper)
F25	P: Pretend that this is Arthur and this is the Sphere, the only thing you can see is the middle part (folding paper to show the curviness of a Sphere and using her hand to show the eye of Arthur) Like us we can't see the spirits in heaven and stuff because they are waiting above us, it is for him he can only see certain directions where we see all around but we can't see under the ground or above space and stuff.
F26	**R: Why, do you think, it was so difficult for the Square to understand the word 'upwards' and why did he keep saying the word 'northwards'? Is 'upwards' and 'northwards' the same thing?** P: The way he is directing people or when people direct him is only northwards and southwards and where he doesn't really use above and below because he can't go above and below.
F27	**Draw Area 33H. Can you explain its shapes?** R: So it is a point, a line, and then.... P: A square. K: And then a mega-square. R: What shape is the mega square? K & P: It's a cube.
F28	**R: Where did the Sphere take him?** K: He showed him the 3D. It is kind of the same thing but continues, it came from the point, to the line, to the square with all the lines inside and then it came to the 3D square and then this guy is also thinking about the 4D and the circle who was actually 3D said 'well no I don't think you should be thinking about that' and I think that's exactly what the square was saying.
F29	**R: Describe Spaceland.** P: Floated, it was moving and it was floating around everything P: That thing is all flexible

	Ka: It's also, it was like everything bending round. P: Like jelly Ka: The kind of jelly that was all woppelling, kind of jelly, you could make other shapes with it.
F30	**R: What types of shapes existed in Spaceland?** Ka: There were cubes. P: Spirals sort of thing Ka: Jelly R: How do we call those shapes? Ka & P: 3D.
F31	**R: What do you mean by '3-dimensional shapes'?** Ka: Cube.
F32	**R: What do you mean by '3-dimensional shapes'?** P: 3D shape stands up, it is a shape that is not flat, it stands up right and you can hold it, you can break it into parts and it has got edges and sides some ...it's got sides and it's got faces and vertices and corners
F33	**R: Why although the Circles knew that 3 dimensions exist, they did not want people to discover it?** P: So they wanted rule over all the dimensions.
F34	**R: What is the film about?** K: It's about the third dimension and other dimensions. There was more than the basic dimensions.
F35	P: Like Flatland because probably might know about Flatland but they know their dimension better than Flatland, I think everyone has heard all the dimensions but they only know their dimension and where they are better than any of the other dimension think and that is why the king of Lineland he said he didn't know about Pointland or Flatland or the 3rd dimension, he just knew about his dimension where he was that was Lineland.
	Did you learn anything by watching this film? What? K: I learned that there's a possibility of thousands of dimensions

F36	really P: Me too. **R: Anything else?** K: I learned that there was a 0-dimension, and 1-dimension and a 4th dimension P: And a 3rd dimension and maybe a 2nd dimension. **R: Anything else?** P: There's a question I want to ask. Are there more dimensions? A 5th dimension, a 6th dimension, maybe a 9th dimension?
F37	**Describing the 4th dimensional shape:** K: But we kind of prove it in there, because there was a square and then a little square and goes and it looks from different angles it looks like different types of shape, like pulled out P: And brought back in. R: You mean the 4th dimensional shape? K: Yeah it was pulled out and it went in and out. P: It was like (she starts drawing small cube inside a big cube) cube, like a normal square K: It was a normal before and then this and that P: It was like a normal square with more cubes inside it. K: Yeah more cubes inside it. It went in different dimensions. P: It was like that...there was a big square with a square inside it. K: I think it is a touchable shape. You see a telescope you can close it and you can open it as well, out of it, you see like a telescope inside one round circle, and then you can open it up, and there has to be fourth dimensions. P: There's a square inside there which made that square and a square in side that and a square inside that and it goes on and on until it's nothing.
	R: Do you want to add anything about your views on the idea of dimension? P: A place with different shapes 3D, 2D and 4D. K: A different place with a different quantity of faces, for example the

F38	telescope it has lots of sockets K: Sockets equal dimension. P: A place that there are different categories of shapes like a 3D dimension or a 4D dimension K: Different styles R: What is Style? You mean different type? Of what? K: Dimension? K: Point, Line, Square, Cube...how do you call a 4D shape?
Nicholas & Chelsea **F39**	**R:How would you describe the world of Flatland to someone?** N: It's a country. C: It's a 2D country. N: It's flat. Everything is flat. N: But when you said someone comes along...is it a 3D shape that comes along or a 0D shape that tries to convince him?
F40	**R: What did you like and what did you dislike from this world?** N: I dislike the priests that are mean towards the people. I like the way they sleep and they've got a little clock and the clock goes like that, and the clock goes like line, every line is the hour.
F41	C: I think that just because the priest is a circle, even though it is a priest it doesn't mean he has to boss everyone around N: Isn't it a 0D shape in a 2D world? (They argued that because it did not have corners)
F42	**R: How does this world differ from our world?** N: We are not flat. C: That is a 2D world. **R: How is our world?** N: Our world is kind of 3D. C: We stand up and we don't have to like...we go everywhere. N: Actually the world is 0 dimension and we are 3 dimensions. C: We are 3D because we actually stand up and the world is 0

	dimensional because it can't stand up, it has got no edges and it doesn't have any corners.
F43	**R: What shape would a child of a square be?** C: Hexagon. N: Is it a baby...no...is it what a square makes? Pentagon or hexagon...not sure...
F44	**R: How is the social class of people formed?** N: Oh... the more corners you have, the more angles you are and the more clever, the more smart you are C: I think, so if you were an octagon and you can eight sides and the same amount of corners and I think the more edges you have, and the more corners, that be how smart you would be.
F45	**R: Before you said, the more sides you have the cleverer you are. Is the circle clever?** N: It might be actually, because it is 2D but it is actually shaped as 0 dimensional. So it might be clever to spot all those people up, but he is not clever in a knowing kind of way. C: About as smart as a triangle and a square, because if it is smart as a triangle it would be the same as a triangle and a square. He could be less smart than the triangle because the triangle has 3 sides and the circle doesn't have any so it could be just about as smart as the triangle or just about less smart.
F46	**R: What is a dimension?** N: It's different categories really...it's hexagons, squares, prisms **R: So it's different categories of what?** N: Different categories of...shapes?
F47	While defining dimension: **R: You also added 0D Chelsea?** C: Yes.
	R: How is a 'square' created? N: With 3 lines of 3 units. **R: And how can we create a 3D shape?** N: With the square.

F48	**R: How? We put 3 lines of 3 units and we make the square. How can we make the 3D shape?** N: By using the three lines of 3 units **R: How many times?** N: Three because it is 3D so you might have to use it 3 times.
F49	**R: What shape, do you think, a square moving can create?** N: A pentagon or a hexagon. So a square creates...like it gives birth to... **R: A unit if it is moving it creates a line, and a line if it's moving creates a square, a square moving creates what?** N: Three units? No, the square creates maybe a hexagon or a 3-dimensional shape.
F50	A moving square: C: I don't know what 3-dimensional shape it would make, but it would probably make just normally like in a 2D shape it would probably make a hexagon but with a 3D shape it... **R: Why a hexagon?** N: Maybe because, it might be a shape with 7 sides, like you said about 3 units, it might be a clue that because the shape has 4 sides or corners and you add like a 3 sides like have 3 units it might be like a seven sides or so, which would probably be three dimensions
F51	**R: What did the Square dream of?** C: He had a 0-dimensional dream, and with the lines they could only see left and right, they couldn't see up so every time he went up, he had to go, and look at him from the left.
F52	**R: How many places did he visit?** N: Two. C: One. N: Yeah, one. One place but he met two people. One of them was that freaky little light ME that he kept singing MeMeMe.
	R: Why did the king of Pointland not give any attention to the Square? N: He did not know such thing, he thinks that length in space is the

F53	same as left and right but it isn't and he explained it to him , the space is not just your line, and he didn't know, because he was just a line with two...
F54	**R: Why the king of Pointland keep repeating ME and why did he not like the word 'TWO'?** C: When he said there are two of us, he did not like it, because when he was the only one there he thought it was nice and relaxing and he had the whole place to himself so then all that ever was, was himself and no one else so he was probably enjoying it being by himself and then if a second person comes along it would probably annoy him. **R: So did he actually realise there was someone there?** C: No.
F55	**R: How is the world of Lineland?** N: He didn't really know....he could only move left and right, he thought that left and right was the same as the length in Space and then Arthur, he explained he wasn't and then told me about below and above which he couldn't see because he could only see left and right. **R: So, did the line understand was below and above Chelsea?** C: No because if he could only knew left and right, that means he could only see left and right, but we can go left, right and look up and down, and with the line because it is straight, it would probably be really hard to look up and to look down.
F56	**R: Why did the King of Lineland become aggressive with the Square?** C: Because the square was telling the line about above and below and the line thought he was being aggressive just because he couldn't see up and down and only actually explaining and he knew that if you couldn't see anyone by hear people it would be up and down.
F57	**R: How did the Square become visible to the King of Lineland?** N: Because it was above and below, and then he went down where the Line was so he could see him and then when he went below he thought it was dead and then when he thought back and he thought it was a monster.
	R: Imagine you are the square in Flatland and a Sphere comes. What do you see?

F58	C: I only see him as a circle. And because you are flat when you are trying to look at it, you only see like a circle, he is actually round and his height is from up and down, and unlike other circles he can go rolling round, and with the priests they can't do that, the can't roll around because they are flat. **R: Do you agree Nicholas?** N: Yes, mostly.
F59	**R: Do you agree that the square sees the Sphere as a circle?** N: Yes, that's only he can see because he is like that (he puts his head on the table showing a bugs eye view).
F60	**R: As he sees the circle, he also sees the circle getting smaller and bigger. How can that happen?** N: Because the circle is 0 dimensional, and he is in 2 dimension... C: Because he is from Space, he can unlike what other shapes are doing, he can come from Space and then go back and don't come back. And when he goes to space and then he comes back to flatland, that's when he gets bigger it's when he is actually there (in Flatland), when he is smaller that's probably when he goes back (to space) and then when he reappears in Flatland
F61	**R: How, do you think, the Sphere was able to 'see' through everything in Flatland?** N: Because he can go up, down and he could see everything. If it was just flat he could see everything in Flatland, he could go up so he can see everything from below him.
F62	**R: Why, do you think, it was so difficult for the Square to understand the word 'upwards' and why did he keep saying the word 'northwards'? Is 'upwards' and 'northwards' the same thing?** N: It is actually the word upwards because north, east, south, west so he thought that upwards or forward because he knows north, east, west and south, he probably thought it was northwards but he actually doesn't know the real meaning which is upwards.
F63	**R: Why, do you think, it was so difficult for the Square to understand the word 'upwards' and why did he keep saying the word 'northwards'? Is 'upwards' and 'northwards' the same thing?** C: Because he is a flat shape, he is a 2D shape, known to him is like up, known to the Sphere it is just like in front of him, and then the square thinks that, the Sphere is kind of explaining that northwards is not the same as upwards. Upwards is going up like that (showing

	up with her hand) and northwards it is just in front of you and because the square is flat he can't see up or anything.
F64	**R: Why was the Square surprised when he left his world?** N: Well he could see things from above that he couldn't see when he was flat. C: When he was in the 3D world, Space world, he was actually able to look up and down, and do something he was not able to do in Flatland.
F65	**R: What types of shapes existed in Spaceland?** C: There was shapes that just constantly go round just in 3D. They were all 3D shapes because they were all in the third dimension. C: You couldn't find any 2D shapes there because it is the third dimension and not the second dimension. It is in Spaceland instead of Flatland, and in Flatland there are only 2D shapes but in Spaceland there's only 3D shapes
F66	**R: What is the difference between 2D and 3D shapes?** C: 2D shapes are always flat and 3D shapes are always standing up.
F67	**R: How did the Square vanish from court?** N: Because of Spherious. C: Because Spherious pulled him down himself and then he was back in Spaceworld.
F68	**R: What did you like most about this film?** N: The way how it was flat, because how could you like have a flatland, because you are flat, you are like that (bug's eye view) 2D it is like you are cross-eyed, they don't have any bends or corners you are just a square like that so you see like cross-eyed, because that's how big you see like that (showing cross-eyed with his hand) but when you move your eyes you see this his left and this his right (turning his head)
F69	**R: What did you like most about this film?** C: I liked in the dream, when he went to Spaceland, and I thought it was quite clever the way they made the film that it went flat, and it went from their point of view, I thought it was clever the way they

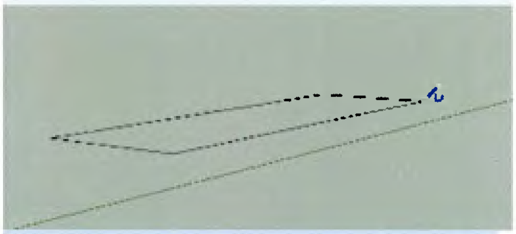
	changed that quite quickly.
F70	**R: Did you learn anything by watching this film? What?** N: what I knew about circle is that it wouldn't go into 3-dimensional and 2-dimensional, but I did know that it would go to 0-dimensional, there's lot of things, lots of things about shapes. C: Like, I don't know how to put this...like how 2D shape goes to 0-dimensional as well as 2D and 3D, it went to 0D as well.
F71	**R: What is a dimension?** N: A dimension could be, it could be if it is 2D it is not just flat, 2D or 3D it is kind of the same thing but it is a different shape, a different type, a different category. Well, dimensions can be different things if it is 2 dimension, 3 dimension or 0 dimension, especially 0 dimension because 0 does not make corners and 2D and 3D do.
F72	**R: You wrote that everything is flat or stood up. What is a point? Is it flat or stood up?** C: Flat because when we saw it in the movie it was flat so on top of a normal shape it would be flat as well. **R: The line is flat or stood up?** C: Flat. N: Yeah Flat. **R: So everything is flat or stood up depending on the dimension?** N & C: Yeah.
F73	**Why was the Square surprised when he left his world?** P: Basically the same as Kai, he is quite shocking because he hadn't seen a 3D shape before or a 3-dimensional world besides from his.

Appendix 2: Transcript of Pair D from Situation IV

Nosakhare (N) and Mya (M)

1. **TASK 1: Building your neighbourhood**
2. R: What is the difference between this neighbourhood (2D) and that one (3D)?
3. M: This is like a plan.
4. N: This is 2D and this is 3D, and this one is like a plan.
5. R: What do you mean like a plan? Floor?
6. N: Yeah floor.
7. R: And what's the difference in 2D and 3D?
8. Both: 2D is something flat
9. N: And 3D you can see the edges, you can hold it, you can feel the corners
10. M: You can feel it.
11. N: If it is 2D you can feel only 3 sides, like a triangle you can only feel three sides, but if it is 3D, like a pyramid you can feel all the sides, like you can feel the bottom and you can have plenty of corners and edges.
12. R: Do you agree Maya?
13. M: Yes.
14. R: You said 2D and 3D. What does the "D" stand for?
15. M: Dimension
16. R: What is dimension?
17. M: I think there is something to do with your eyes.
18. R: Like?
19. M: 3D is when your eyes see all of it, and 2D when your eyes see the flat.
20. N: I think that it means you can only see one side of it while in 3D you can see most of all the sides. When it is 2D it would be the same thing on the same side, but when it is 3D it is different on each side. And 2D would be symmetrical while 3D isn't....or some of them are. If it is 3D and it is flat like in 2D then you will be able to see the reflection...
21. R: I don't understand that. Say it again. If it is a 3D shape...
22. N: And it is flat like that, then if you put the mirror line you can see both sides of it, unless if it is something you can hold completely.
23. R: Can a 3D shape be flat?
24. N: No.
25. M: So basically no...
26. N: It's only 2D shapes.
27. R: You are going to draw your own neighbourhood now. These are the tools you can use. What do you think they do?
28. N: They change...I think some of them can customise it and some of them can turn it.
29. R: What does each tool do?
30. N: One of them, like this one (look around) you can look around like if it in one place and there is another side you can look at, I think that's the one you can look around the whole place.

31. M: I think “line” is to draw lines and “rectangle” you can draw a rectangle and then a circle (showing the circle tool)
32. N: And then “move and copy” that means like turn the thing and move it and then “text tool” you can write.
33. R: The “orbit” tool?
34. M: Isn’t that you go around the place and…
35. N: Yeah that one.
36. R: And the push/pull tool?
37. M: Is like make it 3D, like bring it up
38. N: Yeah when it is flat 2D and then you push it you make it , it will turn 3D.
39. Designing starts [6:50]
40. Mya started by using the rectangle tool.



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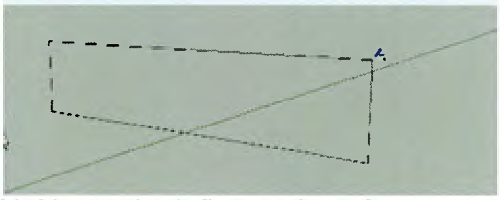
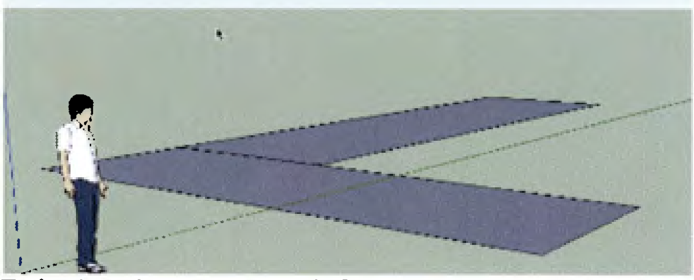
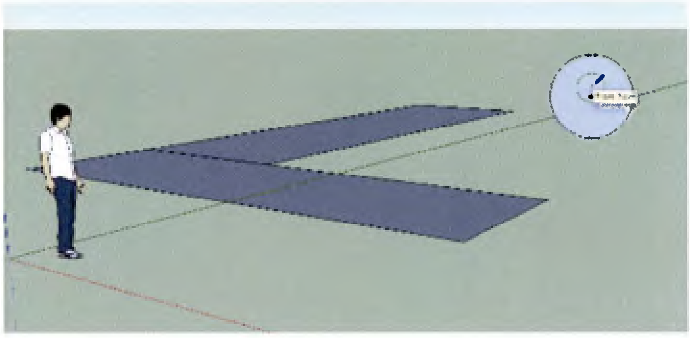
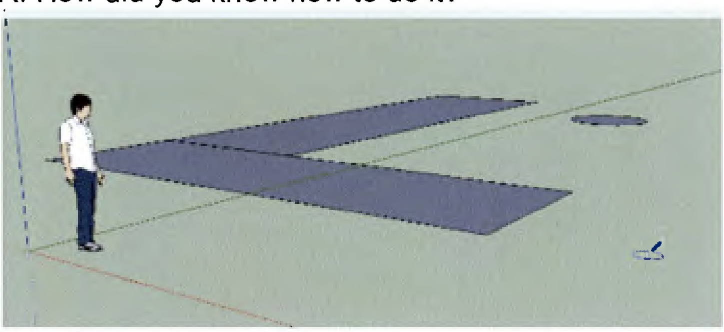
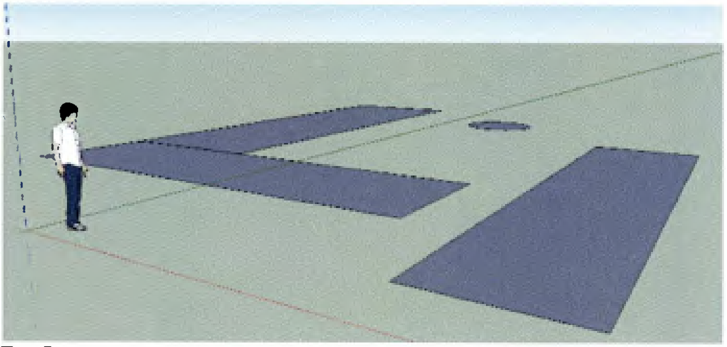
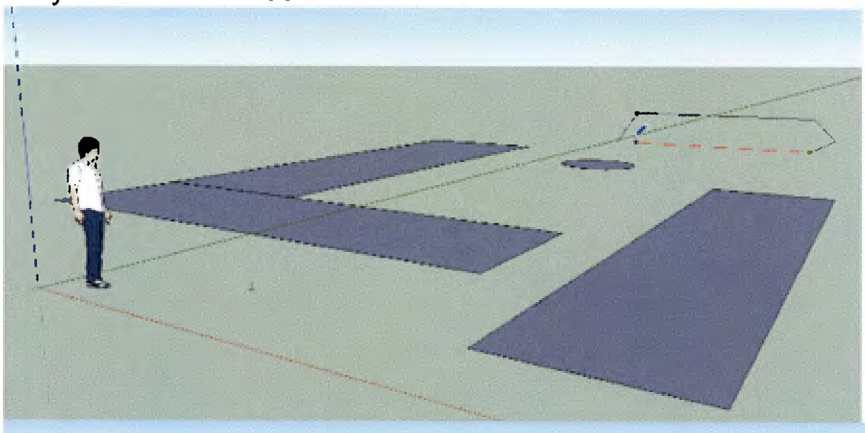
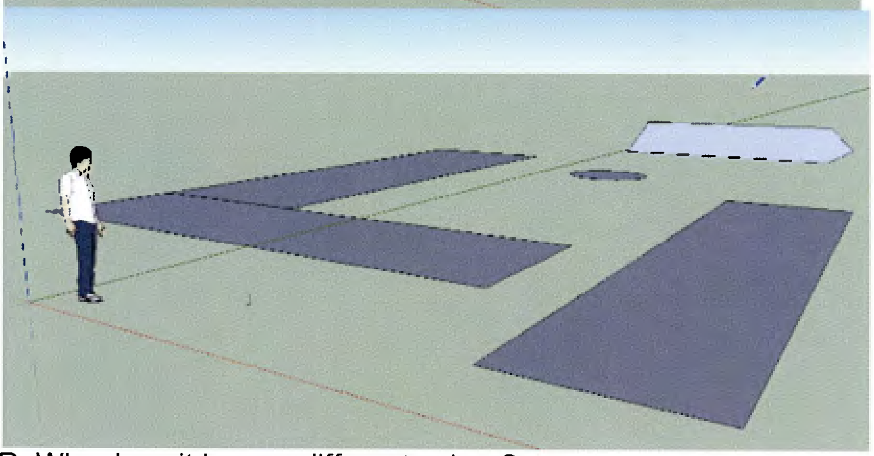


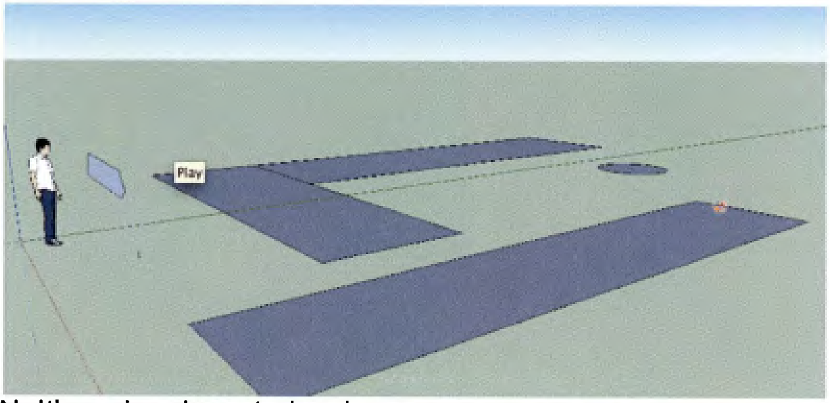
42. N: No, make it flat, make it flat.
43. They made a flat rectangle.
44. Both: That’s a street.
45. They made another one.



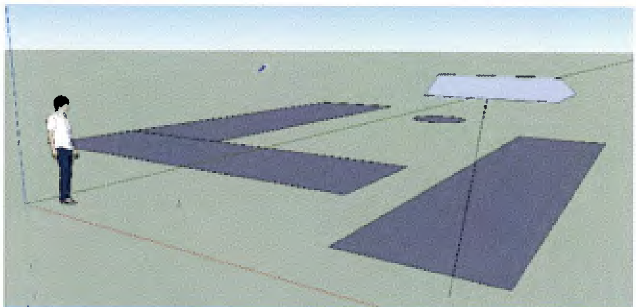
46.
47. R: Is that the street again?
48. N: Yes.
49. M: Let’s do a circle. What should that be?
50. N: A building

51. 
52. N: Ohh I made it wrong.
53. R: Why is it wrong?
54. N: I wanted to make it flat.
55. R: And what is that?
56. M: That is coming towards you.
57. R: Have you noticed the difference in colours?
58. Both: Yeah.
59. R: Why do you think they have different colours?
60. M: That one is down (dark grey) so not a lot light is coming to it, that one is up then there is light.
61. They deleted the circle. They hovered until they found the blue circle and they drew their circle.
62. N: Let's do it here.
63. R: How did you know how to do it?
64. 
65. R: Is it like you want it now?
66. Both: Yeah...
67. R: Why?
68. N: Because when we finish we can pull it up and make it 3D.
69. [10:00]
70. N: Let's make a big supermarket.
71. They used the rectangle tool to create one:

72. 
73. R: Can you use the line tool to create one more shape?
74. They used the line tool.
75. 
76. 
77. R: Why does it have a different colour?
78. M: Because the light is facing.
79. N: Because it is bending.
80. M: It is kind of 3D but it is flat.
81. R: Use orbit to go around.
82. They used orbit to see that what they have created is not in the place they thought.



83.
84. N: It's a sign. I wanted a sign.
85. R: But it is a flying sign.
86. N: I am going to put something there.
87. R: Yeah put something there.
88. They used the line tool to draw a line.
89. M: Yeah right at the edges.
90. They used orbit to see that the line was not in the place they thought it was:



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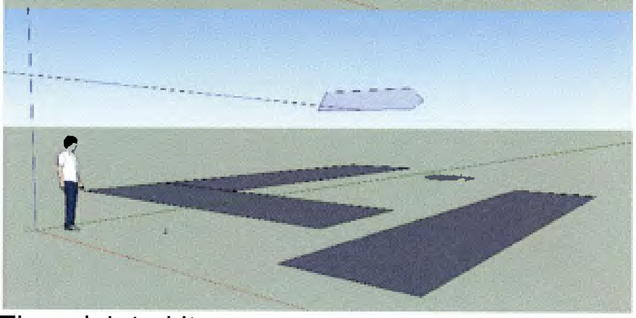
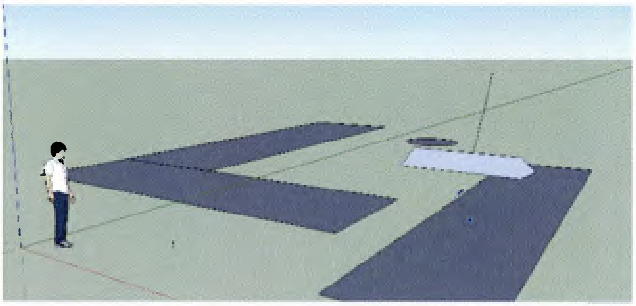
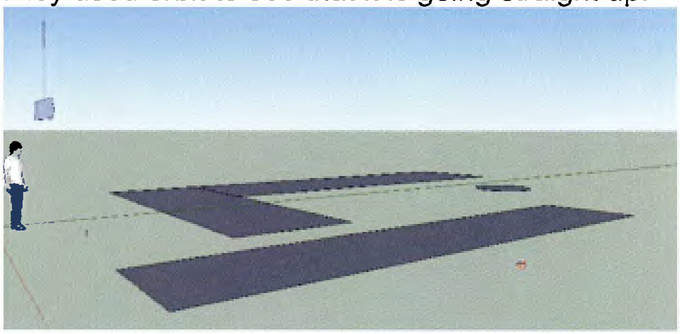
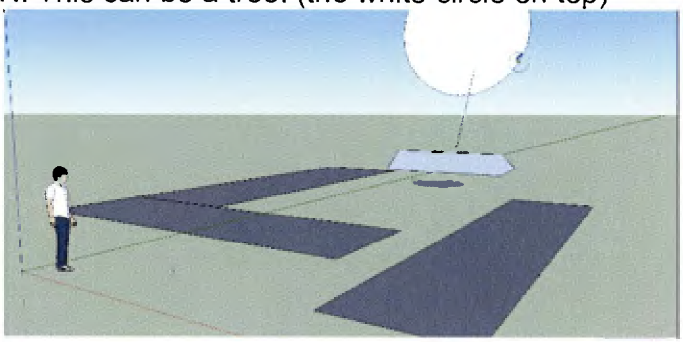
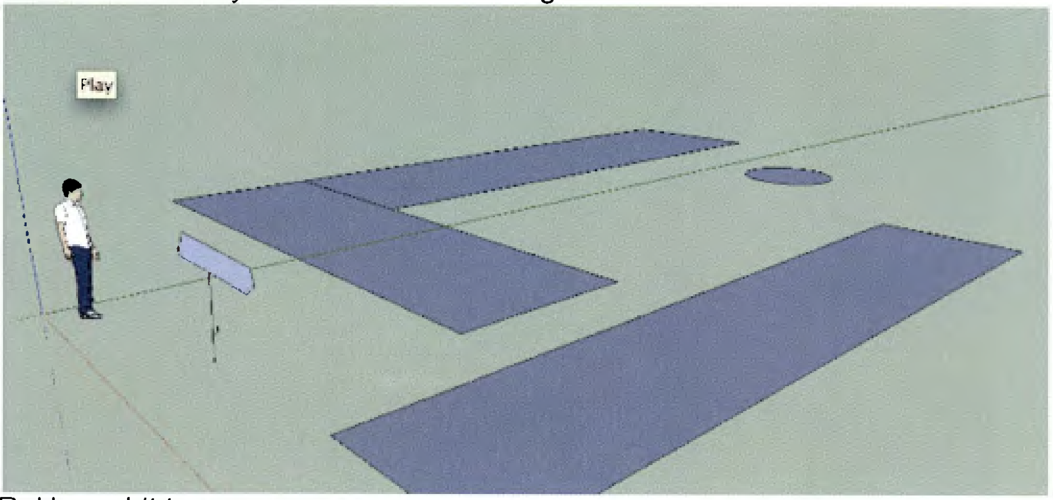


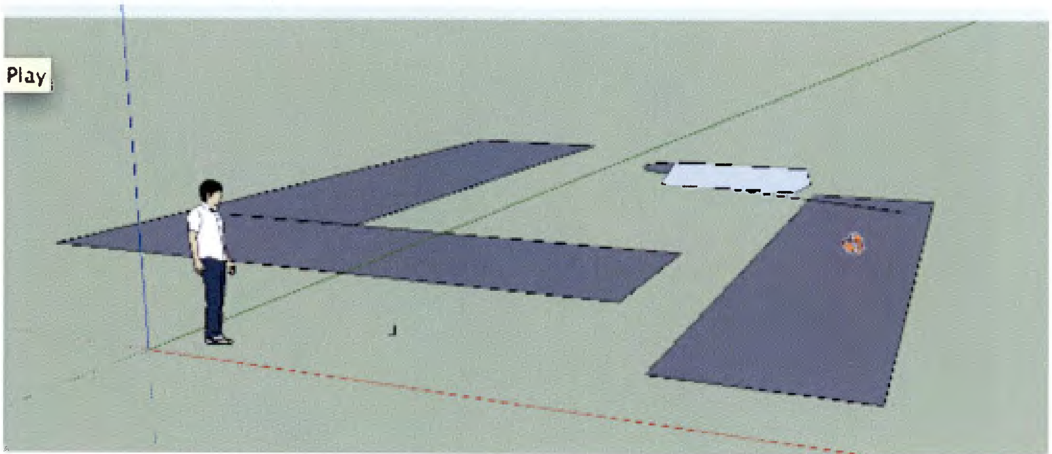
92.
93. They deleted it.
94. N: I think if we do it the other way. Let's see what happens. Let's do it the other way.

95. 
96. They used orbit to see that it is going straight up.
97. 
98. N: That would be good actually, we should leave that.
99. M: We need to put something on the top to hang it with.
100. N: This can be a tree. (the white circle on top)
101. 
102. N: Just delete it.... (they deleted the circle)
103. M: Can you just click on it and move the thing?
104. R: Yes you can do it.
105. They tried it but it went larger...
106. R: Click on the surface not on the sides.
107. They moved it.

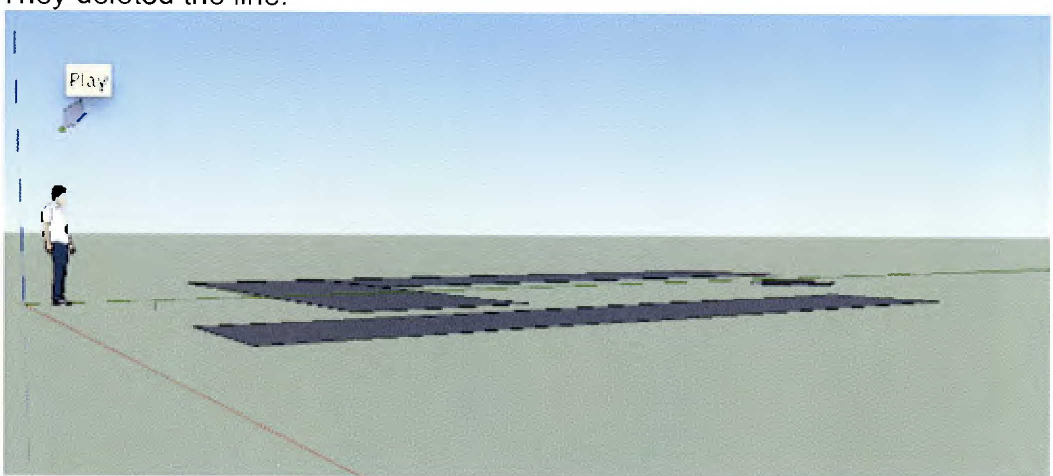
108. They tried to move the line but it was connected on the sign. They deleted it and they drew a new one facing down.



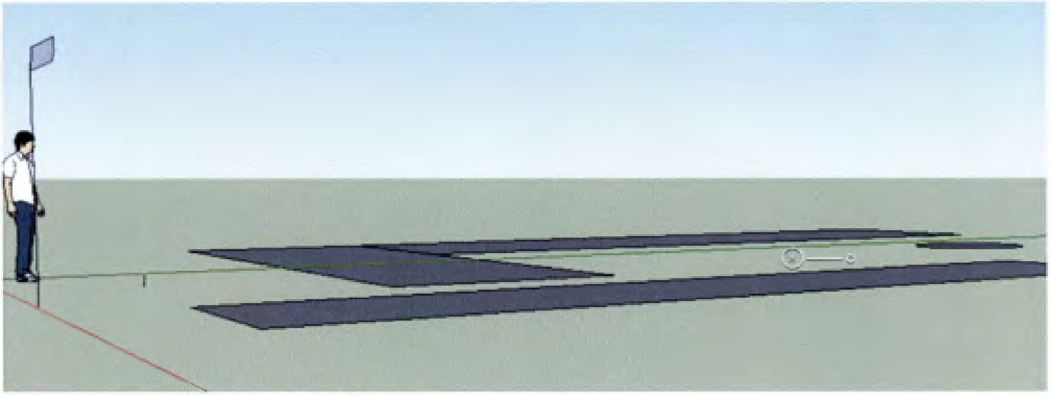
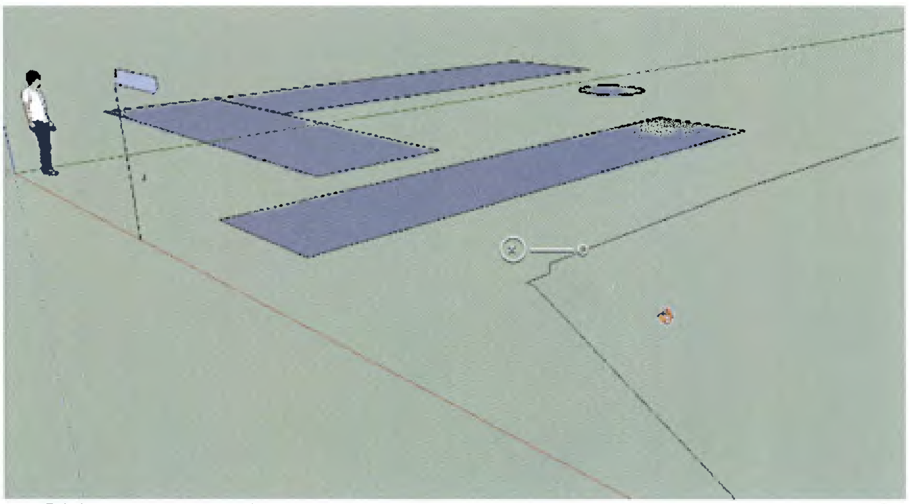
109. R: Use orbit to see…



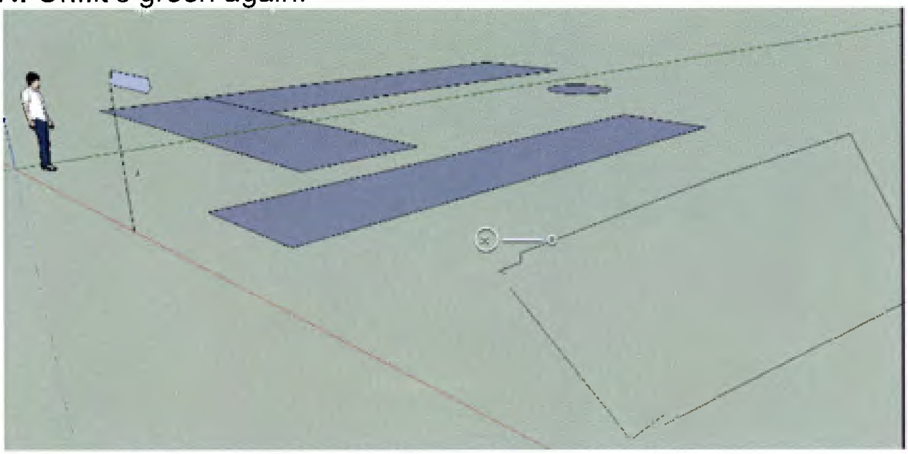
110.
111. M: I think we have to make a longer line.
112. N: Let's turn it this way… (they change the perspective in order to make it right)
113. They deleted the line.



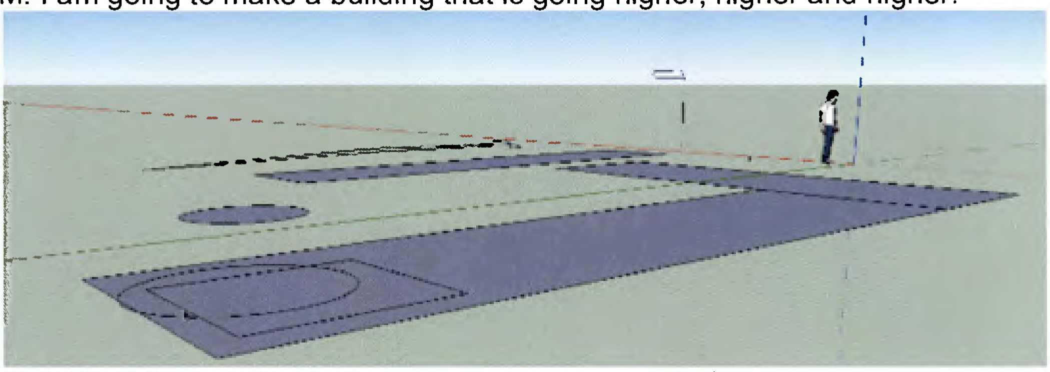
114. N: Now make it down. (the ending of the line was on the red line)

115. 
116. They used orbit to see that it was correct.
117. Both: Yes!
118. R: Now I want you to make one more building by using lines. Because that was actually your building but it turned out to be a sign.
119. M: I think we should build something on the street. Like a park.
120. They started drawing lines. Some of the lines were green like drawing.
121. N: What is happening? Why are they green?
122. R: Yeah, why are they green?
123. N: I don't know.
124. They continue drawing more lines.
125. R: What colour is it now?
126. M: Blue.
127. Both: Red.
128. N: Ohh I know because it is in different directions. Ohh it's black again.
129. M: Like when it goes north, south, east, west, when it goes to these directions it has a colour, when it is going diagonal it is black.
130. R: What are the directions?
131. M: North, south, east and west.
132. At the same time, Nosakhare continued drawing and he drew a line underneath:
133. 
134. N: Ohh my God! What did I do?
135. M: If we had a square (tool) it would have been easier.

136. N: Please be straight this time.
137. N: Oh..it's green again.

138. 

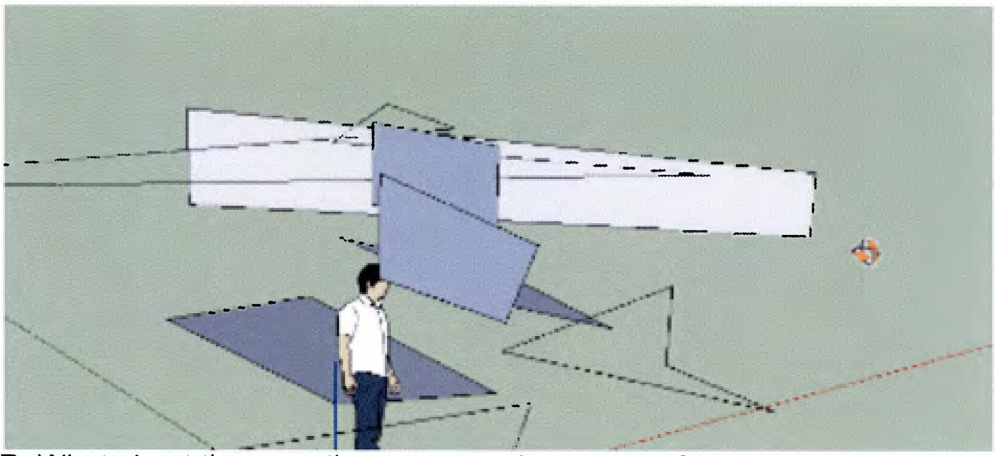
139. R: It is not coloured. You see the other ones are coloured.
140. M: Because it is line.
141. N: When we used a rectangle and a circle it was coloured because it was an object, because with that we just put it there and we make the shape while here we draw it. That's why.
142. [End of video part I]
143. [Start of video part II]
144. R: OK use orbit to see how is the shape you made.
145. N: It looks good actually…
146. R: But why it doesn't have a colour?
147. N: Because we used the rectangle for this and that's why it got colour but we drew that, that's why it doesn't have a colour.
148. In the meantime, Mya is drawing some more shapes on the street.
149. M: I am going to make a building that is going higher, higher and higher.

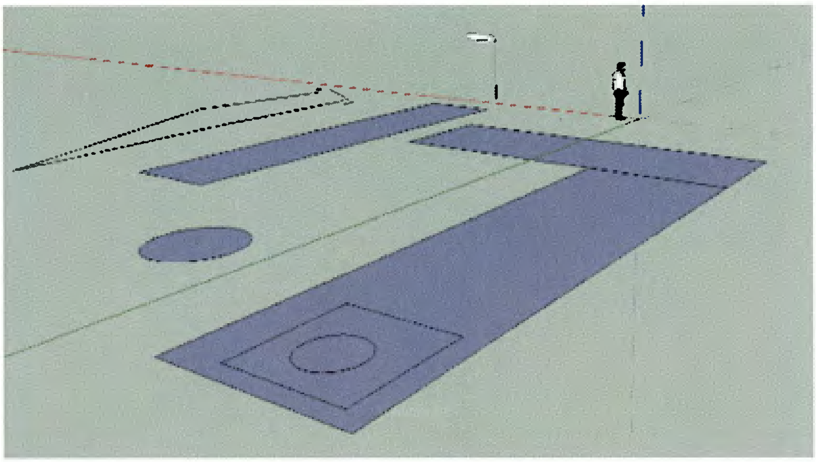


150. She deleted the circle on the street because she didn't like it.

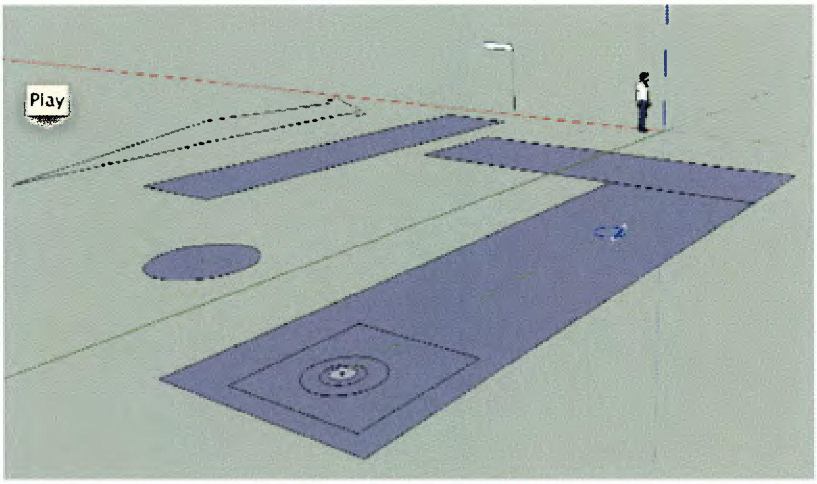
151. NEW TASK: Incomplete shapes
152. R: What do you see here?
153. N: Shapes, lines
154. M: Non-connected rectangles.
155. N: Some…shapes….not complete shapes.
156. R: Complete them then.
157. They completed them.
158. R: What do you see?

159. M: Lots of rectangles but some of them are coloured and some of them are not.
160. N: I think that the reason why they didn't…I think the ones they had the colour in, I think they used the rectangle one (rectangle tool) and then undid a part, a little bit of the shape. And then the ones that aren't coloured, they drew those ones (line tool) but they didn't complete them.
161. R: What do you think Mya?
162. M: I think that the ones that are really dark are at the bottom, and then it goes up and up and right at the top there's no colours.
163. N: Because of the light.
164. M: Yeah.
165. N: Because this is really dark and it goes lighter, this is higher than that, this is the same level as that, and that is lower than that, that's why is darker.
166. M: Because with orbit if we go all the way underneath these will be on the top and if we go all the way that then these would be at the top.
167. R: So you think that the ones that do not have colour at all are the ones which are on the top?
168. N: Yes we think, let's try.
169. They used orbit to be surprised.
170. R: What do you see now? What about the ones that do not have colours?
171. M: They are twisting…and turning
172. R: The ones that are coloured?
173. N: They are just great!
174. M: They don't get longer
175. N: That one is.
176. M: Yeah that one is because it was short before. (a rectangle that looked "shorter" before orbit)
177. R: So what do you think about the ones that are coloured and the ones that are not?
178. M: I think it is what Nosakhare said, that it used to be like rectangles and then cut off and when we did the last line it turned into a proper rectangle.
179. R: Anything else?
180. M: It has something to do those lines as well I think.
181. R: What does it has to do with the lines?
182. M: I think it has to do when it is in a different….more that way or to the right it changes which way the shape turns…
183. R: Do the coloured ones change?
184. N: No, they don't change…sometimes they go long or short, but if you change it to a different angle you can see that that's white and this is black but if you change it it's white and that is black. It is changing the angle of it, that's why, black again…I think it is because of the light and how much light there is.
185. M: When it changes it shows the different levels of light.
186. N: Because the light is coming from this way, that's why it is light and that one is dark.

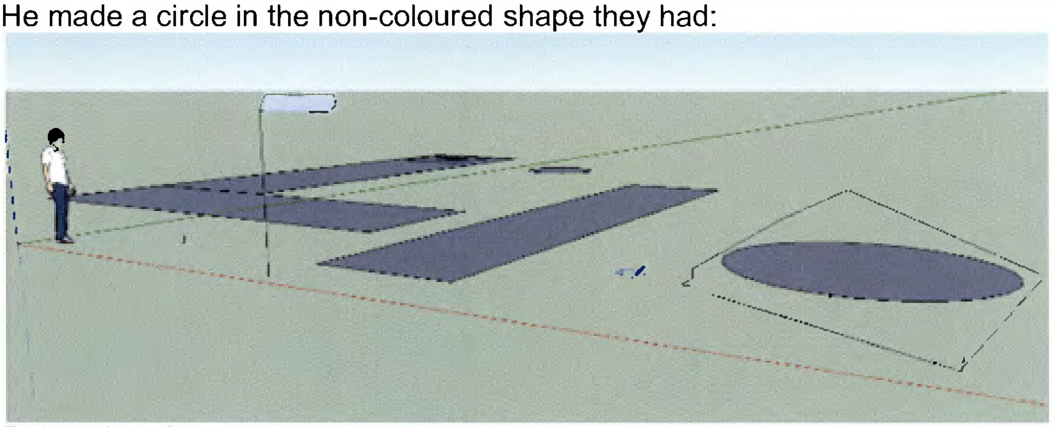
187. 
188. R: What about the ones that are not coloured at all?
189. N: I think those ones can change the natural shape, because look...
190. M: They are in the middle
191. R: What do you mean "they are in the middle"?
192. M: Like this is dark and this is light (top-bottom), and they are in the middle and in the middle there is no colour.
193. R: Nosakhare do you think?
194. N: Something the same as Mya...like this is something the drew (line tool). If they did the shape like this and they drew half of it like they did before then ...I am not sure how to explain this...maybe because it is in the middle...because you can see the light this is light and this is dark...the ones they drew they don't have any colour in them because they drew it, it is not like they put it down like all these ones...they just drew it how they wanted to draw. If you use orbit you can see that they change shapes, I think but I am not sure that it has to do with the sun as well that makes it change the shape I think.
195. [10:08]
196. **BACK TO THE NEIGHBOURHOOD TASK**
197. R: Did you finish or do you want to do something else?
198. M: We want to finish this (the new building they started on the road)
199. They created a circle into the square on the road, and they used the "move tool" to move it to the centre of the square.



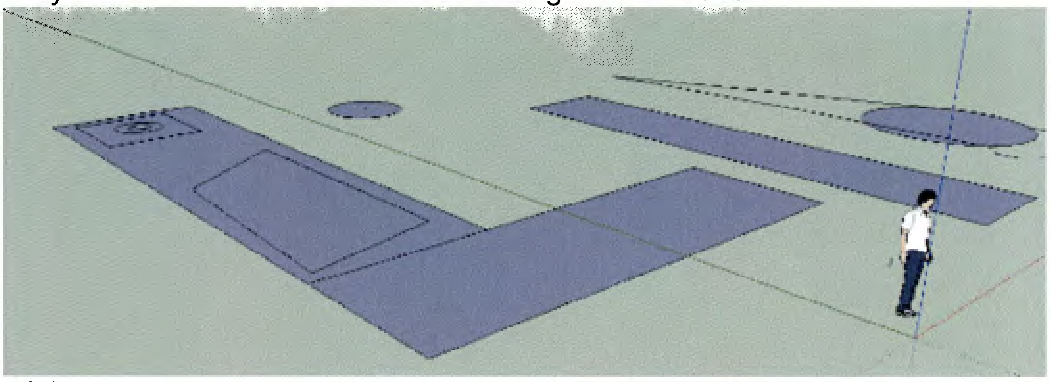
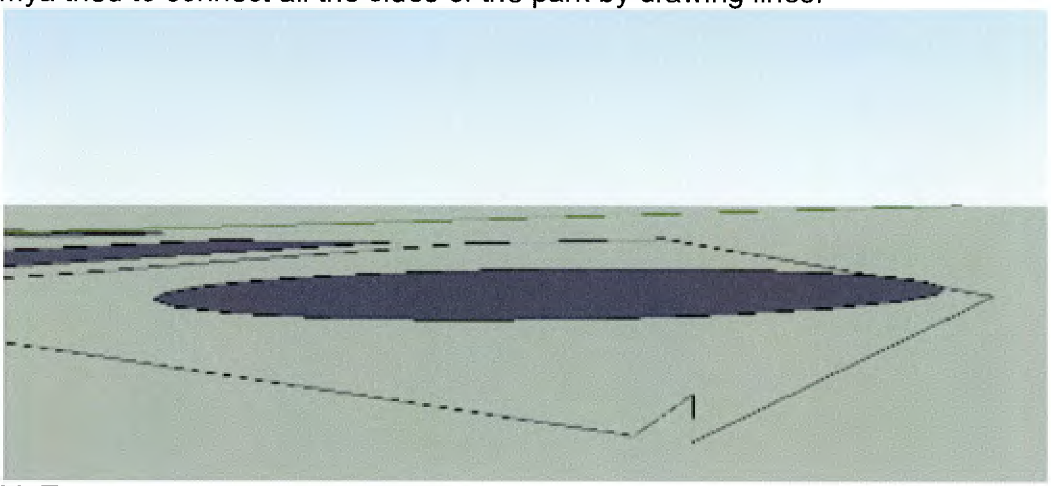
200.
201. They drew two more circles in the middle of the other circle. The second circle drawn had a white colour.

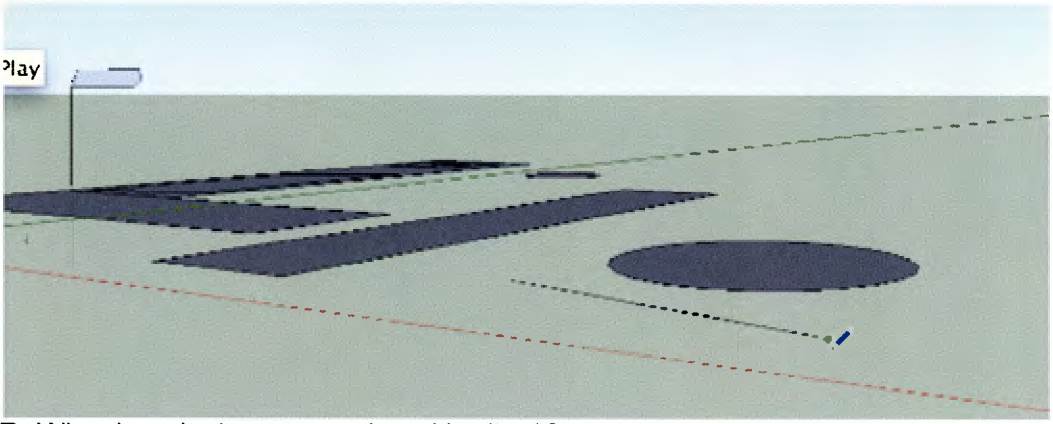


202.
203. R: Why do you think it has a white colour?
204. N: That's odd.
205. M: Ohh…maybe it has to do with the 3D, that is 3D but it is 2D maybe.
206. R: OK do you want to do something else?
207. N: Yeah a pond.
208. He made a circle in the non-coloured shape they had:

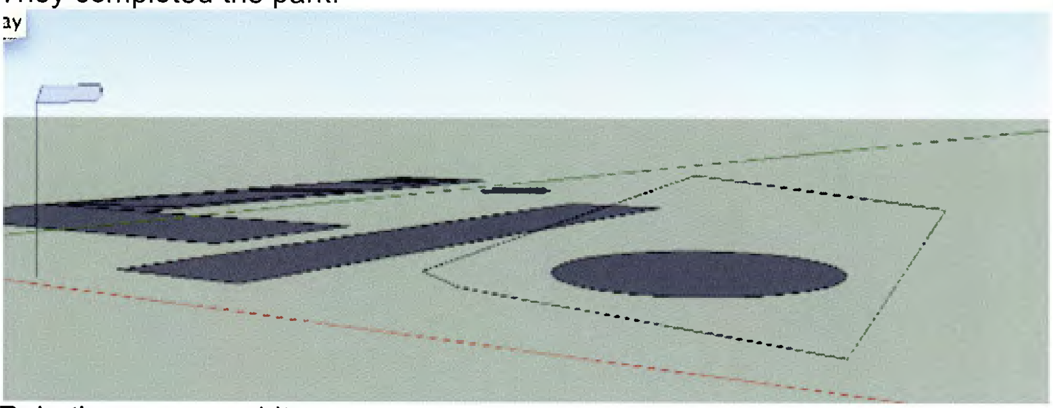


209.
210. R: Is it done?
211. Both: Yes.

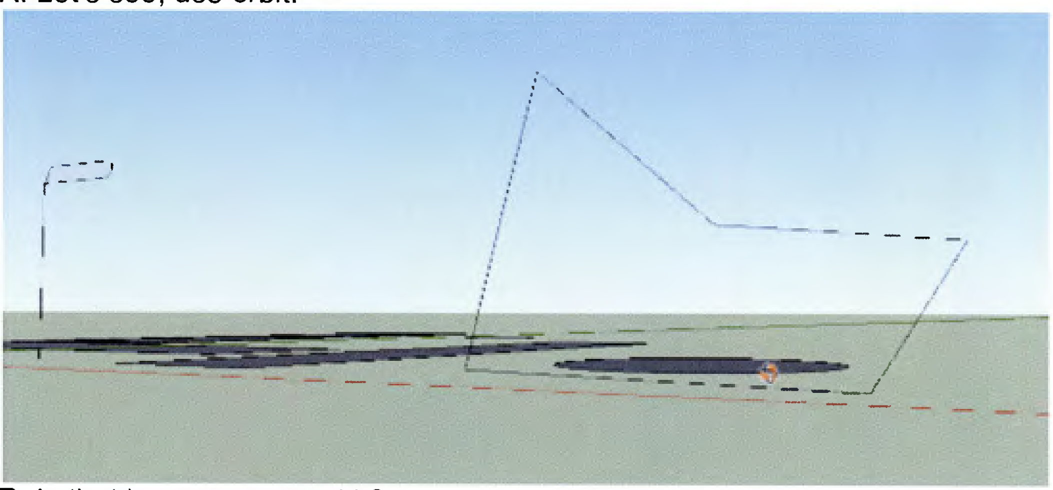
212. R: Make one more shape with the line tool.
213. They used the line tool to create a rectangle on the road.
214. 
215. While using orbit, I noticed the strange look of the park.
216. R: Oh look what is going on there to the park…
217. M: I don't know…
218. N: It's got funny…
219. M: I think it is because of the different angle so when it moved it looks…
220. R: Yeah but why the other shapes are not moving and just the park?
221. N: Because I drew it.
222. M: Yeah we drew it…
223. N: Like the other ones we saw (rectangles-orbit tasks), it changed when we moved it, so that's why.
224. Mya tried to connect all the sides of the park by drawing lines.
225. 
226. M: To make it connected…
227. R: Let's go around the park again to see how it looks. Is it good?
228. Both: No.
229. R: Do you want to try it again?
230. N: Yes, only the park. Undo the park. I want to keep the circle though.
231. They deleted the lines.
232. R: Now use the line tool.
233. M: If you go on top it might be easier (to draw it from the top view)



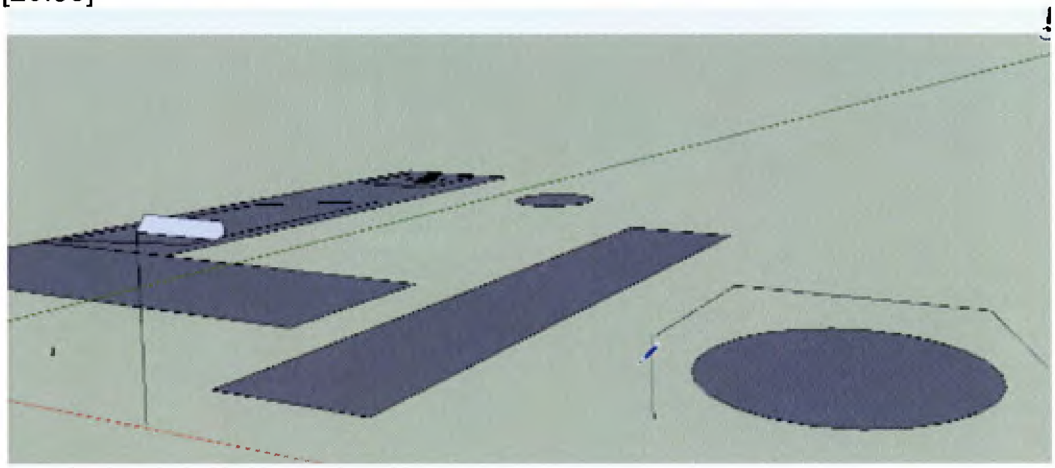
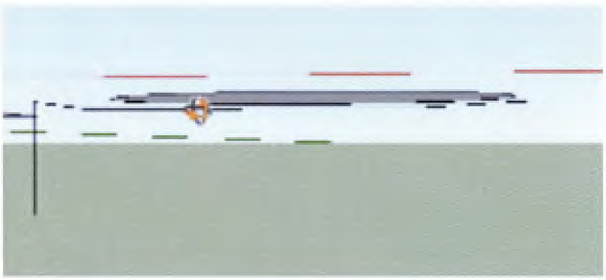
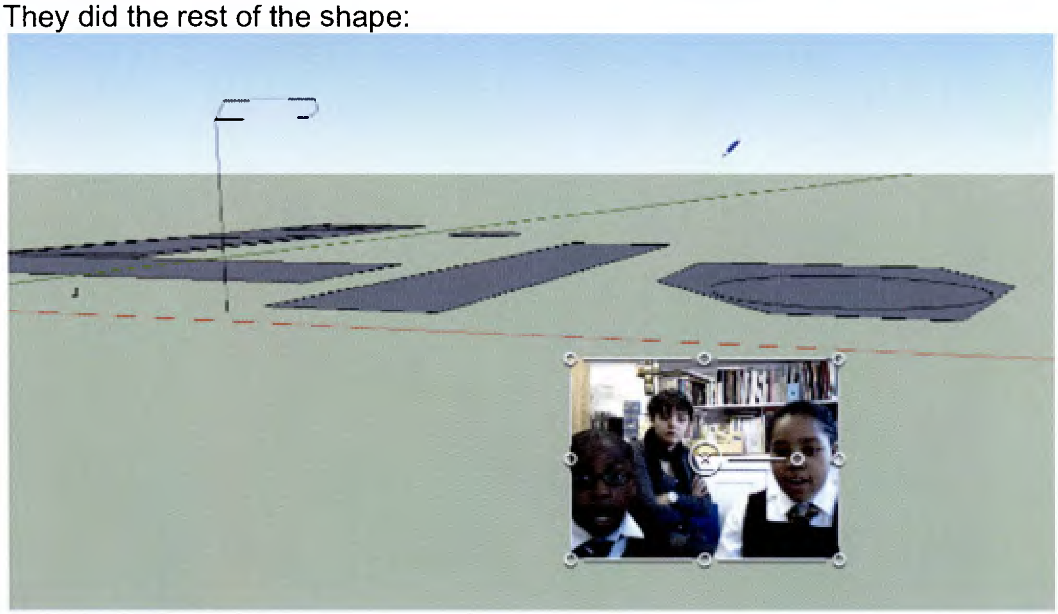
234.
235. R: Why does it change a colour (the line)?
236. M: I think it has to do with these lines when it is going to the direction of those lines.
237. R: Which lines?
238. M: These lines
239. R: The red?
240. M: The red and the green.
241. They completed the park:



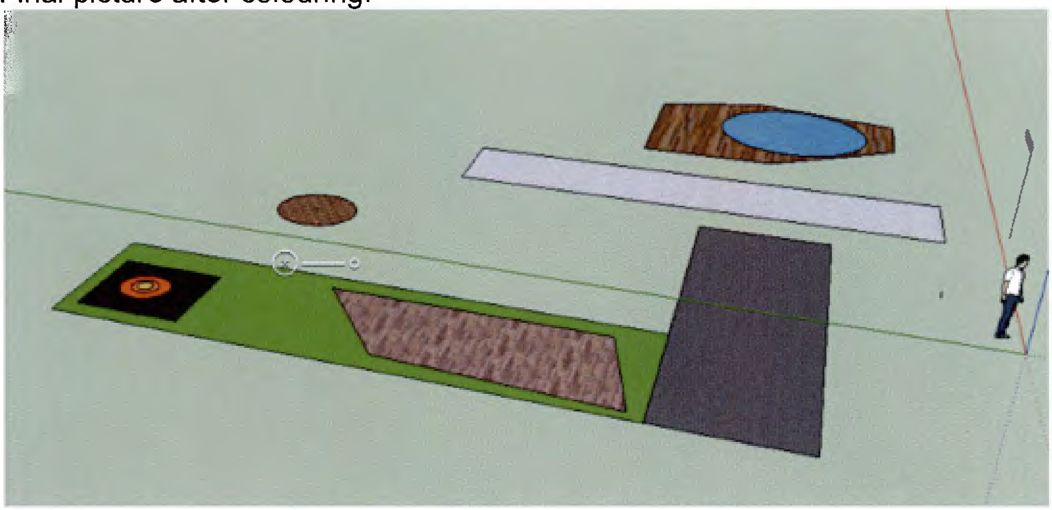
242.
243. R: Let's see, use orbit.



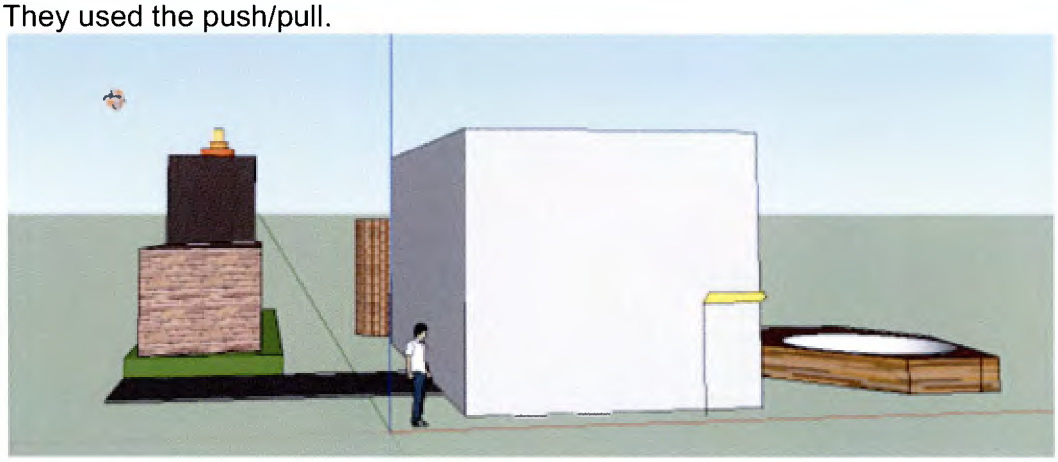
244.
245. R: Is that how you wanted it?
246. Both: No.

247. N: Let's change it.
248. They deleted all the lines.
249. M: Oh I have an idea. I think that if you do an odd square or rectangle like that it might work.
250. She created a shape around the circle.
251. [20:06]
252. 
253. She made a green line, then a blue, then a red, and then a black.
254. M: Wait, wait this is down. You have to delete that line. Look (she uses orbit to show him that the blue line was "down")
255. 
256. M: Look that's down. You have to do lines like that, diagonal lines.
257. They did the rest of the shape:
258. 

259. [21:26]
260. R: It got a colour.
261. M: It has to be that diagonal lines that work on this, maybe not lines that go down, that go straight.
262. R: But isn't that straight?
263. N: Oh...yeah...maybe it is only straight and no diagonal lines then.
264. R: Why do you think it got a colour?
265. N: I think they turned up the sun.
266. R: It is in the same place as before...
267. N: I think I've got it. I think there's parallel, I think we've got some lines that are the same because look that's the same and that's the same (showing the various pairs of parallel lines on the shape). Even though it is not of the same length, it is of the same type of shape that can make the same length...I am not sure about the sun...actually we can forget about the sun.
268. Colouring starts.
269. Final picture after colouring:

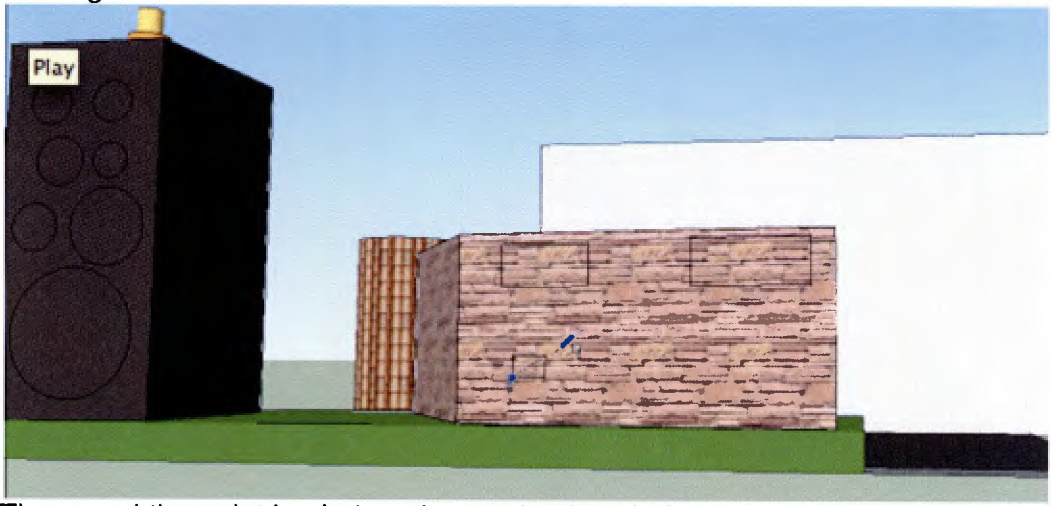


270.
271. [29:38]
272. R: Now you can use the push/pull.
273. They used the push/pull.

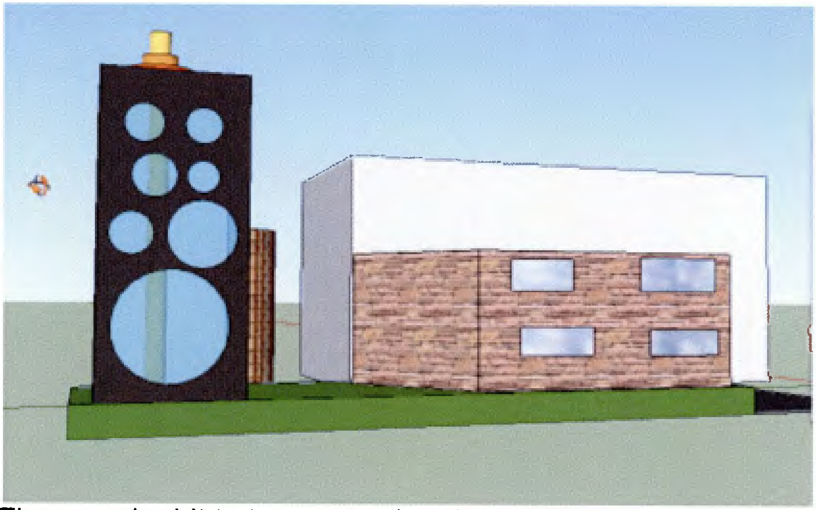


274.
275. R: Do you want to put any details in it?
276. N: Yes, I want to put windows.

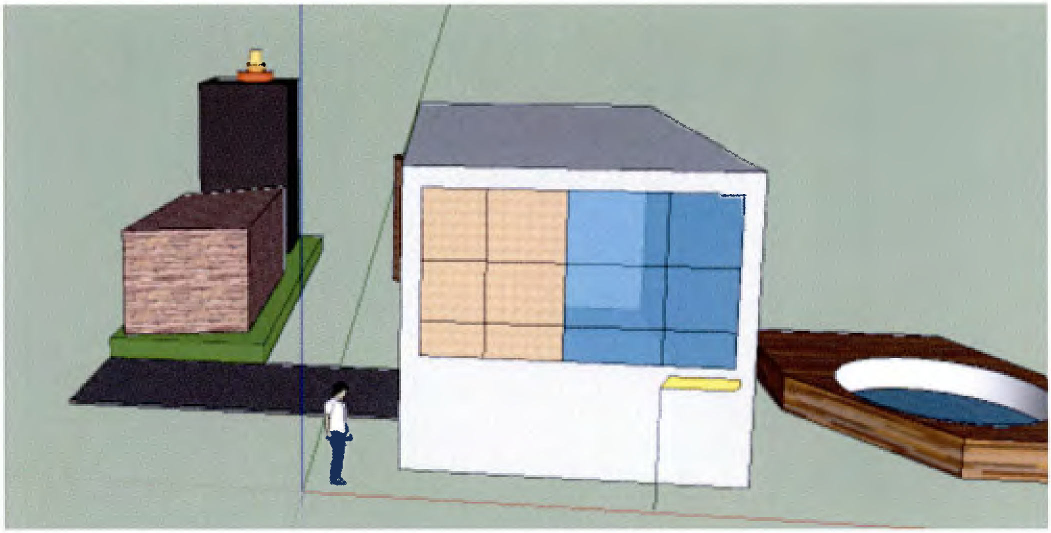
277. They used the circle tool to create some circles-windows on the brown building.

278. 

279. They used the paint bucket again to paint the windows.
280. M: Ohh it's see through
281. N: That's a lift!

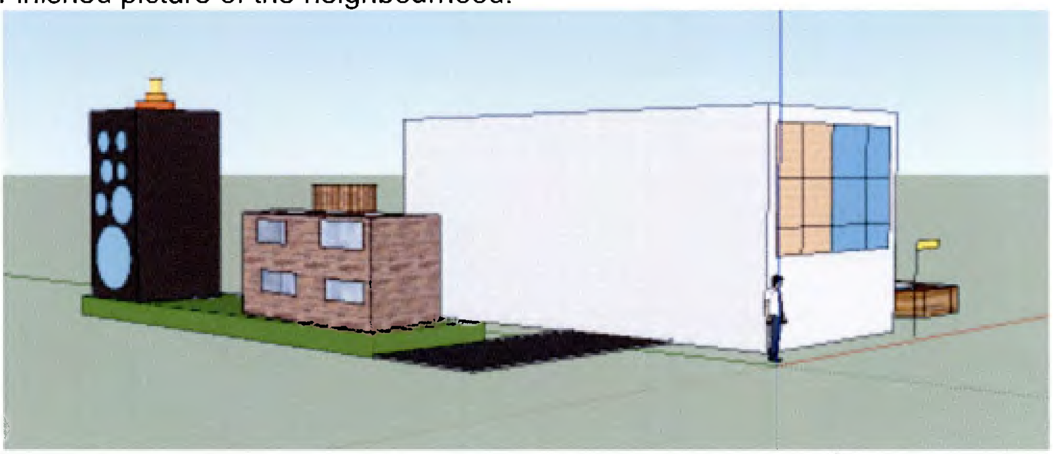
282. 

283. They used orbit to turn around and put windows on other buildings as well.
284. [40:00]



285.

286. Finished picture of the neighbourhood:



287.

288. R: What's the difference between this (their 2D neighbourhood) and this (their 3D neighbourhood)?

289. M: You have added colour

290. Both: You've made it like 3D.

291. R: And what's the difference between 3D and the previous one?

292. M: Because 3D you can actually see like the edges and everything inside

293. N: Like look at the top, if you are standing right in front of it you can see the top. If you see a 2D shape it is flat, you can only see the flat top, there not like a big…it is like that you can only see that bit but if you raise it like that (make it 3D) you can actually see a big piece compared to that.

294. M: Like this table is like that (showing the table) if you look here you see just that (the surface) but if you look underneath you see the whole thing.

295. R: So what is that table?

296. Both: 3D.

297. R: And what can be 2D in that table?

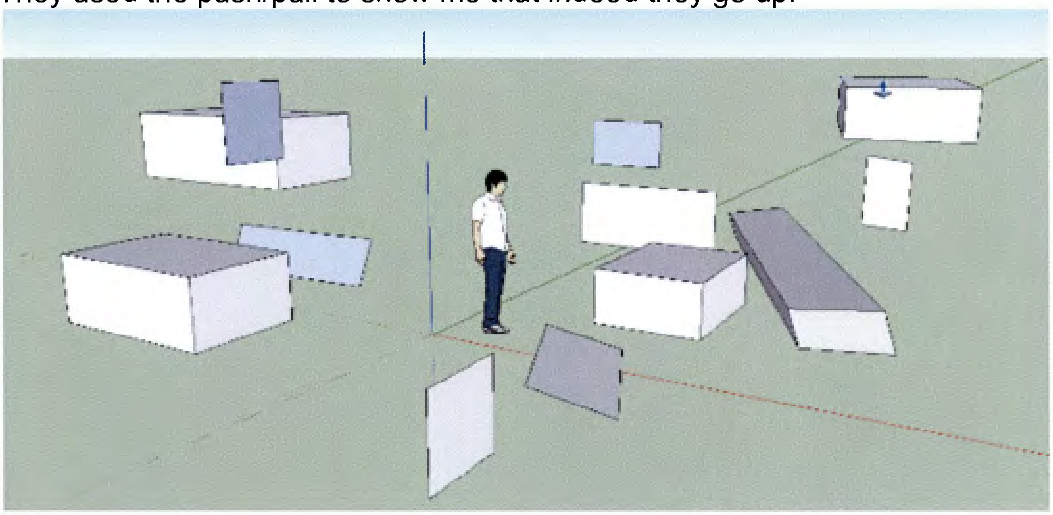
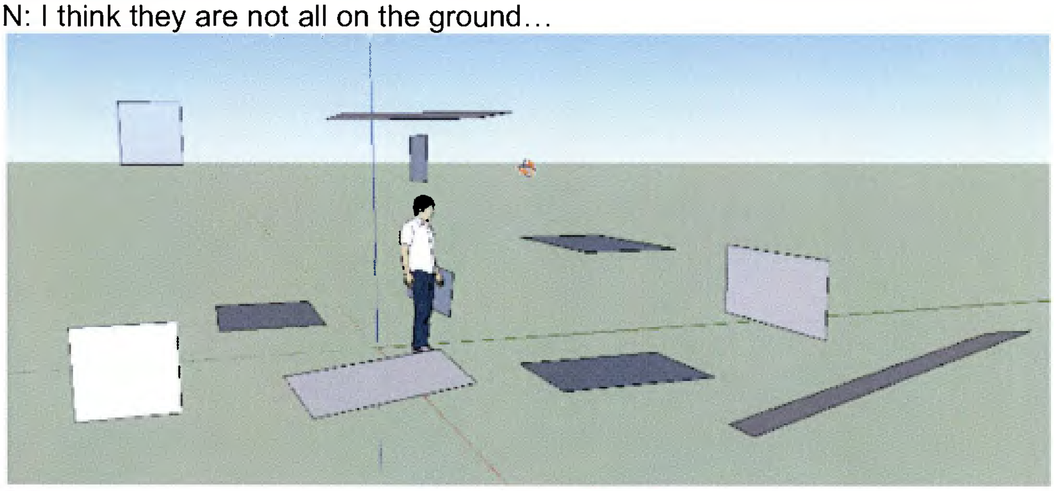
298. M: That (showing the surface)

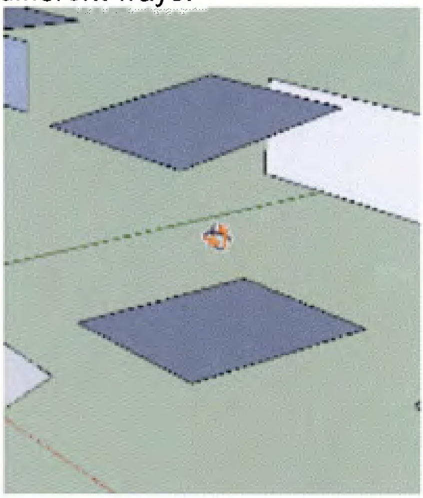
299. R: Oh the surface?
300. Both: Yeah the surface.
301. N: The surface is always 2D
302. M: But everything else is 3D.
303. R: What shapes exist in this world (2D)?
304. N: Circles, rectangle and drawing.
305. R: What do all these shapes have in common?
306. M: Lines
307. N: They are 2D, they are flat.
308. R: And what shapes exist in this world (3D)?
309. N: I would say cylinders, cuboids
310. M: Sphere, cube
311. R: Do they have any shapes in common with the previous one? (the 2D)
312. Both: No.
313. N: Except from that one
314. M: Yeah the flat shapes.
315. N: If you look on top it would have all of it in common but when you look at all around ....
316. R: You mean if I look at it from the top (I use orbit to show the top view) it will have everything in common?
317. N: Yeah look at it...
318. They put the top view to both the 2D and the 3D neighbourhoods.
319. M: That one got more colour in it
320. R: Yes but if this was also with colours, would it look the same?
321. N: Yeah, it would might.
322. R: What tips do you have to give to any new students about this task?
323. N: Look, focus on it, look straight what they see, look at differences and think, they have to think before they draw because they can't draw something that they do not know what there is.
324. M: Take a notice of all the lines that you draw, and after you draw lines, look through orbit and see how you drew it.
325. R: Imagine this man walking in this neighbourhood (2D). What does he see?
326. N: Everything flat, boring. Just pavement.
327. M: He can step on everything.
328. R: So is everything like a pavement?
329. M: Yes flat.
330. N: Surfaced.
331. M: Everything is on the same height and then when it comes to this...(3D).
332. R: What if a bug walks in this neighbourhood. What does it see?
333. M: He would probably think it's fun because he could walk anywhere he wanted
334. N: He would walk on anything he wanted to.
335. M: He wouldn't have to climb, do any hard work
336. N: Just walk on everything. But the bad thing about it is that you can't learn, only sleep, you can't go to work.
337. R: What's the difference between a bug and a human? Will they see the same?

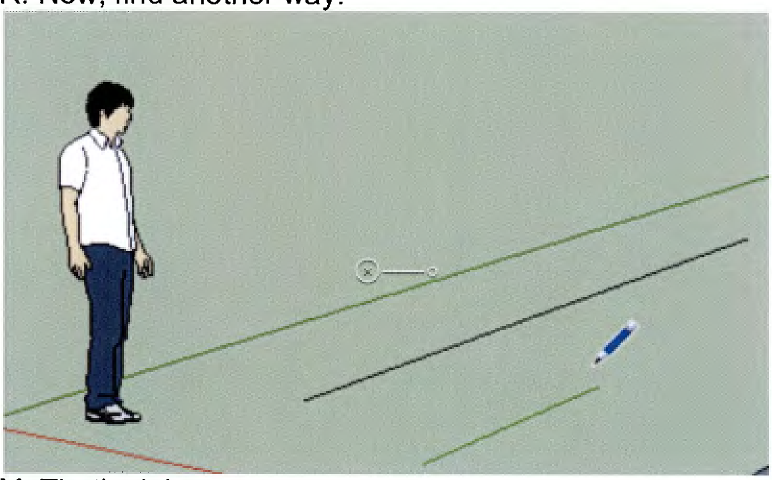
338. N: No.
339. M: Well they see the same things as us but when they see things are bigger and things to us are small.
340. N: Like if I am standing here, I think this is a small world and then if there was a bug, it would say that this is a big world for us that we can live in because the humans they don't work out with these flat things.
341. R: What if a man walks in this world? (3D)
342. M: He would think it's really fun, there's lots of stuff to do, you can work, shopping, live…
343. N: You can live your life.
344. [50:00]
345. R: What's the difference between this world and that one? (2D\3D)
346. N: This world will be way too fun.
347. M: More likely to stay alive.
348. N: And the bug would like this world (2D) and not this world (3D) because if there are people they would be afraid of them, and they would be like things they don't enjoy and they would run away, run back to where they actually live.
349. R: How would a bug see?
350. M: He wouldn't be able to see really because everything is very high
351. N: High, big, you can fly on top of them.
352. M: And he has to be using his wings all day, and get tired and tired.
353. R: Which part of your neighbourhood was the most difficult to do?
354. N: The pond.
355. R: The pond?
356. N: Yes, definitely.
357. R: Why?
358. M: Because of the lines.
359. N: Of the lines because we kept doing it, and it kept shaking and shaking and shaking?
360. R: Shaking? What do you mean?
361. N: Not shaking, like going down.
362. M: Going different directions
363. N: You wanted it to go straight, we were doing it straight and when we turned it all the way around it just blurring against. Look at that (showing how the pond looks now) now is perfect. I think we got symmetry line you know.
364. [51:58]

365. **NEW TASK: Look around vs Orbit**
366. R: You have seen your neighbourhood with orbit…
367. Both: Yeah.
368. R: Now I want you to look at it with the look around tool and tell me what is the difference between the look around and orbit.
369. N: You can only see one side of it.
370. R: Do you agree Mya?

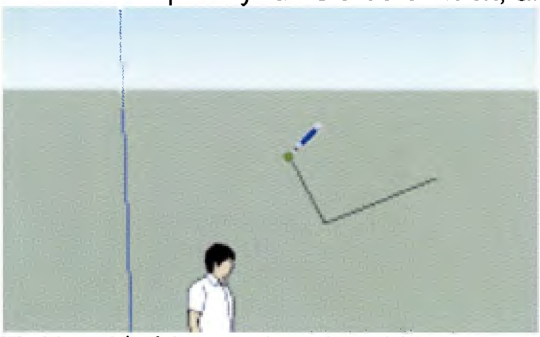
371. M: Yes and you can zoom in and out. (they zoom in and out)

372. NEW TASK: Rectangles – orbit
373. R: What do you see here?
374. M: Different colours of rectangles
375. N: Different colours of different rectangles
376. M: In different angles and places and stuff.
377. R: And why do you think they have different colours.
378. M: Because of the different ways that they have been put like that way and that way and that way (showing different ways with her hands)
379. R: Which do you think they will go up with the push/pull?
380. Both: This one, and that one and that one (showing the dark grey rectangles)
381. They used the push/pull to show me that indeed they go up.
382. 
383. R: How do you know which ones they are?
384. M: Because they are flat on the ground, they are flat.
385. N: The other ones are not flat.
386. After that I did undo so that they will be back to their initial form before the push/pull.
387. R: You said that they are all on the ground....
388. N: I think they are not all on the ground...
389. 

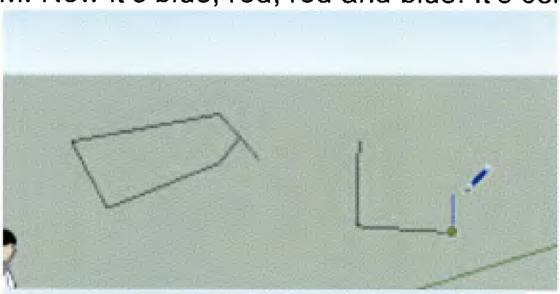
390. R: Is this on the ground? (the dark grey rectangle on the top)
391. Both: No.
392. R: But it is going up…
393. M: It is because they are going up. In different angles they go in different ways.
394. 
395. R: What is similar between this and this? (the two rectangles above) They both go up…But this is not on the floor (the top one)
396. N: Oh I got it. I has to be flat, it doesn't matter if it is on the ground or not, it just has to be flat.
397. R: What about this? (a light grey vertical rectangle shown above) This is not flat? It looks flat to me…(I used orbit to show them that it is flat)
398. N: It is flat but it is like straight like that (showing a vertical direction with his hands) Instead of being this way (showing vertical movement with a paper) it has to be that way (showing the "flat" position of a paper)

399. TASK 2: Carefully structured problems
A. Create a rectangle.
400. R: Create a rectangle.
401. They used the rectangle tool to create a dark grey rectangle.
402. R: Now, find another way.
403. 
404. M: That's right.

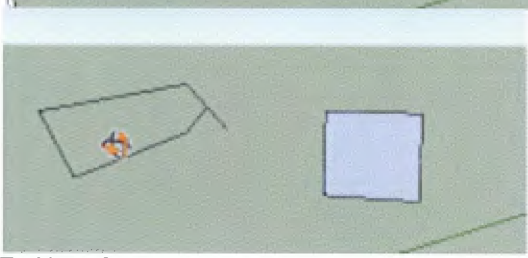
405. R: How do you know that is right?
406. M: Because of the green because you know that the green goes that way. If you turn it to a different dimension…wait not dimension…if you turn it to a different way with orbit then that would be a real proper good blue and not just dots….
407. Their shape created was coloured.
408. R: Yeah but wait…before, you said that when you used lines (the line tool) is not coloured.
409. M: Yeah..
410. R: And now you used the lines and it is coloured.
411. M: It's because the green and the blue, if you do it by using the green and the blue then…
412. N: The red, the green and the blue.
413. R: Try it.
414. M: For example if you do that is black, and black



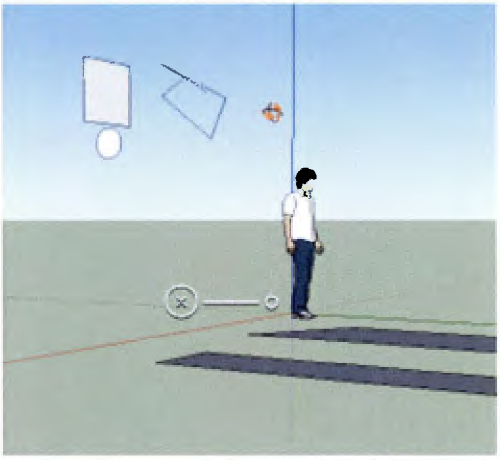
415.
416. M: Now it's blue, red, red and blue. It's coloured.

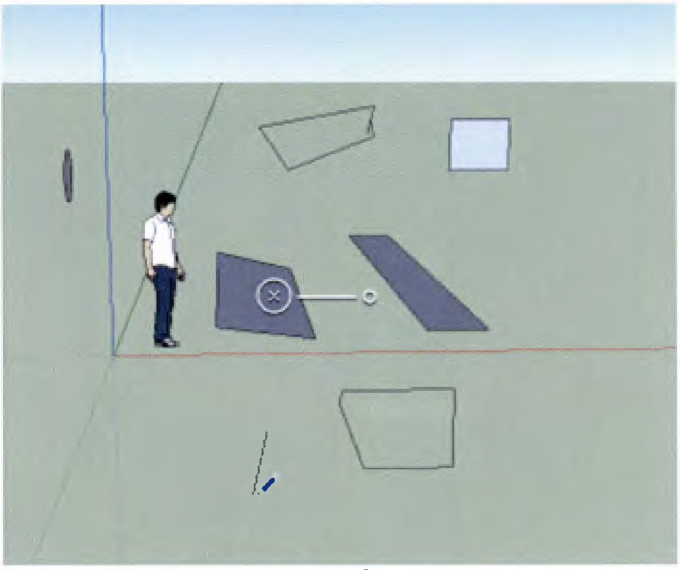


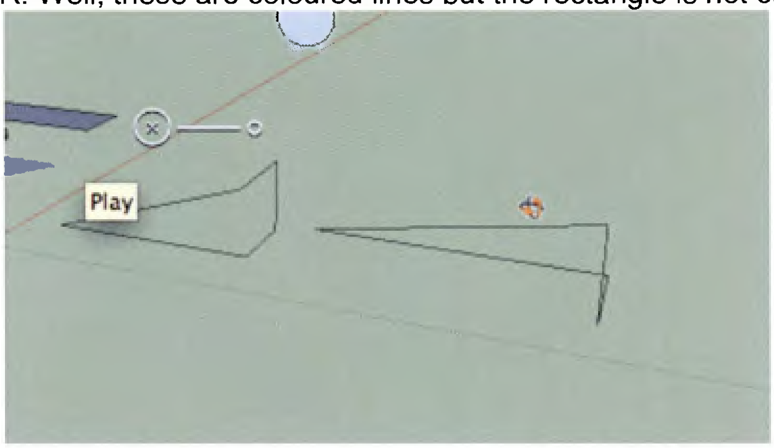
417.



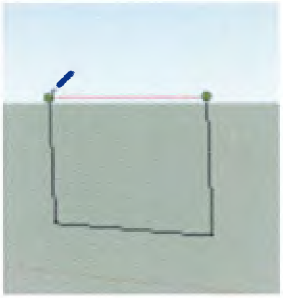
418.
419. R: Why?
420. N: Because if it is coloured it would make a coloured shape but when…
421. R: Use orbit to see the shapes.

422. 
423. M: It's diagonal! (the non-coloured shape). The axes!
424. R: What about the axes?
425. R: Is that a rectangle as you thought?
426. N: No.
427. R: Why?
428. [1:00:00]
429. M: Because it looks a rectangle on the ground but it is not a real rectangle, it is in the air. But it's because you got it turned it looks like that but when you have it flat then it would be on the ground.
430. R: Yes but why it doesn't have a colour?
431. N: Because it wasn't drawn properly.
432. R: What do you mean "it wasn't drawn properly"?
433. N: No, not wasn't drawn properly...because when I was watching while Mya was drawing, I saw that when she drew it, it was black and black and black (the lines) while when she did it here it was different colours, it was red and blue.
434. R: What does the red mean?
435. M: It means south
436. N: South, west, east and north.
437. R: Let's do one more.
438. N: Red, blue, red....(They created the rectangle at the bottom in which the last line was black that's why it didn't get a colour)

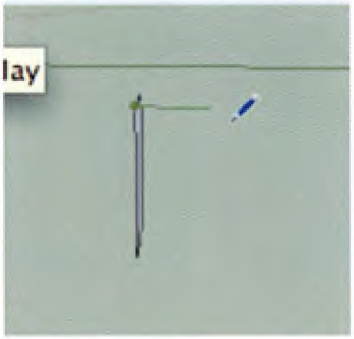
439. 
440. R: Why this is not coloured?
441. M: Because one of the lines was not coloured.
442. They started a new one next to it.
443. I drew a rectangle but by using three different colours of lines. So it looked perfect, created by coloured lines but it wasn't coloured!
444. R: Well, these are coloured lines but the rectangle is not coloured

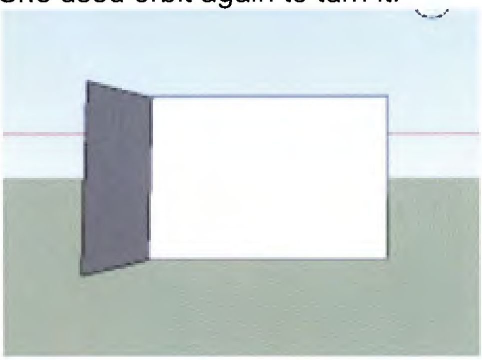
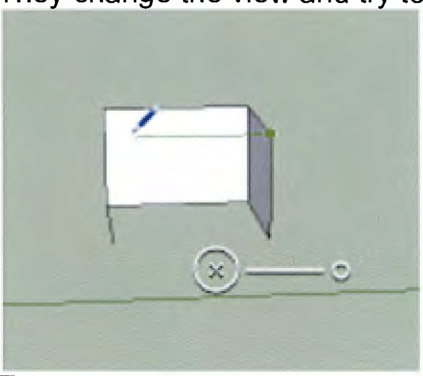
445. 

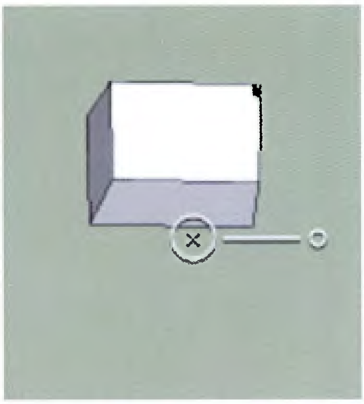
446. N: It's not symmetry! (He then uses orbit) See! It's bending.
447. R: Why? It was coloured (I mean the lines)!
448. N: I think the reason why, it's because they have to be two same colours. Because that's what happened with that (a previous coloured rectangle), it was red there and blue there.
449. R: Try to do it.
450. N: Blue, red, blue red! Yes! I've told you! (It turned a white colour)

451. 
452. M: So it has to be two of the same colour.
453. R: Do it with green and blue.
454. They did it.
455. N: I told you! I told you!
456. R: What do the colours have to do with the three lines (the axes)? Draw a green line. (They drew a green line). That's the relation between the green line and the line you drew?
457. M: Because it is going that way and that way…
458. R: You mean that if it was red, it would go that way? (showing the direction of the red axis). Try.
459. N: Yeap
460. They also drew a blue one which was going to the direction of the blue axis.
461. N: Yes, that's why.
462. They drew all the three lines: red, blue and green.
463. 
464. R: So, what are the two ways of creating a rectangle?
465. Both: With line and rectangle.

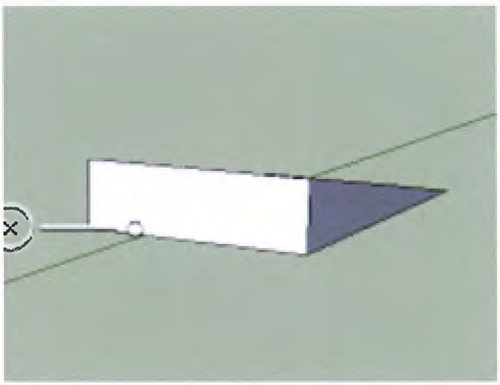
466. B. Create a cube.
467. R: Now your task is to create a cube.
468. They drew a square and then push/pull.
469. R: Well done. Try another way.
470. M: Do it with lines and then push/pull.
471. They drew a square with lines and then push/pull.
472. R: Now do it without using the push/pull tool.
473. M: I want to make a square but with the sides….
474. Mya makes a square with lines. Then she uses orbit to turn to the side view and from there to draw the other side:

475. 

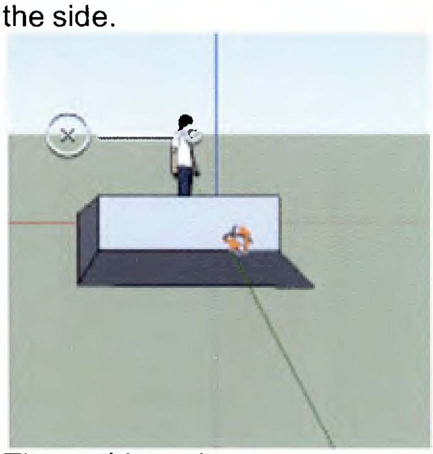
476. She used orbit again to turn it.
477. 
478. M: Two more sides.
479. She is facing a problem of how to estimate the new intersection point of the new sides.
480. [1:11:36]
481. They change the view and try to draw the sides:
482. 
483. They made the bottom part.



484.
485. N: Let's try the top one (part)
486. They tried to estimate the length of the lines and do the top bit and then completed the sides.
487. R: Well done. Now use just the rectangle tool to create a cube.
488. They created a flat dark grey rectangle. And then tried to bring a vertical side to it. They find a difficulty in defining the edges of the side that's why they cannot get it vertical but only horizontal. They finally made the side:



489.
490. They used orbit to turn it around and after some attempts to finally make the side.



491.
492. Then orbit again.
493. N: Yes, we have done it.
494. M: It is not actually a proper cube (it is a cuboids).
495. They made it.

496. R: Now, before you did it with the lines. What colour of lines did you use?
497. M: Red and green.
498. R: Red and green only to make the cube?
499. N: No, red, green and blue to make a cube.
500. [1:20:09]
501. R: So for making a cube you need 3 colours? Because if you remember for creating a rectangle you said you need 2 colours.
502. M: No, I think we need 2 colours. I remember when we did the first square I only used green and red…
503. R: Yes but when you created the cube?
504. M: We used all the different colours.

505. C. Axes: The three circles

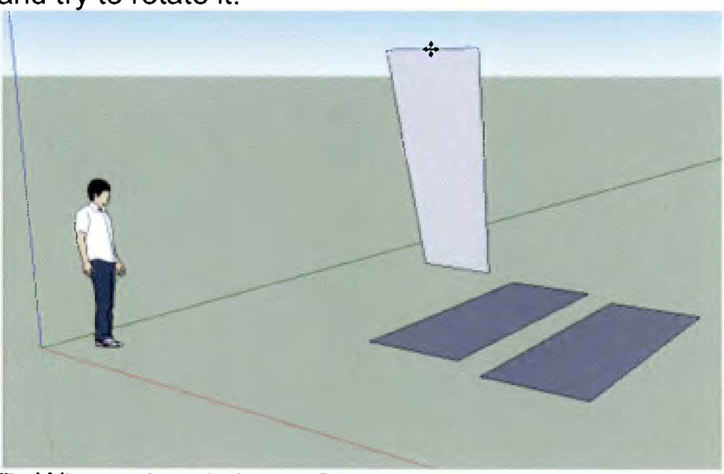
506. R: What do you see here?
507. Both: Circles.
508. R: In what are they similar?
509. M: They are in the air.
510. R: All of them?
511. M: These two are. I know something that is different about them. You know this one because it is at the bottom is darker, this one at the top is lighter and that one is kind of in the top and the middle it is kind of light dark.
512. R: What other differences are there?
513. M: This one looks longer because it is on the floor, this one looks round, that one looks smaller than that one. And, if you turn your head it looks like a face.
514. R: You said it looks longer because it is on the floor. Why?
515. M: Because…
516. N: It looks like the pond we drew.
517. M: Yeah, when you look at things, if you look at things and there are straight then it looks flat but if you turn it around then you can see all the different parts to it and that make it look bigger. (She is showing the movement by holding a 3D cup). If you look at this this way (front view) you think it is flat but if you look at this way then you see that it is going up and it is going down.
518. R: If I use the push/pull to them where will they go?
519. N: This one (the ground one) will go up in the air, this one (the middle one) sideways and that one (the top one) that way (showing the negative direction of the previous one)
520. R: So the same as the previous one?
521. M: No
522. N: I don't think so…
523. R: So one on the right and the other on the left? What do you mean?
524. M: Because that one (the top one) even though they are next to each other, if you turn this way a bit (turning herself) you could see that, that one is in front of that one.
525. R: So?
526. M: Actually I think that both would go the same way.

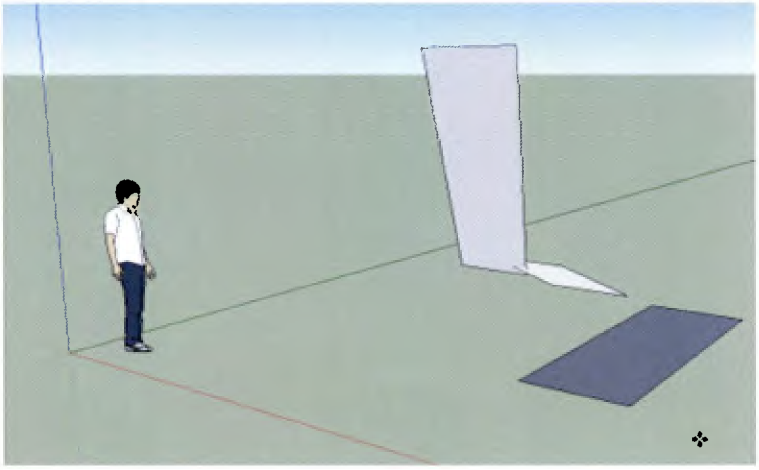
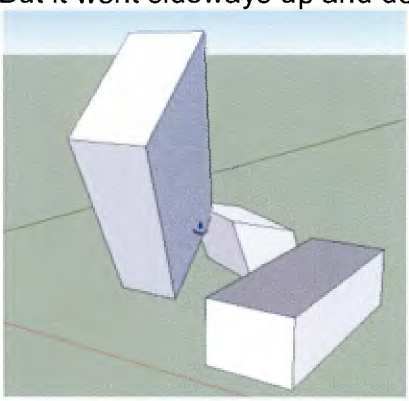
527. R: Let's see.
528. Use of push/pull.
529. R: Is that up (the ground one)?
530. Both: Yeah.
531. Then I do the top one.
532. N: I knew it! I knew it! And that one that way (the middle). I knew it!
533. M: I said that in the beginning!
534. R: And this can go that way as well (showing the negative directions to all the three of them). So, how many ways are there?
535. M: Three.
536. N: Six.
537. R: Which are the ways?
538. N: This way, that way, that way so five.
539. R: OK. It's up and down (showing them the cylinder) What other ways are there?
540. M: Forward and backwards
541. N: Left and right
542. R: So how many are they?
543. M: Six.
544. N: Actually they are six.

545. NEW TASK: Circles 2
546. R: Look at these. What do you see?
547. Both: Circles.
548. N: Circles in the air and on the ground. I think that some of them are not on the ground.
549. M: That one looks on the ground but it is actually up in the air.
550. R: Which are they going up with push/pull?
551. N: This one, this one that one and that one.
552. M: And that one (showing all the dark grey ones)
553. R: How do you know?
554. M: Because they look like it. It is like before with the mouse (the computer mouse). It looks flat but when you really look at it, it has some height.
555. R: But why this is going up? (The dark grey circle on the top)
556. M: I think because of the thinness, if it was turned then you know one way or the other way, but…and most of the time they would go up and down or left and right.
557. R: Which are they going left and right?
558. They show the white ones.
559. R: What I want you to do is to press the circle tool and hover around. Observe the colours.
560. Both: Red, green, blue...blue…
561. R: Which others are they going to be blue?
562. M: That one is going to be blue and that one is going to be blue (showing the other dark grey ones)

563. R: This? (showing a light grey one)
564. Both: Green.
565. R: And this? (showing a white one)
566. Both: Red.
567. N: And that's blue. (showing a dark grey one)
568. R: Show me a red.
569. They showed me.
570. R: Show me another red.
571. They showed it again.
572. R: Show me a green.
573. They showed me.
574. R: Show me another green.
575. They showed it again.
576. R: How do you know?
577. N: Because if you use orbit and turn it then you will see they are different. It is like the square…
578. R: Which square? The lines (the axes)?
579. M: Yeah, because if you imagine it, if you look at it properly, then you would you see that this line is going that way, and that line is the line of the edge…(
580. R: So is it like a square or a cube?
581. M: A cube.
582. R: Did you see the cube Nosakhare?
583. N: Em…
584. R: Mya explain the cube to him.
585. M: You see like…this is the bottom (showing the space between the green and the red axes). It is not a whole cube but a part of it. This is the middle (origin point) and then you go down down, down (across the green line) that's the edge, that's like down, that's the corner (showing the angle created by the blue and the green) and that's the other line (the red axis).
586. [1:30:45]
587. R: What about the colours? These are blue (showing them the dark grey circles). Where are they going with push/pull?
588. M: That way up.
589. N: Up.
590. I used the push/pull to show that indeed they went up.
591. R: What colour were these again?
592. Both: Blue.
593. R: What's the relation between blue and the axes.
594. M: Because the blue goes up
595. N: Straight up (showing the direction of the blue axis on the software)
596. M: And then the green…
597. R: Where does the green go?
598. M: That way (showing the green axis)
599. R: Where does the red go?
600. M: That way. (showing the red axis)
601. R: So, what did you learn from this?

602. M: About the X axis and about how the circles would go the same way..like forward and backwards...
603. R: How is the blue going?
604. Both: Up and down.
605. R: Like which line?
606. N: Like the straight
607. M: It can go north to south.
608. R: What about the green?
609. N: I think north-west.
610. M: No, west and east.
611. N: And the red is like diagonal lines.

612. NEW TASK: Turning rectangles 2
613. R: What do you see here?
614. N: Three squares on the ground.
615. R: Yes, three rectangles on the ground. Are they the same?
616. M: Yes, same size, same colour.
617. R: What colour are they?
618. Both: Black.
619. R: Press the move tool and press on the short side of the first rectangle and try to rotate it.
620. 
621. R: What colour is it now?
622. Both: Greyish.
623. R: Rotate the second one in the same way so that it gets a different colour than the other two.

624. 
625. R: Why do they have different colours?
626. M: Because they are in different angles.
627. N: Yes, different angles.
628. R: How is the colour formed?
629. M: Because of the light, so you know where the light is coming from.
630. R: If I use the push/pull on this where is it going to go? (the first right)
631. Both: Up or down.
632. R: This? (the middle one)
633. M: That way, the green way…
634. It didn't. It went up.
635. N: Oh..that's weird.
636. R: And that? (the third one)
637. M: This way
638. But it went sideways up and down.
639. 
640. **TASK 3: Exposing views about 1D and 0D worlds/objects**
641. R: What's the different now between this one and this one? (2D/3D neighbourhoods)
642. N: That one is 2D and that one is 3D.
643. R: What's the difference between 2D and 3D?
644. N: 3D you can actually see, touch, feel it, like hold and 2D is flat, surfaced.
645. M: 2D is the surface is not the actual thing.

646. R: How would 1D look like?
647. N: I've never heard of 1D before.
648. R: If it existed how would it be?
649. N: Nothing.
650. R: You know the difference between 2D and 3D. How would 1D be?
651. N: One side.
652. R: One side of what?
653. N: If 2D is flat...how is 1D...one side of the shape.
654. M: But this is just like 4D but it is not 4D. Do you know what 4D is? If you watch it in the cinema it is like this but you can hear it. I mean feel it. It is not just coming out to you, you can also feel it in the chair.
655. R: So what do you think 1D would be?
656. M: I think the sound, just this.
657. R: Just a sound? No shapes?
658. M: No shapes.
659. R: How would a shape in 1D look like? You said before it would be like a side...(I refer to Nosakhare)
660. N: One side of a shape.
661. R: And how would it look like? Would it be a triangle, a circle, a line?
662. N: If it is on one side of a shape and how would the other side be? So it cannot be one side of a shape...I would say...2D is flat, 1D must be like...[no answer]
663. R: How many lines did I use to create the 3D?
664. N: Lines?
665. R: The axes. How many lines/axes did I use (3D)? What colour of axes did I use?
666. N: You used like the green
667. M: And the red
668. N: And the blue...
669. R: I used the blue, the red...did I use the green as well?
670. M: Yes.
671. R: Here (in 2D), did I use the red?
672. M: Yes.
673. R: The green?
674. N: Yes.
675. R: The blue?
676. Both: Not the blue.
677. [1:40:28]
678. R: So, how many lines did I use here?
679. M: All of them.
680. R: So how many are they?
681. Both: Three.
682. R: Here? (2D)
683. Both: Two.
684. R: How many lines am I going to use for 1D?
685. N: One.
686. R: So how can it be one line?
687. M: I think it is going to be the green line because it gets pushed up.
688. R: What do you think?

689. N: Yes.
690. R: Just the green line?
691. Both: Yes.
692. N: Or maybe just the blue line. It has to be straight and not that bending so it has to be blue.
693. R: And how would the shapes look like in 1D?
694. M: Like that line. It would be that line on its own.
695. R: Just a line?
696. M: Just lines.
697. R: Just lines on the line?
698. Both: Yeah.
699. R: How would 0D be?
700. Both: Nothing.

701. TASK 4: Revising
702. R: So we talked about 3D, you said 4D, 2D, 1D, 0D. What is D?
703. Both: Dimension.
704. R: And what do we mean by dimension?
705. N: Sight, what you see, sight.
706. M: What you see.
707. R: What's the difference between a square and a cube?
708. N: A square you could see, a square is like flat, a square is a 2D shape. Cuboids is a cube, a cube is a 3D shape.
709. R: And what's the difference between the two?
710. M: A square has all the same sides but a rectangle has two of the same size and two not. A cube has all the same sides.
711. R: What shapes exist in this world? (2D)
712. M: Circles, triangles
713. N: Squares, rectangles, triangles.
714. R: What do they have in common?
715. M: They are all flat, and all 2D shapes
716. N: And all patterned and coloured.
717. R: What shapes exist in this world? (3D)
718. N: Cuboids.
719. M: All 3D shapes.
720. R: How can we create a line?
721. N: Just draw a line.
722. R: A square?
723. Both: Using 4 lines.
724. R: Or by using the software?
725. N: The rectangle tool.
726. R: How can we create a cube?
727. N: Lines.
728. M: Using rectangles.
729. N: Lines and push/pull.
730. R: Give me an example of a 2D shape.
731. N: Square
732. M: Circle
733. R: Give me an example of a 3D shape.

734. N: Cylinder
735. M: Sphere
736. R: Give me an example of a 2D world.
737. Both: That (my 2D neighbourhood example)
738. M: Books.
739. R: Give me an example of a 3D world.
740. M: The whole entire world.
741. N: All the planets, because they are all spheres.